Chapter 4

AN IMPROVED EXPONENTIAL ESTIMATOR OF
FINITE POPULATION MEAN IN SIMPLE RANDOM
SAMPLING
4.1 Introduction

As usual, \( \bar{Y} \) and \( \bar{X} \) denote the unknown population mean of study variable \( y \) and known population mean of auxiliary variable \( x \) respectively. Our aim is to estimate population mean \( \bar{Y} \). As considered in the previous chapters, take a simple random sample without replacement of size \( n \) from the given finite population of size \( N \). Again as usual, \( (\bar{y}, \bar{x}) \) denote the sample means of the bivariate \( (y, x) \).

For correlated variables \( y \) and \( x \), Watson (1937) suggested the ordinary regression estimator of \( \bar{Y} \) as

\[
\bar{y}_r = \bar{y} + b_{xy} (\bar{X} - \bar{x}), \tag{4.1}
\]

where \( b_{xy} \) is the regression coefficient of \( y \) on \( x \) in the sample.

For positively correlated variables \( y \) and \( x \), Cochran (1940) introduced the ordinary ratio estimator of \( \bar{Y} \) as

\[
\bar{y}_r = \bar{y} \frac{\bar{X}}{\bar{x}}. \tag{4.2}
\]

For negatively correlated variables \( y \) and \( x \), Robson (1957) and Murthy (1964) gave the ordinary product estimator of \( \bar{Y} \) as

\[
\bar{y}_p = \bar{y} \frac{\bar{X}}{\bar{x}}. \tag{4.3}
\]

After this fundamental work in the theory of survey sampling, researchers have started the task of generalizing these estimators. We are mentioning some of the generalizations of these estimators carried out by a number of authors in the following table:
Table 4.1: List of different estimators of $\bar{Y}$

<table>
<thead>
<tr>
<th>S. No.</th>
<th>Authors</th>
<th>Corresponding Estimators</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Srivastava (1967)</td>
<td>$\bar{y}_1 = \bar{y} \left( \frac{\bar{X}}{\bar{x}} \right)^\alpha$, $\alpha$ is arbitrary constant number.</td>
</tr>
<tr>
<td>2</td>
<td>Walsh (1970) and Reddy (1973)</td>
<td>$\bar{y}_2 = \bar{y} \frac{\bar{X}}{\alpha \bar{x} + (1-\alpha) \bar{X}}$, $\alpha$ is arbitrary constant number.</td>
</tr>
<tr>
<td>3</td>
<td>Srivastava (1971)</td>
<td>$\bar{y}_3 = \bar{y} h(u)$, with $u = \frac{\bar{x}}{\bar{X}}$ and $h(u)$ is a parametric function satisfying certain regularity conditions.</td>
</tr>
<tr>
<td>4</td>
<td>Chakrabarty (1979)</td>
<td>$\bar{y}_4 = (1-\alpha) \bar{y} + \alpha \bar{y} \frac{\bar{X}}{\bar{x}}$, $\alpha$ is arbitrary constant number.</td>
</tr>
<tr>
<td>5</td>
<td>Sahai (1979)</td>
<td>$\bar{y}_5 = \bar{y} \frac{\alpha \bar{X} + (1-\alpha) \bar{x}}{\alpha \bar{x} + (1-\alpha) \bar{X}}$, $\alpha$ is arbitrary constant number.</td>
</tr>
<tr>
<td>6</td>
<td>Srivenkataramana and Tracy (1979)</td>
<td>$\bar{y}<em>{6 (1)} = \bar{y} \left( \frac{1+\frac{n}{N}}{N} \right) \frac{\bar{X} - \frac{n}{N} \bar{x}}{\bar{X}}$ and $\bar{y}</em>{6 (2)} = \bar{y} \frac{n}{N} \bar{X} + \left( 1 - \frac{n}{N} \right) \frac{\bar{x}}{\bar{X}}$, where $y$ and $x$ are respectively positively and negatively correlated variables.</td>
</tr>
<tr>
<td>7</td>
<td>Srivastava (1980)</td>
<td>$\bar{y}_7 = h(\bar{y},u)$, with $u = \frac{\bar{x}}{\bar{X}}$ and $h(\bar{y},u)$ is a parametric function satisfying certain regularity conditions.</td>
</tr>
<tr>
<td>8</td>
<td>Srivenkataramana and Tracy (1980)</td>
<td>$\bar{y}_8 = \bar{y} \frac{\alpha - \bar{x}}{\alpha - \bar{X}}$, $\alpha$ is arbitrary constant number.</td>
</tr>
<tr>
<td>9</td>
<td>Srivenkataramana (1980)</td>
<td>$\bar{y}_9 = \bar{y} \frac{N \bar{X} - n \bar{x}}{(N-n) \bar{X}}$.</td>
</tr>
<tr>
<td>10</td>
<td>Ray and Sahai (1980)</td>
<td>$\bar{y}<em>{10 (i)} = \bar{y} \frac{\phi \bar{X} + \theta \bar{x}}{\bar{x} + (\phi + \theta - 1) \bar{X}}$ and $\bar{y}</em>{10 (ii)} = \bar{y} \frac{\phi \bar{X} + \theta \bar{x}}{\bar{x} + (\phi + \theta - 1) \bar{X}}$ and</td>
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\[
\bar{y}_{10(2)} = \bar{y} \left[ \frac{\phi \bar{x} + \theta \bar{X}}{\bar{x} + (\phi + \theta - 1)\bar{x}} \right], \quad \theta \text{ and } \phi \text{ are non-negative constants such that } 0 \leq \theta < 1.
\]

<table>
<thead>
<tr>
<th>11</th>
<th>Ray and Singh (1980)</th>
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<tbody>
<tr>
<td></td>
<td>[ \bar{y}_{11(i)} = \left[ \bar{y} + \bar{x}^{-\frac{1}{2}} - \bar{X}^{-\frac{1}{2}} \right] \left( \frac{\bar{X}}{\bar{x}} \right) \text{ and} ]</td>
</tr>
<tr>
<td></td>
<td>[ \bar{y}_{11(2)} = \left[ \bar{y} + \bar{x}^{-\frac{1}{2}} - \bar{X}^{-\frac{1}{2}} \right] \left( \frac{\bar{X}}{\bar{x}} \right) ]</td>
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<tr>
<th>12</th>
<th>Pandey (1980)</th>
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<tbody>
<tr>
<td></td>
<td>[ \bar{y}_{12} = \bar{y} \left[ \theta \left( \frac{\bar{X}}{\bar{x}} \right)^{\alpha} + (1-\theta) \left( \frac{\bar{x}}{\bar{X}} \right) \right], \quad \theta \text{ and } \alpha \text{ are arbitrary constant numbers.} ]</td>
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<tr>
<td></td>
<td>[ \bar{y}_{13} = (1-\alpha) \bar{y} + \alpha \frac{\bar{x}}{\bar{X}}, \quad \alpha \text{ is arbitrary constant number.} ]</td>
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<thead>
<tr>
<th>14</th>
<th>Bedi and Hajela (1984)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>[ \bar{y}_{14} = \alpha \left[ \bar{y} + \beta^* \left( \bar{X} - \bar{x} \right) \right], \quad \beta^* \text{ is regression coefficient of } y \text{ on } x \text{ in population and } \alpha \text{ is arbitrary constant number.} ]</td>
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<tr>
<th>15</th>
<th>Kaur (1985) and Prasad (1986)</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>[ \bar{y}_{15} = \alpha \bar{y} + \beta^* \left( \bar{X} - \bar{x} \right), \quad \beta^* \text{ is regression coefficient of } y \text{ on } x \text{ in population and } \alpha \text{ is arbitrary constant number.} ]</td>
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<thead>
<tr>
<th>16</th>
<th>Jain (1987)</th>
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<tbody>
<tr>
<td></td>
<td>[ \bar{y}_{16} = \alpha \bar{y} + (1-\alpha) \left( \bar{X} - \bar{x} \right), \quad \alpha \text{ is arbitrary constant number.} ]</td>
</tr>
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<tr>
<th>17</th>
<th>Singh and Shukla (1987)</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>[ \bar{y}_{17} = \bar{y} \left[ \frac{(A_i + C_i) \bar{x} + f' B_i \bar{x}}{(A_i + f' B_i) \bar{x} + C_i \bar{x}} \right], \quad f' = \frac{n}{N}, ]</td>
</tr>
<tr>
<td></td>
<td>[ A_i = (d-1)(d-2), B_i = (d-1)(d-4), ]</td>
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<tr>
<td></td>
<td>[ C_i = (d-2)(d-3)(d-4) \text{ and } d \text{ is a non-negative arbitrary constant number.} ]</td>
</tr>
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<table>
<thead>
<tr>
<th></th>
<th>Authors</th>
<th>Year</th>
<th>Equation Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>18</td>
<td>Sahoo and Swain</td>
<td>1987</td>
<td>$\bar{Y}_{18(1)} = \bar{Y}\left(\frac{\bar{X}}{\alpha}\right)^{\hat{\alpha}}$, and</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>$\bar{Y}_{18(2)} = \bar{Y}\frac{\bar{X}}{\hat{\alpha}\bar{x} + (1 - \hat{\alpha})\bar{X}}$, $\hat{\alpha}$'s are respective consistent estimators of suitable values of constant $\alpha$ for estimators $\bar{Y}_1$ and $\bar{Y}_2$.</td>
</tr>
<tr>
<td>19</td>
<td>Sampath and Durairajan</td>
<td>1988</td>
<td>$\bar{Y}_{19} = \left[\bar{Y} + \bar{X}^\alpha - \bar{X}^\alpha\right] \left(\frac{\bar{X}}{\bar{x}}\right)$, $\alpha$ is arbitrary constant number.</td>
</tr>
<tr>
<td>20</td>
<td>Rao</td>
<td>1991</td>
<td>$\bar{Y}_{20} = \alpha\bar{Y} + \beta(\bar{X} - \bar{x})$, $\alpha$ and $\beta$ are arbitrary constant numbers.</td>
</tr>
<tr>
<td>21</td>
<td>Bahl and Tuteja</td>
<td>1991</td>
<td>$\bar{Y}<em>{21(1)} = \bar{Y}</em>{Re} = \bar{Y} \exp\left(\frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}}\right)$ and</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\bar{Y}<em>{21(2)} = \bar{Y}</em>{pe} = \bar{Y} \exp\left(\frac{\bar{X} - \bar{X}}{\bar{X} + \bar{x}}\right)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\left(\bar{Y}<em>{Re}$ and $\bar{Y}</em>{pe}$ are the same estimators as already defined in chapters 2 and 3 respectively)</td>
</tr>
<tr>
<td>22</td>
<td>Tripathi and Singh</td>
<td>1992</td>
<td>$\bar{Y}_{22} = \bar{Y} \frac{\alpha + \beta\bar{x}}{\alpha + \beta\bar{X}}$, $\alpha$ and $\beta$ are arbitrary constant numbers.</td>
</tr>
<tr>
<td>23</td>
<td>Dubey and Singh</td>
<td>2001</td>
<td>$\bar{Y}_{23} = \alpha\bar{Y} + \beta\bar{X} + (1 - \alpha - \beta)\bar{X}$, $\alpha$ and $\beta$ are arbitrary constant numbers.</td>
</tr>
<tr>
<td>24</td>
<td>Singh and Espejo</td>
<td>2003</td>
<td>$\bar{Y}_{24} = \bar{Y} \left[\alpha\frac{\bar{X}}{\bar{x}} + (1 - \alpha)\frac{\bar{x}}{\bar{X}}\right]$, $\alpha$ is arbitrary constant number.</td>
</tr>
<tr>
<td>25</td>
<td>Khoshnevisan et al.</td>
<td>2007</td>
<td>$\bar{Y}_{25} = \bar{Y} \left[\frac{a\bar{X} + b}{\alpha(a\bar{x} + b) + (1 - \alpha)(a\bar{X} + b)}\right]^\gamma$, $a(\neq 0)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>and $b$ are either known real numbers or function of known parameters of auxiliary variable $x$, $\gamma$ is a known constant and $\alpha$ is an arbitrary constant number.</td>
</tr>
<tr>
<td>26</td>
<td>Singh et al.</td>
<td>2008</td>
<td>$\bar{Y}_{26} = \bar{Y} \left[\alpha \exp\left(\frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}}\right) + (1 - \alpha) \exp\left(\frac{\bar{x} - \bar{X}}{\bar{x} + \bar{X}}\right)\right]$, $\alpha$ is arbitrary constant number.</td>
</tr>
</tbody>
</table>
Note that all estimators considered above (except $\bar{y}_{14}$ and $\bar{y}_{15}$) utilize exactly the same information of auxiliary variable $x$ in the form of its known population mean $\bar{X}$. The estimators $\bar{y}_{14}$ and $\bar{y}_{15}$ utilized information of population mean of auxiliary variable and population regression coefficient of $y$ on $x$. It should also be noted that the estimators $\bar{y}_{14}$, $\bar{y}_{15}$, $\bar{y}_{16}$, $\bar{y}_{20}$ and $\bar{y}_{23}$ are not particular members of the general class of estimators $\bar{y}_7 = h(\bar{y}, u)$ because they do not satisfy one of regularity condition $h(\bar{Y}, 1) = \bar{Y}$ for this class. On the other hand, all remaining estimators of $\bar{Y}$ considered in this chapter are the particular members of general class of estimators $\bar{y}_7 = h(\bar{y}, u)$ because they satisfy all regularity conditions for this class. It was proved by the researchers that most of the existing estimators considered in this chapter are either less efficient or at the most equally efficient as compared to the ordinary regression estimator $\bar{y}_b$, up to first order of approximation. It was also shown that regression type estimators $\bar{y}_{14}$, $\bar{y}_{15}$ and $\bar{y}_{20}$ are always more efficient than ordinary regression estimator $\bar{y}_b$, up to first order of approximation. Whereas, the regression type estimators $\bar{y}_{16}$ and $\bar{y}_{23}$ are more efficient than ordinary regression estimator $\bar{y}_b$, under certain conditions, up to first order of approximation.

When the population mean $\bar{X}$ of auxiliary variable $x$ is known in advance then we proposed an improved regression type exponential estimator of $\bar{Y}$ in the next section i.e. Section 4.2 of the present chapter. The expressions of bias and mean square error of the proposed estimator are obtained as in Section 4.2. It has been found that the proposed estimator is further more efficient than most of all the existing estimators, up to first order of approximation, as discussed in Section 4.3. Theoretical results obtained have been verified by taking some empirical populations from the literature, in the last section i.e. Section 4.4 of this chapter.
4.2 Proposed estimator and its properties

If the population mean $\bar{X}$ of auxiliary variable $x$ is known in advance then on getting motivation from Rao (1991) and Bahl and Tuteja (1991), we propose the following ratio cum regression type exponential estimator of $\bar{Y}$:

$$\hat{y}_o = \left[ \omega_1 \bar{y} + \omega_2 \left( \bar{X} - \bar{x} \right) \right] \exp \left( \frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}} \right),$$

(4.4)

where $\omega_1$ and $\omega_2$ are any constants and their values are suitably chosen.

For $\omega_1 = 1$ and $\omega_2 = 0$, the proposed estimator $\hat{y}_o$ reduces to the estimator $\bar{y}_{21}$ of $\bar{y}$. On deleting the exponential term from the proposed estimator $\hat{y}_o$, it reduces to the estimator $\bar{y}_{20}$.

As usual, we take $\varepsilon = \frac{\bar{y}}{Y} - 1$ and $\eta = \frac{\bar{X}}{X} - 1$. Under simple random sampling without replacement (SRSWOR), we have the following expectations:

$$E(\varepsilon) = E(\eta) = 0, \quad E(\varepsilon^2) = fC_x^2, \quad E(\eta^2) = fC_x^2, \quad E(\varepsilon\eta) = f\rho_{xy}C_xC_x,$$

where $f = \frac{1}{n} - \frac{1}{N}$ and other notations are same as that of previous chapters.

The proposed estimator $\hat{y}_o$ can be rewritten as

$$\hat{y}_o = \left[ \omega_1 \bar{Y} (1 + \varepsilon) - \omega_2 \bar{X} \bar{\eta} \right] \exp \left[ -\frac{\eta}{2} \left( 1 + \frac{\eta}{2} \right)^{\frac{1}{2}} \right].$$

(4.5)

On retaining only the terms up to second degree of $\varepsilon^2$ and $\eta^2$, we have

$$\hat{y}_o = \left[ \omega_1 \bar{Y} (1 + \varepsilon) - \omega_2 \bar{X} \bar{\eta} \right] \exp \left[ -\frac{1}{2} \eta + \frac{3}{8} \eta^2 \right].$$

(4.6)

Again on retaining only terms up to second degrees of $\varepsilon^2$ and $\eta^2$, we have...
\[
\hat{y}_o - \bar{Y} = \bar{Y} \left[ (\omega_1 - 1) + \omega_1 \epsilon - \frac{\omega_1}{2} (\eta + \epsilon \eta) + \frac{3}{8} \omega_1 \eta^2 \right] + \omega_2 \bar{X} \left( \frac{\eta^2}{2} - \eta \right). \tag{4.7}
\]

Thus, up to first order of approximation, bias of the proposed estimator is

\[
\text{Bias} (\hat{y}_o) = E (\hat{y}_o) - \bar{Y}
= \bar{Y} \left[ (\omega_1 - 1) + f \omega_1 \frac{C_x}{2} \left( \frac{3}{4} C_s - \rho_{y_s} C_y \right) \right] + \omega_2 \bar{X} f \frac{C_s^2}{2}, \quad \text{(on using (4.7))} \tag{4.8}
\]

Again, up to first order of approximation, mean square error of the proposed estimator is

\[
\text{MSE} (\hat{y}_o) = E (\hat{y}_o - \bar{Y})^2
= \bar{Y}^2 \left[ (\omega_1 - 1)^2 + \omega_1^2 E (\epsilon^2) + \frac{1}{4} \omega_1^2 E (\eta^2) - \omega_1 (\omega_1 - 1) E (\epsilon \eta) + \frac{3}{4} \omega_1 (\omega_1 - 1) E (\eta^2) \right. \\
\left. - \omega_1^2 E (\epsilon \eta) \right]
+ \omega_2^2 \bar{X}^2 E (\eta^2) + 2 \omega_2 \bar{X} \bar{Y} \left[-\omega_1 E (\epsilon \eta) + \frac{\omega_1}{2} E (\eta^2) + \frac{1}{2}(\omega_1 - 1) E (\eta^3) \right].
\]

(on using result (4.7) and retaining terms only up to second degrees of \( \epsilon^s \) and \( \eta^s \))

\[
\therefore \text{MSE} (\hat{y}_o) = \bar{Y}^2 \left[ (\omega_1 - 1)^2 + \omega_1^2 f L_4 - \frac{1}{2} f \omega_1 \left( L_2 + \frac{C^2_s}{2} - \rho_{y_s} C_y C_s \right) \right] + \omega_2^2 \bar{X}^2 f C_s^2
\]
\[
+ 2 \omega_2 \bar{X} \bar{Y} f \left( \omega_1 L_2 - \frac{C_s^2}{2} \right), \tag{4.9}
\]

where \( L_4 = C_y^2 + C_x^2 - 2 \rho_{y_s} C_y C_x \) and \( L_2 = C_x^2 - \rho_{y_s} C_y C_x \).
Theorem 4.2.1: The $MSE(\hat{Y}_o)$ is optimum (or minimum) for

$$\omega_1 = \omega_1^* = \omega_{1(\text{opt})} = -C_s^2 \left[ 2 - \frac{1}{2} fL_2 + \frac{f}{2} \left( \frac{C_s^2}{2} - \rho_{ys} C_y C_s \right) \right] \left[ 2fL_2 - (1 + fL_4) C_s^2 \right],$$

(4.10)

$$\omega_2 = \omega_2^* = \omega_{2(\text{opt})} = \frac{\bar{Y} \left[ L_2 \left[ 2 + \frac{1}{2} fL_2 + \frac{f}{2} \left( \frac{C_s^2}{2} - \rho_{ys} C_y C_s \right) \right] - C_s^2 (1 + fL_4) \right]}{2\bar{X} \left[ fL_2 - (1 + fL_4) C_s^2 \right]},$$

(4.11)

and its minimum value, up to first order of approximation, is given by

$$\text{Min.}MSE(\hat{Y}_o) = \frac{f\bar{Y}^2 C_s^2 \left( 1 - \rho_{ys}^2 \right)}{1 + fC_s^2 \left( 1 - \rho_{ys}^2 \right)} - \frac{f^2 \bar{Y}^2 C_s^2 \left[ 4C_y^2 (1 - \rho_{ys}^2) + \frac{C_s^2}{4} \right]}{16 \left[ 1 + fC_s^2 \left( 1 - \rho_{ys}^2 \right) \right]].$$

(4.12)

Proof: For proof of this theorem, see Appendix A4.1.

Remarks 4.2.1: When $n=N$, that is, $\bar{y} = \bar{Y}$ and $\tau = \bar{X}$, then $\omega_{1(\text{opt})} = 1$ and

$$\omega_{2(\text{opt})} = \frac{-\bar{Y} \left( C_s - 2 \rho_{ys} C_y \right)}{2\bar{X}C_s},$$

which implies further that $\hat{Y}_o = \bar{Y}$. Thus the proposed estimator $\hat{Y}_o$ is a consistent estimator of $\bar{Y}$, under the optimal conditions.

Remarks 4.2.2: The expression for $\text{Min.}MSE(\hat{Y}_o)$ may be rewritten in the following form:

$$\text{Min.}MSE(\hat{Y}_o) = \frac{\text{MSE}(\bar{y}_r)}{1 + \frac{\text{MSE}(\bar{y}_r)}{\bar{Y}^2}} - \frac{fC_s^2 \left[ \text{MSE}(\bar{y}_r) + \frac{f\bar{Y}^2 C_s^2}{16} \right]}{4 \left[ 1 + \frac{\text{MSE}(\bar{y}_r)}{\bar{Y}^2} \right]},$$

$$= \text{MSE}(\bar{y}_r) - \frac{\left[ \text{MSE}(\bar{y}_r) \right]^2}{1 + \frac{\text{MSE}(\bar{y}_r)}{\bar{Y}^2}} \frac{fC_s^2 \left[ \text{MSE}(\bar{y}_r) + \frac{f\bar{Y}^2 C_s^2}{16} \right]}{4 \left[ 1 + \frac{\text{MSE}(\bar{y}_r)}{\bar{Y}^2} \right]}.$$
where

\[ MSE (\bar{y}_{lr}) = fY^2 C_\gamma^2 (1 - \rho_{ys}^2) \]  

is mean square error of ordinary regression estimator \( \bar{y}_{lr} \) of \( \bar{Y} \), up to first order of approximation, and

\[ M_1 = \frac{\left[ MSE (\bar{y}_{lr}) \right]^2}{Y^2} > 0 \quad \text{and} \quad M_2 = \frac{fC_\gamma^2 \left[ MSE (\bar{y}_{lr}) + \frac{fY^2 C_\gamma^2}{16} \right]}{4 \left[ 1 + \frac{MSE (\bar{y}_{lr})}{Y^2} \right]} > 0. \]  

**Remarks 4.2.3:** Since optimum values \( \omega_{1_{\text{opt}}} \) and \( \omega_{2_{\text{opt}}} \), as obtained in (4.10) and (4.11) respectively, are functions of unknown population quantities, in general, so we can estimate these optimum values just by replacing unknown population quantities with their respective analogous consistent estimators based on the same sample. Let \( \hat{\omega}_{1_{\text{opt}}} \) and \( \hat{\omega}_{2_{\text{opt}}} \) are estimated values of \( \omega_{1_{\text{opt}}} \) and \( \omega_{2_{\text{opt}}} \) respectively which can be obtained as discussed above. Now the proposed regression type exponential optimum estimator of \( \bar{Y} \) will take following form:

\[ \hat{Y}_{O_{\text{opt}}} = \left[ \hat{\omega}_{1_{\text{opt}}} \bar{Y} + \hat{\omega}_{2_{\text{opt}}} (\bar{X} - \bar{x}) \right] \exp \left( \frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}} \right). \]  

Srivastava and Jhajj (1983) have shown that on replacing unknown population quantities in the optimum values of constants of an estimator of interest with their respective consistent estimator, mean square error of the estimator of interest remains the same, up to the terms of order \( n^{-1} \) (or up to first order of approximation). Obviously, approximate mean square error of proposed ratio cum regression type exponential optimum estimator of \( \bar{Y} \), as obtained in (4.16), has the same mean square error as given in (4.13) and so it is given by

\[ MSE \left( \hat{Y}_{O_{\text{opt}}} \right) = MSE (\bar{y}_{lr}) - M_1 - M_2. \]  

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4.3 Comparison of proposed estimator with existing ones

To compare efficiencies of various existing estimators considered in this chapter with that of the proposed estimator, we require expressions of mean square errors of these estimators, up to first order of approximation.

We noted that

\[
MSE(\overline{y}_l) = Min.MSE(\overline{y}_1) = Min.MSE(\overline{y}_2) = Min.MSE(\overline{y}_3) = Min.MSE(\overline{y}_4) \\
= Min.MSE(\overline{y}_5) = Min.MSE(\overline{y}_7) = Min.MSE(\overline{y}_8) = Min.MSE(\overline{y}_{13}) = Min.MSE(\overline{y}_{17}) \\
= Min.MSE(\overline{y}_{18 \ (1)}) = Min.MSE(\overline{y}_{18 \ (2)}) = Min.MSE(\overline{y}_{24}) = Min.MSE(\overline{y}_{25}) = Min.MSE(\overline{y}_{26})
\]

(4.18)

Now we shall take following differences of mean square errors.

\[
\text{Var}(\overline{y}) - MSE(\hat{y}_{\theta_{opt}}) = fY^2 \rho_{x^2}^2 + M_1 + M_2 > 0, \text{ always.} \quad (4.19)
\]

\[
MSE(\overline{y}_R) - MSE(\hat{y}_{\theta_{opt}}) = fY^2 (\rho_{x^2} C_y - C_x)^2 + M_1 + M_2 > 0, \text{ always.} \quad (4.20)
\]

\[
MSE(\overline{y}_P) - MSE(\hat{y}_{\theta_{opt}}) = fY^2 (\rho_{x^2} C_y + C_x)^2 + M_1 + M_2 > 0, \text{ always.} \quad (4.21)
\]

\[
MSE(\overline{y}_l) - MSE(\hat{y}_{\theta_{opt}}) = M_1 + M_2 > 0, \text{ always.} \quad (4.22)
\]

\[
MSE(\overline{y}_{6 \ (1)}) - MSE(\hat{y}_{\theta_{opt}}) = fY^2 \left[ \rho_{x^2 C_y - \left( \frac{n}{N} \right) C_x} \right]^2 + M_1 + M_2 > 0, \text{ always.} \quad (4.23)
\]

\[
MSE(\overline{y}_{6 \ (2)}) - MSE(\hat{y}_{\theta_{opt}}) = fY^2 \left[ \rho_{x^2 C_y + \left( \frac{1 - n}{N} \right) C_x} \right]^2 + M_1 + M_2 > 0, \text{ always.} \quad (4.24)
\]

\[
MSE(\overline{y}_9) - MSE(\hat{y}_{\theta_{opt}}) = fY^2 \left[ \rho_{x^2 C_y - \left( \frac{n}{N - n} \right) C_x} \right]^2 + M_1 + M_2 > 0, \text{ always.} \quad (4.25)
\]

\[
MSE(\overline{y}_{10 \ (1)}) - MSE(\hat{y}_{\theta_{opt}}) = fY^2 \left[ \rho_{x^2 C_y - \left( \frac{1 - \theta}{K + \theta} \right) C_x} \right]^2 + M_1 + M_2 > 0, \text{ always.}
\]
\( \forall \phi \text{ and } \theta \), always.  \hfill (4.26)

\[
MSE\left( \bar{y}_{10(2)} \right) - MSE\left( \hat{y}_{O(\text{opt})} \right) = fY^2 \left[ \rho_{yx} C_y + \left( \frac{1 - \theta}{K + \theta} \right) C_x \right]^2 + M_1 + M_2 > 0, \quad \forall \phi \text{ and } \theta, \text{ always.} \hfill (4.27)
\]

\[
MSE\left( \bar{y}_{11(1)} \right) - MSE\left( \hat{y}_{O(\text{opt})} \right) = f \left[ \rho_{yx} \bar{Y} C_y - \left( \bar{Y} - \frac{1}{2} \bar{X}^2 \right) C_x \right]^2 + M_1 + M_2 > 0, \text{ always.} \hfill (4.28)
\]

\[
MSE\left( \bar{y}_{11(2)} \right) - MSE\left( \hat{y}_{O(\text{opt})} \right) = f \left[ \rho_{yx} \bar{Y} C_y - \left( \bar{Y} + \frac{1}{2} \bar{X}^2 \right) C_x \right]^2 + M_1 + M_2 > 0, \text{ always.} \hfill (4.29)
\]

\[
MSE\left( \bar{y}_{12} \right) - MSE\left( \hat{y}_{O(\text{opt})} \right) = f\tilde{Y}^2 \left[ \rho_{yx} C_y + \left\{ 1 - \theta (\alpha + 1) \right\} C_x \right]^2 + M_1 + M_2 > 0, \hfill (4.30)
\]

\[
\text{Min.MSE}\left( \bar{y}_{14} \right) - MSE\left( \hat{y}_{O(\text{opt})} \right) = \text{Min.MSE}\left( \bar{y}_{24} \right) - MSE\left( \hat{y}_{O(\text{opt})} \right) = M_2 > 0, \text{ always.} \hfill (4.31)
\]

\[
\text{Min.MSE}\left( \bar{y}_{15} \right) - MSE\left( \hat{y}_{O(\text{opt})} \right) = \left[ MSE\left( \bar{y}_{y} \right) \right]^2 \left[ \frac{f \rho^2 C_y^2}{\left\{ 1 + f C_y^2 \left( 1 - \rho_{yx}^2 \right) \right\} \left\{ 1 + f C_y^2 \right\} } \right] + M_2 > 0, \text{ always.} \hfill (4.32)
\]

\[
\text{Min.MSE}\left( \bar{y}_{16} \right) - MSE\left( \hat{y}_{O(\text{opt})} \right) = MSE\left( \bar{y}_{b} \right) \left[ \frac{S_y^2}{S_x^2 + S_y^2 + 2 \rho_{yx} S_x S_y} - 1 \right] + M_1 + M_2 > 0, \quad \text{for } \rho_{yx} < -\frac{S_y}{2S_x}. \hfill (4.33)
\]

\[
MSE\left( \bar{y}_{19} \right) - MSE\left( \hat{y}_{O(\text{opt})} \right) = f \left[ \rho_{yx} \bar{Y} C_y - \left( \bar{Y} - a \bar{X}^a \right) C_x \right]^2 + M_1 + M_2 > 0, \quad \forall a, \text{ always} \hfill (4.34)
\]
\[ \text{MSE}\left(\bar{y}_{21}^{(1)}\right) - \text{MSE}\left(\hat{\bar{y}}_{\text{opt}}\right) = fY^2\left(\rho_{xy}C_y - \frac{C_x}{2}\right)^2 + M_1 + M_2 > 0, \text{ always.} \quad (4.35) \]

\[ \text{MSE}\left(\bar{y}_{21}^{(2)}\right) - \text{MSE}\left(\hat{\bar{y}}_{\text{opt}}\right) = fY^2\left(\rho_{xy}C_y + \frac{C_x}{2}\right)^2 + M_1 + M_2 > 0, \text{ always.} \quad (4.36) \]

\[ \text{MSE}\left(\bar{y}_{22}\right) - \text{MSE}\left(\hat{\bar{y}}_{\text{opt}}\right) = fY^2\left[\rho_{xy}C_y + \left(\frac{bX}{a+bX}\right)C_x\right]^2 + M_1 + M_2 > 0, \forall a \text{ and } b, \]

\[ \text{always.} \quad (4.37) \]

\[ \text{Min.MSE}\left(\bar{y}_{23}\right) - \text{MSE}\left(\hat{\bar{y}}_{\text{opt}}\right) = \frac{\left[MSE\left(\bar{y}_{lr}\right)\right]^2\left[1 - \frac{1}{\bar{Y}^2 - \bar{X}}\right]}{\left[1 + \frac{MSE\left(\bar{y}_{lr}\right)}{\bar{Y}^2}\right]\left[1 + \frac{MSE\left(\bar{y}_{lr}\right)}{(\bar{Y} - \bar{X})^2}\right]} + M_2 > 0, \]

\[ \text{for } \bar{X} > 2\bar{Y} > 0. \quad (4.38) \]

From results (4.18) to (4.38), we can see that the proposed estimator has lesser mean square error as compared to that of most of the existing estimators. So the proposed estimator \( \hat{\bar{y}}_{\text{opt}} \) is always more efficient than all existing estimators except estimators \( \bar{y}_{16} \) and \( \bar{y}_{23} \). The superiority of the proposed estimator over these two estimators \( \bar{y}_{16} \) and \( \bar{y}_{23} \) is conditional as shown in results (4.33) and (4.38), respectively.

### 4.3 Numerical illustration

To study the relative performance of different estimators considered in this chapter, we have taken eight empirical populations used by other authors in the literature. The source of population and the values of requisite population parameters are given in Table 4.2.

The percent relative efficiencies (PREs) of all estimators considered in this chapter for all the eight populations are given in Table 4.3.
Table 4.2: Parameters of the populations

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Serial Number of Populations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>I</td>
</tr>
<tr>
<td>N</td>
<td>10</td>
</tr>
<tr>
<td>n</td>
<td>4</td>
</tr>
<tr>
<td>( \bar{Y} )</td>
<td>52</td>
</tr>
<tr>
<td>( \bar{X} )</td>
<td>200</td>
</tr>
<tr>
<td>( C_y )</td>
<td>0.1562</td>
</tr>
<tr>
<td>( C_x )</td>
<td>0.0458</td>
</tr>
<tr>
<td>( \rho_{yx} )</td>
<td>-0.94</td>
</tr>
</tbody>
</table>

Note: For Populations No. IV and V, we interchange the variables \( y \) and \( x \) as given in the source of population. For the remaining populations, we take the same variables as given in source of population.
### Table 4.3: PREs of estimators

<table>
<thead>
<tr>
<th>Estimators</th>
<th>PREs of estimators for population</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>I</td>
</tr>
<tr>
<td>$\bar{y}$</td>
<td>100.00</td>
</tr>
<tr>
<td>$\bar{y}_R$</td>
<td>61.0793</td>
</tr>
<tr>
<td>$\bar{y}_P$</td>
<td>187.0094</td>
</tr>
<tr>
<td>$\bar{y}_{br}$</td>
<td>859.1065</td>
</tr>
<tr>
<td>$\bar{y}_{6 (1)}$</td>
<td>81.0207</td>
</tr>
<tr>
<td>$\bar{y}_{6 (2)}$</td>
<td>142.8152</td>
</tr>
<tr>
<td>$\bar{y}_9$</td>
<td>71.13866</td>
</tr>
<tr>
<td>$\bar{y}_{11 (1)}$</td>
<td>64.91546</td>
</tr>
<tr>
<td>$\bar{y}_{11 (2)}$</td>
<td>61.0609</td>
</tr>
<tr>
<td>$\bar{y}_{14}$</td>
<td>859.4725</td>
</tr>
<tr>
<td>$\bar{y}_{15}$</td>
<td>859.4713</td>
</tr>
<tr>
<td>$\bar{y}_{16}$</td>
<td>102.4383</td>
</tr>
<tr>
<td>$\bar{y}_{21 (1)}$</td>
<td>77.0942</td>
</tr>
<tr>
<td>$\bar{y}_{21 (2)}$</td>
<td>134.0712</td>
</tr>
<tr>
<td>$\bar{y}_{23}$</td>
<td>859.1517</td>
</tr>
<tr>
<td>$\hat{y}_{O(opt)}$</td>
<td>859.5432</td>
</tr>
</tbody>
</table>
Interpretation and conclusion:

From Table 4.3, we have made the following observations:

1. The proposed estimator \( \hat{\bar{y}}_{O\text{opt}} \) is more efficient than all the estimators except estimators \( \bar{y}_{16} \) and \( \bar{y}_{23} \) in all the eight populations (Sr. No. I to VIII).

2. The conditions of results (4.33) and (4.38), are satisfied by first five populations (Sr. No. I to V), so the proposed estimator \( \hat{\bar{y}}_{O\text{opt}} \) becomes more efficient than estimators \( \bar{y}_{16} \) and \( \bar{y}_{23} \) for these five populations.

3. For the population No. VI, the condition of result (4.38) is satisfied but the condition of result (4.33) is not satisfied. So, the proposed estimator \( \hat{\bar{y}}_{O\text{opt}} \) is also more efficient than estimator \( \bar{y}_{23} \) but less efficient than estimator \( \bar{y}_{16} \) for this population.

4. For population No. VII, condition of result (4.33) is satisfied but condition of result (4.38) is not satisfied. So, the proposed estimator \( \hat{\bar{y}}_{O\text{opt}} \) is also more efficient than estimator \( \bar{y}_{16} \) but less efficient than estimator \( \bar{y}_{23} \) for this population.

5. For the population No. VIII, conditions of both the results (4.33) and (4.38) are not satisfied so the proposed estimator \( \hat{\bar{y}}_{O\text{opt}} \) does not become more efficient than both estimators \( \bar{y}_{16} \) and \( \bar{y}_{23} \) for this population.

Thus, we recommend the usage of our proposed estimator \( \hat{\bar{y}}_{O\text{opt}} \) to estimate unknown population mean \( \bar{Y} \) when population mean \( \bar{X} \) is known in advance and the very simple conditions of results (4.33) and (4.38) are satisfied.
Appendix A4.1

Proof of Theorem 4.2.1

To minimize \( \text{MSE} (\hat{y}_o) \), we have the following two normal equations:

\[
\frac{\partial}{\partial \omega_i} \{ \text{MSE} (\hat{y}_o) \} = 0 ; i = 1,2.
\]

Differentiating partially (4.9) with respect to \( \omega_1 \) and equating to zero, we have

\[
2\bar{Y} \omega_1 (1 + fL_1) + 2\bar{X} \omega_2 fL_2 = \bar{Y} \left[ 2 + \frac{1}{2} fL_2 + \frac{f}{2} \left( \frac{C_i^2}{2} - \rho_{x_i} C_y C_x \right) \right]
\]

Again, differentiating partially (4.9) with respect to \( \omega_2 \) and equating to zero, we have

\[
\bar{Y}fL_2 \omega_1 + \bar{X}fC_i^2 \omega_2 = \bar{Y} \frac{C_i^2 f}{2}
\]

On solving these two normal equations simultaneously for \( \omega_1 \) and \( \omega_2 \), we obtained the optimum values \( \omega_1 = \omega_1^{\text{opt}} = \omega_{\text{opt}}(\omega_1) \) and \( \omega_2 = \omega_2^{\text{opt}} = \omega_{\text{opt}}(\omega_2) \) as given in (4.10) and (4.11) respectively.

To obtain minimum value of \( \text{MSE} (\hat{y}_o) \), substitute these optimum values in (4.9), that is, taking \( \omega_1 = \omega_1^{\text{opt}} \) and \( \omega_2 = \omega_2^{\text{opt}} \) in (4.9), we get
\[ \text{Min. MSE} \left( \hat{\nu}_o \right) = Y^2 \left[ (\alpha^*_o - 1)^2 + f \omega^*_1 \omega^*_2 \left( L_1 + \frac{1}{2} f \omega^*_1 \left( L_2 + \frac{C^2}{2} - \rho_{xy} C_s C_x \right) \right) + f \omega^*_2 \bar{X}^2 C_s^2 + 2 f \omega^*_1 \bar{X} \left( \omega^*_2 L_2 - \frac{C^2}{2} \right) \right] \]  

(A4.1.1)

Noting that

\[
\omega^*_1 - 1 = \frac{-C^2_s \left\{ 2 - \frac{1}{2} f L_2 + \frac{f}{2} \left( \frac{C^2}{2} - \rho_{xy} C_s C_x \right) \right\} + 2 \left\{ f L_2^2 - (1 + f L_1) C_s^2 \right\}}{2 \left[ f L_2^2 - (1 + f L_1) C_s^2 \right]} 
\]

(A4.1.2)

\[
\omega^*_1 \omega^*_2 = -\frac{C^2_s Y}{X} \frac{\left\{ 2 + \frac{f L_2}{2} + \frac{f}{2} \left( \frac{C^2}{2} - \rho_{xy} C_s C_x \right) \right\} - f L_2 \left\{ 2 + \frac{1}{2} f L_2 + \frac{f}{2} \left( \frac{C^2}{2} - \rho_{xy} C_s C_x \right) \right\} - C_s^2 (1 + f L_1) \}}{4 \left[ f L_2^2 - (1 + f L_1) C_s^2 \right]^2} 
\]

(A4.1.3)

Also note the following results (on substituting the values of \( L_1 \) and \( L_2 \)):

\[
fl_1^2 - (1 + f L_1) C_s^2 = -C^2_s \left[ 1 + f C_s^2 \left( 1 - \rho_{xy}^2 \right) \right] 
\]

(A4.1.4)

\[
2 - \frac{1}{2} \frac{f L_2}{2} + \frac{f}{2} \left( \frac{C^2}{2} - \rho_{xy} C_s C_x \right) = 2 - \frac{1}{4} f C_s^2 
\]

(A4.1.5)

\[
\left[ f L_2^2 - (1 + f L_1) C_s^2 \right] \left[ 1 - \frac{1}{4} f C_s^2 \right] = -C^2_s \left[ \left\{ 1 + f C_s^2 \left( 1 - \rho_{xy}^2 \right) \right\} - \frac{1}{4} f C_s^2 \right] - \frac{1}{4} f^2 C_s^2 C_s^2 \left( 1 - \rho_{xy}^2 \right) 
\]

(A4.1.6)

Now using (4.10), (4.11), (A4.1.2) and (A4.1.3) in (A4.1.1), we get
\[ \text{Min.MSE} \left( \hat{\gamma}_0 \right) = \bar{Y}^2 \left[ -C_i^2 \left( \frac{1}{2} \beta \mathcal{L}_2 + \frac{f}{2} \left( \frac{C_i^2}{2} - \rho_{yy}, C, x \right) - \mathcal{F}_2 \right) + 2 \left\{ \mathcal{F}_2 \left( 1 + \mathcal{F}_1 \right) C_i^2 \right\} \right] \\
+ \frac{f}{2} \left( \frac{C_i^2}{2} - \rho_{yy}, C, x \right) - \mathcal{F}_2 \left[ \frac{1}{4} \left[ \frac{f}{2} \left( \frac{C_i^2}{2} - \rho_{yy}, C, x \right) - \mathcal{F}_2 \right] \right] \\
- \frac{1}{2} \left( \frac{f}{2} \left( \frac{C_i^2}{2} - \rho_{yy}, C, x \right) - \mathcal{F}_2 \right) \left[ \frac{1}{4} \left[ \frac{f}{2} \left( \frac{C_i^2}{2} - \rho_{yy}, C, x \right) - \mathcal{F}_2 \right] \right] \\
+ \frac{1}{2} \left( \frac{f}{2} \left( \frac{C_i^2}{2} - \rho_{yy}, C, x \right) - \mathcal{F}_2 \right) \left[ \frac{1}{4} \left[ \frac{f}{2} \left( \frac{C_i^2}{2} - \rho_{yy}, C, x \right) - \mathcal{F}_2 \right] \right] \\
+ 2 \bar{X} f C_i^2 \left[ \frac{1}{4} \left[ \frac{f}{2} \left( \frac{C_i^2}{2} - \rho_{yy}, C, x \right) - \mathcal{F}_2 \right] \right] \\
+ \frac{1}{2} \left( \frac{f}{2} \left( \frac{C_i^2}{2} - \rho_{yy}, C, x \right) - \mathcal{F}_2 \right) \left[ \frac{1}{4} \left[ \frac{f}{2} \left( \frac{C_i^2}{2} - \rho_{yy}, C, x \right) - \mathcal{F}_2 \right] \right] \\
- \frac{1}{2} \left( \frac{f}{2} \left( \frac{C_i^2}{2} - \rho_{yy}, C, x \right) - \mathcal{F}_2 \right) \left[ \frac{1}{4} \left[ \frac{f}{2} \left( \frac{C_i^2}{2} - \rho_{yy}, C, x \right) - \mathcal{F}_2 \right] \right] \\
\]
\[
\begin{align*}
\text{Min. MSE}\left(\hat{v}_{o}\right) & = \frac{(1 + fL_1)C_s^4 \left\{ \left( 2 + \frac{1}{2} fL_2 + f \left( \frac{C_s^2}{2} - \rho_{y,x} C_s, C_s \right) \right) - fL_2 \right\}^2}{4 \left( fL_2^2 - (1 + fL_1)C_s^2 \right)^2} \\
& + fC_s^2 \left[ (fL_2^2 - C_s^2 (1 + fL_1))^2 - L_2^2 \left\{ 2 + \frac{1}{2} fL_2 + f \left( \frac{C_s^2}{2} - \rho_{y,x} C_s, C_s \right) \right\} - fL_2 \right]^2 \\
& + 4 \left[ fL_2^2 - (1 + fL_1)C_s^2 \right] + 4C_s^2 \left\{ 2 + \frac{1}{2} fL_2 + f \left( \frac{C_s^2}{2} - \rho_{y,x} C_s, C_s \right) \right\} - fL_2 \right]^2 \\
& \frac{C_s^2 \left( \frac{1}{2} fL_2 + f \left( \frac{C_s^2}{2} - \rho_{y,x} C_s, C_s \right) \right) - fL_2 \right\}^2}{2 \left( fL_2^2 - (1 + fL_1)C_s^2 \right)^2} \\
& - fC_s^2 \left[ L_2 \left\{ 2 + \frac{1}{2} fL_2 + f \left( \frac{C_s^2}{2} - \rho_{y,x} C_s, C_s \right) - fL_2 \right\} + \left\{ fL_2^2 - C_s^2 (1 + fL_1) \right\} \right] \\
& + \frac{C_s^2 \left( \frac{1}{2} fL_2 + f \left( \frac{C_s^2}{2} - \rho_{y,x} C_s, C_s \right) - fL_2 \right\}^2}{2 \left( fL_2^2 - (1 + fL_1)C_s^2 \right)^2} \\
\end{align*}
\]
\[
\frac{\text{Min. MSE} \left( \hat{y}_o \right)}{\bar{Y}^2} = \frac{-C^2 \left[ \left\{ 2 + \frac{1}{2} fL_2 + \frac{f}{2} \left( \frac{C_i^2}{2} - \rho_{y_i} C_i, C_i \right) \right\} - fL_2 \right]^2 + 4 \left[ fL_2^2 - C^2 \left( 1 + fL_1 \right) \right] \left( 1 + \frac{1}{4} fC_i^2 \right)}{4 \left[ fL_2^2 - (1 + fL_1) C_i^2 \right]} \\
\]

\[
\Rightarrow \quad \frac{\text{Min. MSE} \left( \hat{y}_o \right)}{\bar{Y}^2} = \frac{C^2 \left[ \left\{ 2 + \frac{1}{2} fL_2 + \frac{f}{2} \left( \frac{C_i^2}{2} - \rho_{y_i} C_i, C_i \right) \right\} - fL_2 \right]^2 + 4 \left[ fL_2^2 - (1 + fL_1) C_i^2 \right] \left[ 1 - \frac{1}{4} fC_i^2 \right]}{4 \left[ fL_2^2 - (1 + fL_1) C_i^2 \right]} \\
\]

(A4.1.7)
On using (A4.1.4), (A4.1.5) and (A4.1.6) in (A4.1.7), we get the following

\[
\min \text{MSE} \left( \hat{y}_{i0} \right) = \frac{C_i^2 \left( 2 - \frac{1}{4} fC_i^2 \right)^2 - 4C_i^2 \left[ fC_y^2 \left( 1 - \rho_{yx}^2 \right) \left( 1 - \frac{1}{4} fC_i^2 \right) + \left( 1 - \frac{1}{4} fC_i^2 \right) \right]}{-4C_i^2 \left[ 1 + fC_y^2 \left( 1 - \rho_{yx}^2 \right) \right]}
\]

\[
= \frac{4C_i^2 + \frac{1}{16} f^2 C_i^6 - fC_i^4 - 4C_i^2 \left( 1 - \frac{1}{4} fC_i^2 \right) fC_y^2 \left( 1 - \rho_{yx}^2 \right) - 4C_i^2 + fC_i^4}{-4C_i^2 \left[ 1 + fC_y^2 \left( 1 - \rho_{yx}^2 \right) \right]}
\]

\[
= \frac{fC_i^2 \left( 1 - \rho_{yx}^2 \right)}{1 + fC_y^2 \left( 1 - \rho_{yx}^2 \right)} - \frac{f^2 C_i^4 C_y^2 \left( 1 - \rho_{yx}^2 \right) + \frac{1}{16} f^2 C_i^6}{4C_i^2 \left[ 1 + fC_y^2 \left( 1 - \rho_{yx}^2 \right) \right]}
\]

\[
\Rightarrow \min \text{MSE} \left( \hat{y}_{i0} \right) = \frac{f\overline{Y} C_i^2 \left( 1 - \rho_{yx}^2 \right)}{1 + fC_y^2 \left( 1 - \rho_{yx}^2 \right)} - \frac{f^2 \overline{Y}^2 C_i^2 \left[ 4C_i^2 \left( 1 - \rho_{yx}^2 \right) + \frac{C_i^2}{4} \right]}{16 \left[ 1 + fC_y^2 \left( 1 - \rho_{yx}^2 \right) \right]}
\]

Hence Theorem 4.2.1 is proved.