CHAPTER 6

FOURIER TRANSFORM AND ITS INVERSE
FOR FUNCTIONS OF BICOMPLEX VARIABLES
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6.1 Introductory remarks.

The theory of bicomplex numbers is a matter of active research for quite a long time since the seminal work of Segre (cf. [43]) in search of a special algebra. The algebra of bicomplex numbers are widely used in the literature as it becomes a viable commutative alternative (cf. [44] & [45]) to the non-commutative skew field of quaternions introduced by Hamilton (cf. [25]) (both are four-dimensional and generalization of complex numbers). The commutativity in the former is gained at the cost of the fact that the ring of these numbers contains zero-divisors and so cannot form a field (cf. [41]). However, the novelty of commutativity of bicomplex numbers is that the later can be recognized as the complex numbers with complex coefficients as its immediate effect and so there are deep similarities between the properties of complex and bicomplex numbers (cf. [40]). Many recent developments have aimed to achieve different algebraic (cf. [24], [49] & [51]) and geometric (cf. [8], [59] & [60]) properties of bicomplex numbers, the analysis of bicomplex functions (cf. [23], [37], [42] & [56]) and its applications on different branches of physics (such as quantum physics, high energy physics, bifurcation and chaos etc.) (cf. [30], [38] & [58]) to name a few.

In two recent developments (cf. [5] & [31]) efforts have been done to extend the Laplace transform and its inverse transform in the bicomplex Communicated, see [6].
variables from their complex counterpart. In their procedure the idempotent representation of the bicomplex variables plays a vital role. Actually these idempotent components are complex valued and the bicomplex counterpart simply is their combination with idempotent hyperbolic numbers. The Laplace transform of these idempotent complex variables within their regions of convergence are taken and then the bicomplex version of that transform can be obtained directly by combination of them with idempotent hyperbolic numbers. The region of convergence in the later case will be the union of the respective regions of those idempotent complex variables. Bicomplex version of the inversion of Laplace transform is achieved by employing the residual procedure on both the complex planes in connection to idempotent representation.

In the same spirit we take up the study for existence of Fourier transform, its inverse transform and the region of convergence in bicomplex variables. The Fourier transform{cf. [1] & [29]} is actually a reversible operation employed to transform signals between the spatial (or time) domain and the frequency domain. Most often in the literature $f$ is a real valued function and its Fourier transform $\hat{f}$ is complex valued where a complex number describes both the amplitude and phase of a corresponding frequency component.

In this chapter one of our concern is to extend the Fourier transform in bicomplex variables from its complex version that can be capable of transferring signals from real-valued ($t$) domain to bicomplex frequency ($\omega$) domain. The later should have two idempotent complex frequency components $\omega_1$ and $\omega_2$. Our secondary concern is to develop the inverse Fourier transform from both complex $\omega_1$ and bicomplex $\omega$—domain to $t$-domain.

The organization of the present discussion is as follows:

Section 6.2 introduces a brief preliminaries of bicomplex numbers. In Article §§I, we present the existence and region of convergence of bicomplex version of Fourier transform. Some of its basic properties are extended from complex Fourier transform. Finally, Article §§II deals with the existence of inverse Fourier transform for both the complex and bicomplex variables.
6.2 Bicomplex numbers.

We start with an unconventional interpretation of the set of complex numbers $\mathbb{C}$ in which its members are found by duplication of the elements of the set of real numbers $\mathbb{R}$ in association with a non-real unit $i$ such that $i^2 = -1$ in the form

$$\mathbb{C} = \{z = x + iy : x, y \in \mathbb{R}\}. \quad (6.1)$$

Now if we repeat our duplication process once on the members of $\mathbb{C}$, for the sake of definiteness we first denote the imaginary unit $i$ of (6.1) by $i_1$ resulting

$$\mathbb{C}(i_1) = \{z = x + i_1 y : x, y \in \mathbb{R}\}.$$  

If $i_2$ be a new imaginary unit associated with duplication having the properties

$$i_2^2 = -1; i_1 i_2 = i_2 i_1; a i_2 = i_2 a, a \in \mathbb{R},$$

we can extend $\mathbb{C}(i_1)$ onto the set of bicomplex numbers

$$\mathbb{C}_2 = \{\omega = z_1 + i_2 z_2 : z_1, z_2 \in \mathbb{C}(i_1)\} \quad (6.2)$$

where an additional structure of commutative multiplication is embedded.

Going back to the real variables for $z_1 = x_1 + i_1 x_2$ and $z_2 = x_3 + i_1 x_4$, the bicomplex numbers admit of an alternative representation of the form

$$\omega = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4$$

which is the linear combination of four units: one real unit 1, two imaginary units $i_1, i_2$ and one non-real hyperbolic unit $i_1 i_2 (= i_2 i_1)$ for which $(i_2 i_1)^2 = 1$. In particular, if $x_2 = x_3 = 0$ one may identify bicomplex numbers with the hyperbolic numbers.

However looking onto the algebraic structure of $\mathbb{C}_2$, we can observe that it becomes a commutative ring with unit and $\mathbb{R}, \mathbb{C}(i_1)$ are two subrings embedded within it as
\[ \mathbb{R} \cong \{ z_1 + i_2 z_2 : z_2 = 0, z_1 \in \mathbb{R} \} \subset \mathbb{C}_2 \]

and \( \mathbb{C}(i_1) \cong \{ z_1 + i_2 z_2 : z_2 = 0, z_1 \in \mathbb{C}(i_1) \} \subset \mathbb{C}_2. \)

Interestingly, we may indeed identify the set of complex numbers \( \mathbb{C} \) with duplication of reals associated with imaginary unit \( i_2 \), i.e.

\[ \mathbb{C}(i_2) = \{ z = x + i_2 y : x, y \in \mathbb{R} \} \]

as another possible subring embedding onto \( \mathbb{C}_2 \). Both \( \mathbb{C}(i_1) \) and \( \mathbb{C}(i_2) \) are isomorphic to \( \mathbb{C} \) but are essentially different.

Furthermore for two arbitrary bicomplex numbers \( \omega = z_1 + i_2 z_2 \) and \( \omega' = z'_1 + i_2 z'_2; \quad z_1, z_2, z'_1, z'_2 \in \mathbb{C}(i_1) \), the scalar addition is defined by

\[ \omega \pm \omega' = (z_1 + z'_1) + i_2 (z'_2 + z_2) \]

and the scalar multiplication is governed by

\[ \omega \cdot \omega' = (z_1 z'_1 - z'_2 z_2) + i_2 (z_1 z'_2 + z_2 z'_1). \]

### 6.2.1 Idempotent representation.

We now introduce two bicomplex numbers

\[ e_1 = \frac{1 + i_1 i_2}{2} \quad \text{and} \quad e_2 = \frac{1 - i_1 i_2}{2} \quad (6.3) \]

which satisfy

\[ e_1 + e_2 = 1, \]

\[ e_1 e_2 = e_2 e_1 = 0, \]

\[ e_1^2 = e_1, e_1 = e_1, \]

\[ e_2^2 = e_2, e_2 = e_2. \]

The second requirement indicates that \( e_1, e_2 \) are orthogonal while the last
two indicate them as idempotent. They offer us a unique decomposition of \( \mathbb{C}_2 \) in the following form:

For any \( \omega = z_1 + i_2z_2 \in \mathbb{C}_2 \),

\[
z_1 + i_2z_2 = (z_1 - i_1z_2)e_1 + (z_1 + i_1z_2)e_2 \quad (6.4)
\]

resulting a pair of mutually complementary projections

\[
P_1 : (z_1 + i_2z_2) \in \mathbb{C}_2 \longmapsto (z_1 - i_1z_2) \in \mathbb{C}(i_1)
\]

and \( P_2 : (z_1 + i_2z_2) \in \mathbb{C}_2 \longmapsto (z_1 + i_1z_2) \in \mathbb{C}(i_1) \).

One may at once verify that

\[
P_1^2 = P_1, P_2^2 = P_2, P_1e_1 + P_2e_2 = I
\]

and for any \( \omega_1, \omega_2 \in \mathbb{C}_2 \),

\[
P_k(\omega_1 + \omega_2) = P_k(\omega_1) + P_k(\omega_2)
\]

and \( P_k(\omega_1\omega_2) = P_k(\omega_1)P_k(\omega_2), k = 1, 2 \).

At this stage, we now mention the auxiliary complex spaces of the space of bicomplex numbers which are actually

\[
A_1 = \{ P_1(\omega) : \omega \in \mathbb{C}_2 \}
\]

and \( A_2 = \{ P_2(\omega) : \omega \in \mathbb{C}_2 \} \).

**Bicomplex functions.**

We start with a bicomplex valued function \( f : \Omega \subset \mathbb{C}_2 \longmapsto \mathbb{C}_2 \). The derivative of \( f \) at a point \( \omega_0 \in \Omega \) is defined by

\[
f'(\omega) = \lim_{h \to 0} \frac{f(\omega_0 + h) - f(\omega_0)}{h}
\]

provided the limit exists and the domain is so chosen that \( h = h_0 + i_1h_1 + i_2h_2 + i_1i_2h_3 \) is invertible. It is easy to prove that \( h \) is not invertible only for \( h_0 = -h_3, h_1 = h_2 \) or \( h_0 = h_3, h_1 = -h_2 \).

If the bicomplex derivative of \( f \) exists at each point of its domain then in similar to complex functions, \( f \) will be a bicomplex holomorphic function in \( \Omega \). Indeed if \( f \) can be expressed as
\[ f(\omega) = g_1(z_1, z_2) + ig_2(z_1, z_2), \omega = (z_1 + iz_2) \in \Omega \]

then \( f \) will be holomorphic if and only if \( g_1, g_2 \) are both complex holomorphic in \( z_1, z_2 \) and

\[
\frac{\partial g_1}{\partial z_1} = \frac{\partial g_2}{\partial z_2}, \frac{\partial g_1}{\partial z_2} = -\frac{\partial g_2}{\partial z_1}.
\]

Moreover, \( f'(\omega) = \frac{\partial g_1}{\partial z_2} + ig_2 \frac{\partial g_2}{\partial z_1} \) and it is invertible only when

\[
det \begin{pmatrix} \frac{\partial g_1}{\partial z_1} & \frac{\partial g_1}{\partial z_2} \\ \frac{\partial g_2}{\partial z_1} & \frac{\partial g_2}{\partial z_2} \end{pmatrix} \neq 0.
\]

In the following we take up the idempotent representation of bicomplex numbers which is crucial in a deeper understanding of the analysis of holomorphic functions. Any bicomplex holomorphic function

\[ f : \Omega \subset \mathbb{C}_2 \mapsto \mathbb{C}_2 \]

involving unique idempotent decomposition into two complex valued functions reads as

\[ f(\omega) = f_1(\omega_1)e_1 + f_2(\omega_2)e_2, \ \omega = (\omega_1 e_1 + \omega_2 e_2) \in \Omega. \]

One may then verify in a straightforward way that

\[
\Omega_1 = \{ \omega_1 : \omega \in \Omega \} \subset \mathbb{C}(i_1)
\]
and
\[
\Omega_2 = \{ \omega_2 : \omega \in \Omega \} \subset \mathbb{C}(i_1)
\]

will be the domain of complex-valued functions \( f_1 \) and \( f_2 \) respectively. In view of projection operators \( P_1 \) and \( P_2 \) the above can be represented as

\[
\Omega_1 = P_1(\Omega_1) \implies f_1 = P_1 f
\]
and
\[
\Omega_2 = P_2(\Omega_2) \implies f_2 = P_2 f.
\]

Indeed in case of bicomplex valued holomorphic functions, most often the properties of its idempotent complex valued holomorphic components are
just carried over their bicomplex counterparts. For example, \( f(\omega) \) will be convergent in a domain \( \Omega \) if and only if \( f_1(\omega_1) \) and \( f_2(\omega_2) \) are convergent in their domains \( \Omega_1 = P_1(\Omega_1) \) and \( \Omega_2 = P_2(\Omega_2) \) respectively.

\[ \begin{align*}
\textbf{II. Bicomplex version of Fourier transform:} \\
\text{In this discussion our aim is to extend the Fourier transform } & \mathcal{F}: D \subset \mathbb{R} \rightarrow \mathbb{C}_2 \text{ in bicomplex variables from its complex version and to verify the basic properties in our version those hold good in later case.} \\
\end{align*} \]

6.2.2 Conjecture.

Suppose \( f(t) \) be a real valued function which is continuous for \(-\infty < t < \infty \) and satisfies the estimates

\[
|f(t)| \leq C_1 \exp(-\alpha t), \ t \geq 0, \alpha > 0 \\
\text{and } |f(t)| \leq C_2 \exp(-\beta t), \ t \leq 0, \beta > 0 \tag{6.5}
\]

which guarantees that \( f \) is absolutely integrable on the whole real line.

Now we start with the complex Fourier transform

\[
\mathcal{F} : D \subset \mathbb{R} \rightarrow \mathbb{C}(i_1).
\]

The complex Fourier transform of \( f(t) \) associated with complex frequency \( \omega_1 \) is defined by

\[
\hat{f}_1(\omega_1) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} \exp(i_1\omega_1 t)f(t)dt, \omega_1 \in \mathbb{C}(i_1)
\]

together with the requirement of \( |\hat{f}_1(\omega_1)| < \infty \).

Now for \( \omega_1 = x + i_1y \),
\[ |\hat{f}_1(\omega_1)| = |\int_{-\infty}^{\infty} \exp(i\omega_1 t)f(t)\,dt| \leq \int_{-\infty}^{\infty} |\exp(-yt)f(t)|\,dt \]
\[ = \int_{-\infty}^{0} \exp(-yt)|f(t)|\,dt + \int_{0}^{\infty} \exp(-yt)|f(t)|\,dt \]
\[ \leq C_2 \int_{-\infty}^{0} \exp\{(\beta - y)t\}dt + C_1 \int_{0}^{\infty} \exp\{-(\alpha + y)t\}dt \]
\[ = C_2 \frac{1}{\beta - y} + C_1 \frac{1}{\alpha + y} \]

where we use the estimate (6.5) and the facts

\[ |\exp(i\omega t)| = 1, \]
\[ |\exp(-yt)| = \exp(-yt), \]

and \(\exp(-yt) > 0\).

Then the requirement \( |\hat{f}_1(\omega_1)| < \infty \) only implies that \(-\alpha < y < \beta \). As its consequence \(\hat{f}_1(\omega_1)\) is holomorphic in the strip

\[ \Omega_1 = \{\omega_1 \in \mathbb{C}(i_1) : \text{Re}(\omega_1) < \infty, -\alpha < \text{Im}(\omega_1) < \beta\}. \]

Following similar arguments the complex Fourier transform \(f(t)\) associated with another complex frequency \(\omega_2\) will be

\[ \hat{f}_2(\omega_2) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} \exp(i\omega_2 t)f(t)\,dt, \omega_2 \in \mathbb{C}(i_1) \]

which will be holomorphic in the strip

\[ \Omega_2 = \{\omega_2 \in \mathbb{C}(i_1) : \text{Re}(\omega_2) < \infty, -\alpha < \text{Im}(\omega_2) < \beta\}. \]

Now employing duplication over the complex functions \(\hat{f}_1(\omega_1)\) and \(\hat{f}_2(\omega_2)\) in association with idempotent units \(e_1\) and \(e_2\) we observe that
\[ \hat{f}_1(\omega_1)e_1 + \hat{f}_2(\omega_2)e_2 = \int_{-\infty}^{\infty} \exp(i_1\omega_1 t)f(t)dt.e_1 + \int_{-\infty}^{\infty} \exp(i_1\omega_2 t)f(t)dt.e_2 \]
\[ = \int_{-\infty}^{\infty} \exp\{i_1(\omega_1 e_1 + \omega_2 e_2) t\}f(t)dt \]
\[ = \int_{-\infty}^{\infty} \exp(i_1\omega t)f(t)dt \]
\[ = \hat{f}(\omega) \]

where we use duplication of complex frequencies \( \omega_1 \) and \( \omega_2 \) to obtain bicomplex frequency \( \omega \) as \( \omega = \omega_1 e_1 + \omega_2 e_2 \).

Since \( \hat{f}_1(\omega_1) \) and \( \hat{f}_2(\omega_2) \) are complex holomorphic functions in \( \Omega_1 \) and \( \Omega_2 \) respectively then as its natural consequence the bicomplex function \( \hat{f}(\omega) \) will be holomorphic in the region

\[ \Omega = \{ \omega \in \mathbb{C}_2 : \omega = \omega_1 e_1 + \omega_2 e_2, \omega_1 \in \Omega_1 \ and \ \omega_2 \in \Omega_2 \}. \]

It is worthwhile to mention that the complex valued holomorphic functions \( \hat{f}_1(\omega_1) \) and \( \hat{f}_2(\omega_2) \) are both absolutely convergent in \( \Omega_1 \) and \( \Omega_2 \) respectively. Then it is easy to prove that the region of absolute convergence of \( \hat{f}(\omega) \) will be \( \Omega \).

For better geometrical understanding of the region of convergence of bicomplex Fourier transform, it will be advantageous to use the general four-unit representation of bicomplex numbers. In this occasion, we take conventional representation of \( \omega_1, \omega_2 \in \mathbb{C}(i_1) \) as

\[ \omega_1 = x_1 + i_1 x_2, \omega_2 = y_1 + i_1 y_2; x_1, x_2, y_1, y_2 \in \mathbb{R} \]  \hspace{1cm} (6.6)

where the requirement for \( \omega_1 \in \Omega_1 \) and \( \omega_2 \in \Omega_2 \) implies that \(-\infty < x_1, y_1 < \infty \) and \(-\alpha < x_2 < \beta; -\alpha < y_2 < \beta \). Using these and (6.3), \( \omega \) takes the explicit four-components form

\[ \omega = \frac{x_1 + y_1}{2} + \frac{i_1 x_2 + y_2}{2} + \frac{i_2 y_2 - x_2}{2} + i_1 i_2 \frac{x_1 - y_1}{2} \]
\[ = a_0 + i_1 a_1 + i_2 a_2 + i_1 i_2 a_3 \]  \hspace{1cm} (6.7)
where $a_0, a_1, a_2, a_3 \in \mathbb{R}$.

On the basis of the restrictions on $x_2$ and $y_2$ given in (6.6), the following three possibilities can occur:

1. If $x_2 = y_2$, it is trivial to obtain that

   
   $$-\alpha < a_1 < \beta \text{ and } a_2 = 0,$$

2. For $x_2 > y_2$, one may infer

   
   $$-\alpha - a_2 < a_1 < \beta + a_2$$

where as

   
   $$-\frac{\alpha + \beta}{2} < a_2 < 0$$

and

3. If $x_2 < y_2$ then in similar to previous possibility, we obtain that

   
   $$-\alpha + a_2 < a_1 < \beta - a_2 \text{ and } 0 < a_2 < \frac{\alpha + \beta}{2}$$

where as $-\infty < a_0, a_3 < \infty$ in all the three cases.

Considering all the results, we conclude that

   
   $$-\infty < a_0, a_3 < \infty, \quad -\alpha + |a_2| < a_1 < \beta - |a_2|$$

and

   
   $$0 \leq |a_2| < \frac{\alpha + \beta}{2}$$

and hence the region of convergence of $\hat{f}(\omega)$ (See Figure-1 in Appendix for $a_1 - a_2$ plane section of the region) can be identified as

   
   $$\Omega = \{ \omega = a_0 + i_1 a_1 + i_2 a_2 + i_1 i_2 a_3 \in \mathbb{C}_2 : -\infty < a_0, a_3 < \infty, \quad -\alpha + |a_2| < a_1 < \beta - |a_2| \text{ and } 0 \leq |a_2| < \frac{\alpha + \beta}{2} \}. \quad (6.8)$$
Conversely, the existence of bicomplex Fourier transform $\hat{f}(\omega)$ can be obtained in the following way: If

$$\omega = a_0 + i_1 a_1 + i_2 a_2 + i_1 i_2 a_3 \in \mathbb{C}_2$$

with

$$-\infty < a_0, a_3 < \infty,$$

$$-\alpha + |a_2| < a_1 < \beta - |a_2|$$

and

$$0 \leq |a_2| < \frac{\alpha + \beta}{2}.$$ 

Now expressing $\omega$ in idempotent components as

$$\omega = a_0 + i_1 a_1 + i_2 a_2 + i_1 i_2 a_3 = [(a_0 + a_3) + i_1 (a_1 - a_2)] e_1 + [(a_0 - a_3) + i_1 (a_1 + a_2)] e_2$$

we obtain that

1. $a_2 = 0$ and $-\alpha < a_1 < \beta$ trivially leading to

$$-\alpha < a_1 - a_2 < \beta$$

and $-\alpha < a_1 + a_2 < \beta$,

2. when $a_2 < 0$, from the first inequality of

$$-\alpha - a_2 < a_1 < \beta + a_2$$

we can get that $-\alpha < a_1 + a_2$ where as the last inequality gives that

$$a_1 - a_2 < \beta.$$ 

Following $a_2 < 0$, these results can be interpreted as

$$-\alpha < a_1 + a_2 < a_1 - a_2$$

and $a_1 + a_2 < a_1 - a_2 < \beta$

which in together may be combined into

$$-\alpha < a_1 + a_2 < a_1 - a_2 < \beta$$

and
3. when \( a_2 > 0 \), from the first inequality of

\[-\alpha + a_2 < a_1 < \beta - a_2\]

we can get \(-\alpha < a_1 - a_2\) whereas the last inequality gives

\[a_1 + a_2 < \beta.\]

Following \( a_2 > 0 \) these results can be interpreted as

\[-\alpha < a_1 - a_2 < a_1 + a_2 \quad \text{and} \quad a_1 - a_2 < a_1 + a_2 < \beta\]

which in together can be combined into

\[-\alpha < a_1 - a_2 < a_1 + a_2 < \beta.\]

Hence the result.

Now we are ready to define the Fourier transform for bicomplex variable.

6.2.3 Definition of Fourier transform.

Let \( f(t) \) be a real valued continuous function in \((-\infty, \infty)\) which satisfies the estimates (6.5). The Fourier transform of \( f(t) \) can be defined as

\[
\hat{f}(\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} \exp(i\omega t)f(t)dt, \omega \in \mathbb{C}_2. \tag{6.9}
\]

The Fourier transform \( \hat{f}(\omega) \) exists and holomorphic for all \( \omega \in \Omega \) where \( \Omega \) (given in (6.8)) is the region of absolute convergence of \( \hat{f}(\omega) \).

6.2.4 Existence of Fourier transform.

**Theorem 6.2.1** If \( f(t) \) be a real valued function and is continuous for \(-\infty < t < \infty\) satisfying estimates (6.5) then \( \hat{f}(\omega) \) as defined in (6.9) exists in the region (6.8).
Proof.

\[ \hat{f}(\omega) = \int_{-\infty}^{\infty} \exp(i\omega t)f(t)dt = (\int_{-\infty}^{\infty} \exp(i\omega_1 t)f(t)dt)e_1 + (\int_{-\infty}^{\infty} \exp(i\omega_2 t)f(t)dt)e_2. \]

Both the integrals exist when \(-\alpha < \text{Im}(\omega_1 = x_1 + i_1x_2) < \beta\) and \(-\alpha < \text{Im}(\omega_2 = y_1 + i_1y_2) < \beta\). So \(f(\omega)\) exists for \(\omega = \omega_1 e_1 + \omega_2 e_2 = a_0 + i_1a_1 + i_2a_2 + i_1i_2a_3\) where

\[-\infty < a_0, a_3 < \infty, -\alpha < |a_2| < a_1 < \beta - |a_2| \text{ and } 0 \leq |a_2| < \frac{\alpha + \beta}{2}.\]

This proves the theorem. \(\blacksquare\)

6.2.5 Uniqueness of Fourier transform.

Theorem 6.2.2 If \(f(t)\) and \(g(t)\) have Fourier transforms \(\hat{f}(\omega)\) and \(\hat{g}(\omega)\) respectively and \(\hat{f}(\omega) = \hat{g}(\omega)\), then \(f(t) = g(t)\).

Proof. Let \(\hat{f}(\omega) = \hat{f}_1(\omega_1)e_1 + \hat{f}_2(\omega_2)e_2\) and \(\hat{g}(\omega) = \hat{g}_1(\omega_1)e_1 + \hat{g}_2(\omega_2)e_2\) in their idempotent representations. Now \(\hat{f}(\omega) = \hat{g}(\omega)\) is possible if and only if

\[ \hat{f}_1(\omega_1) = \hat{g}_1(\omega_1) \text{ and } \hat{f}_2(\omega_2) = \hat{g}_2(\omega_2) \]

which implies that

\[ \int_{-\infty}^{\infty} \exp(i\omega_1 t)f(t)dt = \int_{-\infty}^{\infty} \exp(i\omega_1 t)g(t)dt \]

and

\[ \int_{-\infty}^{\infty} \exp(i\omega_2 t)f(t)dt = \int_{-\infty}^{\infty} \exp(i\omega_2 t)g(t)dt \]

i.e.

\[ f(t) = g(t). \]

This completes the proof of the theorem. \(\blacksquare\)
6.2.6 Basic properties of Fourier transform.

- Linearity property.

**Theorem 6.2.3** If the Fourier transforms of \( f(t) \) and \( g(t) \) are \( \hat{f}(\omega) \) and \( \hat{g}(\omega) \) respectively and \( a, b \) are constants then \( \mathcal{F}(af + bg) = a\hat{f} + b\hat{g} \).

**Proof.** Let \( \hat{f} \) and \( \hat{g} \) are both defined for \( \omega = \omega_1 e_1 + \omega_2 e_2 \in \Omega, \omega_1 \in \Omega_1, \omega_2 \in \Omega_2 \) then

\[
\mathcal{F}\{af(t) + bg(t)\} = \int_{-\infty}^{\infty} \exp(i\omega t)\{af(t) + bg(t)\}dt
\]

\[
= \int_{-\infty}^{\infty} \exp\{i_1(\omega_1 e_1 + \omega_2 e_2)t\}\{af(t) + bg(t)\}dt
\]

\[
= [\int_{-\infty}^{\infty} \exp(i_1\omega_1 t)\{af(t) + bg(t)\}dt]e_1
\]

\[
+ [\int_{-\infty}^{\infty} \exp(i_1\omega_2 t)\{af(t) + bg(t)\}dt]e_2
\]

\[
= a[\int_{-\infty}^{\infty} \exp(i_1\omega_1 t)f(t)dt]e_1 + \{\int_{-\infty}^{\infty} \exp(i_1\omega_2 t)f(t)dt\}e_2
\]

\[
+ b[\int_{-\infty}^{\infty} \exp(i_1\omega_1 t)g(t)dt]e_1 + \{\int_{-\infty}^{\infty} \exp(i_1\omega_2 t)g(t)dt\}e_2
\]

\[
= a \int_{-\infty}^{\infty} \exp\{i_1(\omega_1 e_1 + \omega_2 e_2)t\}f(t)dt
\]

\[
+ b \int_{-\infty}^{\infty} \exp\{i_1(\omega_1 e_1 + \omega_2 e_2)t\}g(t)dt
\]

\[
= a \hat{f}(\omega) + b\hat{g}(\omega).
\]

This completes the proof of the theorem. ■

- Shifting property.

**Theorem 6.2.4** If \( \hat{f}(\omega) \) is the Fourier transform of \( f(t) \) then

\[
\mathcal{F}\{f(t - a)\} = \exp(i\omega a)\hat{f}(\omega).
\]
Proof. By definition we get that

$$\mathcal{F}\{f(t - a)\} = \int_{-\infty}^{\infty} \exp(i\omega t) f(t - a) dt.$$  

Now for $t = a + u$, the integral in right hand side is equal to

$$\int_{-\infty}^{\infty} \exp\{i\omega(a + u)\} f(u) du = \exp(i\omega a) \int_{-\infty}^{\infty} \exp(i\omega u) f(u) du = \exp(i\omega a) \hat{f}(\omega).$$

Thus the theorem is established. 

• Scaling property.

Theorem 6.2.5 If $\hat{f}(\omega)$ is the Fourier transform of $f(t)$ then $\mathcal{F}\{f(at)\} = \frac{1}{|a|} \hat{f}\left(\frac{\omega}{a}\right)$ where $a \neq 0$.

Proof. If $a > 0$ then

$$\mathcal{F}\{f(at)\} = \int_{-\infty}^{\infty} \exp(i\omega t) f(at) dt = \frac{1}{a} \int_{-\infty}^{\infty} \exp(i\omega u) f(u) du = \frac{1}{a} \hat{f}\left(\frac{\omega}{a}\right)$$

where we take $at = u$.

If $a < 0$ then for $a = -b$ where $b > 0$, we have

$$\mathcal{F}\{f(at)\} = \int_{-\infty}^{\infty} \exp(i\omega t) f(at) dt = \int_{-\infty}^{\infty} \exp(i\omega t) f(-bt) dt.$$  

Now taking $bt = -u$, the integral becomes that

$$\frac{1}{b} \int_{-\infty}^{\infty} \exp(i\omega u) f(u) du = \frac{1}{b} \int_{-\infty}^{\infty} \exp(i\omega u) f(u) du = \frac{1}{a} \hat{f}\left(\frac{\omega}{a}\right).$$

From the above results we conclude that $\mathcal{F}\{f(at)\} = \frac{1}{|a|} \hat{f}\left(\frac{\omega}{a}\right)$. 

• Convolution theorem.
**Theorem 6.2.6** The Fourier transform of the convolution of two functions $f(t)$ and $g(t)$ where $-\infty < t < \infty$ is the product of their Fourier transforms $\widehat{f}(\omega)$ and $\widehat{g}(\omega)$ respectively i.e.

$$\mathcal{F}\{f(t) \ast g(t)\} = \mathcal{F}\{ \int_{-\infty}^{\infty} f(u)g(t-u)du \} = \widehat{f}(\omega)\widehat{g}(\omega).$$

**Proof.** By definition and by using the method for changing order of integrals in complex analysis [cf.39] and also in view of the Shifting Property we get that

$$\mathcal{F}\{f(t) \ast g(t)\} = \mathcal{F}\{ \int_{-\infty}^{\infty} f(u)g(t-u)du \}$$

$$= \int_{-\infty}^{\infty} \exp(i\omega t)\{ \int_{-\infty}^{\infty} f(u)g(t-u)du \}dt$$

$$= [ \int_{-\infty}^{\infty} \exp(i\omega t)\{ \int_{-\infty}^{\infty} f(u)g(t-u)du \}dt ]e_1$$

$$+ [ \int_{-\infty}^{\infty} \exp(i\omega t)\{ \int_{-\infty}^{\infty} f(u)g(t-u)du \}dt ]e_2$$

$$= [ \int_{-\infty}^{\infty} f(u)\{ \int_{-\infty}^{\infty} \exp(i\omega t)g(t-u)dt \}du ]e_1$$

$$+ [ \int_{-\infty}^{\infty} f(u)\{ \int_{-\infty}^{\infty} \exp(i\omega t)g(t-u)dt \}du ]e_2$$

$$= \int_{-\infty}^{\infty} f(u)\{ \int_{-\infty}^{\infty} \exp(i\omega t)g(t-u)dt \}du$$

$$= \int_{-\infty}^{\infty} f(u)\exp(i\omega u)\widehat{g}(\omega)du,$$

$$= \{ \int_{-\infty}^{\infty} f(u)\exp(i\omega u)du \}\widehat{g}(\omega) = \widehat{f}(\omega)\widehat{g}(\omega).$$

This proves the theorem. □

**Theorem 6.2.7** If $f(t)$ and $t^rf(t)$ for $r = 1, 2, 3, \ldots n$ are all integrable in $-\infty < t < \infty$ then

$$\mathcal{F}\{t^n f(t)\} = (-i_1)^n \frac{d^n}{dx^n}\{\widehat{f}(\omega)\}$$
where $\hat{f}(\omega)$ is the Fourier transform of $f(t)$.

**Proof.** We will prove this theorem by using the method of mathematical induction and differentiation under integral sign. For $n = 1$,

$$
\frac{d}{d\omega} \hat{f}(\omega) = \left[ \frac{\partial}{\partial \omega_1} \hat{f}_1(\omega_1) \right] e_1 + \left[ \frac{\partial}{\partial \omega_2} \hat{f}_2(\omega_2) \right] e_2
$$

$$
= \left[ \frac{\partial}{\partial \omega_1} \int_{-\infty}^{\infty} \exp(i \omega_1 t) f(t) dt \right] e_1
$$

$$
+ \left[ \frac{\partial}{\partial \omega_2} \int_{-\infty}^{\infty} \exp(i \omega_2 t) f(t) dt \right] e_2
$$

$$
= \left[ \int_{-\infty}^{\infty} \frac{\partial}{\partial \omega_1} \{\exp(i \omega_1 t) f(t)\} dt \right] e_1
$$

$$
+ \left[ \int_{-\infty}^{\infty} \frac{\partial}{\partial \omega_2} \{\exp(i \omega_2 t) f(t)\} dt \right] e_2,
$$

by using the Leibnitz rule in complex analysis (cf. [39])

$$
= i_1 \int_{-\infty}^{\infty} tf(t) \{\exp(i \omega_1 t) e_1 + \exp(i \omega_2 t) e_2\} dt
$$

$$
= i_1 \int_{-\infty}^{\infty} tf(t) \exp(i \omega t) dt = i_1 \mathcal{F}\{tf(t)\}
$$

$$
i.e \mathcal{F}\{tf(t)\} = -i_1 \frac{d}{d\omega} \hat{f}(\omega).
$$
Now for $n=2$, in a similar to the case for $n=1$ we get that

\[
\frac{d^2}{d\omega^2} \hat{f}(\omega) = \frac{d}{d\omega} \left[ \frac{d}{d\omega} \hat{f}(\omega) \right] = i_1 \frac{d}{d\omega} \left[ \int t f(t) \exp(i_1 \omega t) dt \right] = i_1 \frac{\partial}{\partial \omega} \int_{-\infty}^{\infty} \exp(i_1 \omega_1 t) t f(t) dt] e_1 + i_1 \frac{\partial}{\partial \omega_2} \int_{-\infty}^{\infty} \exp(i_1 \omega_2 t) t f(t) dt] e_2 = - \int_{-\infty}^{\infty} \exp(i_1 \omega_1 t) t^2 f(t) dt] e_1 - \int_{-\infty}^{\infty} \exp(i_1 \omega_2 t) t^2 f(t) dt] e_2 = - \int_{-\infty}^{\infty} t^2 f(t) \{ \exp(i_1 \omega_1 t) e_1 + \exp(i_1 \omega_2 t) e_2 \} dt = - \int_{-\infty}^{\infty} t^2 f(t) \exp(i_1 \omega t) dt = - \mathcal{F} \{ t^2 f(t) \} = \left( -i_1 \right)^2 \frac{d^2}{d\omega^2} \hat{f}(\omega).
\]

This completes the proof of the theorem. ■

**Theorem 6.2.8** If $f(t)$ and $f^r(t)$ for $r = 1, 2, \ldots, n$ are piecewise smooth and tend to 0 as $|t| \to \infty$ and $f$ with its derivatives of order up to $n$ are integrable in $-\infty < t < \infty$ then

\[
\mathcal{F} \{ f^n(t) \} = (-i_1)^n \frac{d^n}{dx^n} \{ \hat{f}(\omega) \}.
\]
where \( \hat{f}(\omega) \) is the Fourier transform of \( f(t) \) and \( f^r(t) = \frac{d}{dt} f(t) \).

**Proof.** We will prove it using the method of mathematical induction. For \( n = 1 \), we get that

\[
\mathcal{F}\{f'(t)\} = \int_{-\infty}^{\infty} \exp(i\omega t) f'(t) dt \\
= [f(t) \exp(i\omega t)]_{-\infty}^{\infty} - i\omega \int_{-\infty}^{\infty} \exp(i\omega t) f(t) dt \\
= 0 - i\omega \hat{f}(\omega) = -i\omega \hat{f}(\omega).
\]

Similarly for \( n = 2 \),

\[
\mathcal{F}\{f''(t)\} = \int_{-\infty}^{\infty} \exp(i\omega t) f''(t) dt \\
= [f'(t) \exp(i\omega t)]_{-\infty}^{\infty} - i\omega \int_{-\infty}^{\infty} \exp(i\omega t) f'(t) dt \\
= 0 - i\omega \mathcal{F}\{f'(t)\} = (-i\omega)^2 \hat{f}(\omega).
\]

Proceeding with similar arguments, we obtain that

\[
\mathcal{F}\{f^n(t)\} = (-i\omega)^n \hat{f}(\omega).
\]

Thus the theorem is proved. \( \blacksquare \)

**Corollary 6.2.1** If \( f(t) \) is finite, i.e. \( f(t) = 0 \) for \( |t| > T \) and is continuous inside \( |t| \leq T \), then its complex Fourier transform is an entire function. As its consequence

\[
\hat{f}_1(\omega_1) = \int_{-T}^{T} \exp(i\omega_1 t) f(t) dt
\]

and

\[
\hat{f}_2(\omega_2) = \int_{-T}^{T} \exp(i\omega_2 t) f(t) dt.
\]

So the bicomplex Fourier transform \( \hat{f}(\omega) \) exists and converges absolutely within the whole of \( \mathbb{C}_2 \).

Now we can make our notion clear by means of the following examples:
6.2.7 Examples.

Example 6.2.1 If

\[ f(t) = \exp(-a|t|) \text{ for } a > 0 \]

then it satisfies the estimates (6.5) for \( \alpha = \beta = a \) and its complex Fourier transforms are

\[ \hat{f}_1(\omega_1) = \frac{2a}{a^2 + \omega_1^2} \text{ and } \hat{f}_2(\omega_2) = \frac{2a}{a^2 + \omega_2^2}. \]

Both \( \hat{f}_1 \) and \( \hat{f}_2 \) are holomorphic in the strip \(-a < \text{Im}(\omega_1), \text{Im}(\omega_2) < a\). Then the bicomplex Fourier transform will be

\[ \hat{f}(\omega) = \frac{2a}{a^2 + \omega^2} \]

with region of convergence

\[ \Omega = \{ \omega = a_0 + i_1a_1 + i_2a_2 + i_1i_2a_3 \in \mathbb{C}_2 : 0 \leq |a_2| < a, -a + |a_2| < a_1 < a - |a_2| \}. \]

Example 6.2.2 If

\[ f(t) = \exp(-t) \text{ for } t > 0 \]

\[ = 0 \text{ for } t \leq 0 \]

then it satisfies the estimates (6.5) for \( \alpha = 1 \) but \( \beta \) may be any positive number. Here

\[ \hat{f}(\omega) = \frac{1}{1 - i_1\omega} \]

and its region of convergence is

\[ \Omega = \{ \omega = a_0 + i_1a_1 + i_2a_2 + i_1i_2a_3 \in \mathbb{C}_2 : 0 \leq |a_2| < \frac{1 + \beta}{2} \text{ and } a_1 > -1 \text{ for any positive number } \beta \}. \]
Example 6.2.3 If

\[ f(t) = \exp\left(-\frac{t^2}{2}\right) \]

then

\[ \hat{f}(\omega) = \sqrt{2\pi} \exp\left(-\frac{\omega^2}{2}\right). \]

Its region of convergence is

\[ \Omega = \{ \omega = a_0 + i_1 a_1 + i_2 a_2 + i_1 i_2 a_3 \in \mathbb{C}_2 : -\infty < a_1, a_2 < \infty \}. \]

Example 6.2.4 If

\[ f(t) = 1 \text{ for } |t| \leq a \]
\[ = 0 \text{ for } |t| > a \]

then its complex Fourier transforms in both \( \omega_1 \) and \( \omega_2 \) planes are entire functions. Then using the above corollary, we obtain that

\[ \hat{f}_1(\omega_1) = \frac{2}{\omega_1} \sin(a\omega_1) \]

and

\[ \hat{f}_2(\omega_2) = \frac{2}{\omega_2} \sin(a\omega_2) \]

where the singularity at \( \omega = 0 \) is removable. In this case the bicomplex Fourier transform will be \( \hat{f}(\omega) = \frac{2}{\omega} \sin(a\omega) \) and its region of convergence is \( \mathbb{C}_2 \).

Example 6.2.5 Let

\[ f(t) = 0 \text{ for } t < 0 \]
\[ = \exp\left(-\frac{t}{T}\right) \sin(\omega_0 t) \text{ for } t \geq 0 \text{ and } T, \omega_0 > 0. \]
In fact, it represents the displacement of a damped harmonic oscillator. Here from the estimates (6.5), we have \( \alpha = \frac{1}{T} \). Then the complex Fourier transform in \( \omega_1 \)-plane (similar for \( \omega_2 \)-plane) is given by

\[
\hat{f}_1(\omega_1) = \frac{1}{2} \left[ \frac{1}{\omega_1 - \omega_0 + \frac{it}{T}} - \frac{1}{\omega_1 + \omega_0 + \frac{it}{T}} \right]
\]

which is holomorphic in the infinite strip \( \text{Im}(\omega_1) > -\frac{1}{T} \) except for \( \text{Re}(\omega_1) \neq \pm \omega_0 \). In this problem the bicomplex Fourier transform will be

\[
\hat{f}(\omega) = \frac{1}{2} \left[ \frac{1}{\omega + \omega_0 + \frac{it}{T}} - \frac{1}{\omega - \omega_0 + \frac{it}{T}} \right]
\]

with the region of convergence

\[
\Omega = \{ \omega = a_0 + i_1a_1 + i_2a_2 + i_1i_2a_3 \in \mathbb{C}_2 : a_0 \neq 0, \pm \omega_0; a_1 > -\frac{1}{T} \}
\]

where

\[
a_2 = \frac{\text{Im}(\omega_2) - \text{Im}(\omega_1)}{2} \text{ with } \text{Im}(\omega_1), \text{Im}(\omega_2) > -\frac{1}{T}.
\]

\section{II. Fourier inverse transform for bicomplex variables.}

In this section we extend the usual Fourier inverse transform procedure in complex exponential form \{cf. [1] & [29]\} for complex and bicomplex variables.

\subsection{6.2.8 Complex version of Fourier inverse transform.}

We start with the complex version of Fourier inverse transform and in this connection we consider a continuous function \( f(t) \) for \( -\infty < t < \infty \) satisfying the estimates (6.5) possessing the Fourier transform \( \hat{f}_1 \) in complex variable \( \omega_1 = x_1 + i_1 x_2 \) i.e.,

\[
\hat{f}_1(\omega_1) = \int_{-\infty}^{\infty} \exp(i_1 \omega_1 t) f(t) dt
\]

\[
= \int_{-\infty}^{\infty} \exp(i_1 x_1 t) \{ \exp(-x_2 t) f(t) \} dt = \phi(x_1, x_2).
\]
In fact, one may identify \( \phi(x_1, x_2) \) as the Fourier transform of \( g(t) = \exp(-x_2 t)f(t) \) in usual complex exponential form \{cf. [1] & [29]\}.

Towards this end, we assume that \( f(t) \) is continuous and \( f'(t) \) is piecewise continuous on the whole real line. Then \( \hat{f}_1(\omega_1) \) converges absolutely for \(-\alpha < x_2 < \beta\) and

\[
|\hat{f}_1(\omega_1)| < \infty
\]

which implies that

\[
\int_{-\infty}^{\infty} |\exp(i_1 \omega_1 t)f(t)| \, dt = \int_{-\infty}^{\infty} |\exp(i_1 x_1)g(t)| \, dt = \infty \int_{-\infty}^{\infty} |g(t)| \, dt < \infty.
\]

The later condition shows \( g(t) \) is absolutely integrable. Then by the Fourier inverse transform in complex exponential form \{cf. [1] & [29]\},

\[
g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-i_1 x_1 t)\phi(x_1, x_2) \, dx_1
\]

which implies that

\[
f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(x_2 t) \exp(-i_1 x_1 t)\phi(x_1, x_2) \, dx_1.
\]

Now if we integrate along a horizontal line then \( x_2 \) is constant and so for complex variable \( \omega_1 = x_1 + i_1 x_2 \) (which implies \( d\omega_1 = dx_1 \)), the above inversion formula can be extended up to complex Fourier inverse transform

\[
f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\{-i_1 (x_1 + i_1 x_2) t\} \phi(x_1, x_2) \, dx_1
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty+i_1 x_2} \exp(-i_1 \omega_1 t) \hat{f}_1(\omega_1) \, d\omega_1
\]

\[
= \frac{1}{2\pi} \lim_{x_1 \to -\infty} \int_{-x_1+i_1 x_2}^{x_1+i_1 x_2} \exp(-i_1 \omega_1 t) \hat{f}_1(\omega_1) \, d\omega_1. \tag{6.10}
\]
Here the integration is to be performed along a horizontal line in complex $\omega_1$-plane employing contour integration method.

We first consider the case $\text{Im}(\omega_1) = x_2 \geq 0$. We observe that $\hat{f}_1(\omega_1)$ is continuous for $x_2 \geq 0$ and in particular it is holomorphic (and so it has no singularities) for $0 \leq x_2 < \beta$. We now introduce a contour $\Gamma_R$ consisting of the segment $[-R, R]$ and a semicircle $C_R$ of radius $|\omega_1| = R > \beta$ with centre at the origin. Then all possible singularities (if exists) of $\hat{f}_1(\omega_1)$ must lie in the region above the horizontal line $x_2 = \beta$. At this stage we now consider the following two cases:

**Case I:** We assume that $\hat{f}_1(\omega_1)$ is holomorphic in $x_2 > \beta$ except for having a finite number of poles $\omega_1^{(k)}$ for $k = 1, 2, ... n$ therein (See Figure 2 in Appendix). By taking $R \to \infty$, we can guarantee that all these poles lie inside the contour $\Gamma_R$. Since $\exp(-i_1 \omega_1 t)$ never vanishes then the status of these poles $\omega_1^{(k)}$ of $\hat{f}_1(\omega_1)$ is not affected by multiplication of it with $\exp(-i_1 \omega_1 t)$. Then by Cauchy’s residue theorem,

$$\lim_{R \to \infty} \int_{\Gamma_R} \exp(-i_1 \omega_1 t) \hat{f}_1(\omega_1) d\omega_1 = 2\pi i \sum_{\text{Im}(\omega_1^{(k)}) > 0} \text{Re} \{ \exp(-i_1 \omega_1 t) \hat{f}_1(\omega_1) : \omega_1 = \omega_1^{(k)} \}. \quad (6.11)$$

Furthermore as $x_2 \geq 0$, we can get $|\exp(-i_1 \omega_1 t)| \leq 1$ for $\omega_1 \in C_R$ only when $t \leq 0$. In particular for $t < 0$,

$$M(R) = \max_{\omega_1 \in C_R} |\hat{f}_1(\omega_1)| = \max_{\omega_1 \in C_R} |\int_{-\infty}^{0} \exp(i_1 \omega_1 t) f(t) dt|$$

$$\leq C_2 \max_{\omega_1 \in C_R} |\int_{-\infty}^{0} \exp\{ (\beta + i_1 \omega_1) t \} dt| = C_2 \max_{\omega_1 \in C_R} \frac{1}{|\beta + i_1 \omega_1|}$$

$$\leq C_2 \max_{\omega_1 \in C_R} \frac{1}{\beta + |i_1||\omega_1|}$$

where we use the estimate $(6.5)$. Now for $|\omega_1| = R \to \infty$, we obtain that $M(R) \to 0$. Thus the conditions for Jordan’s lemma {cf. [50]} are met and so employing it we get that

$$\lim_{R \to \infty} \int_{C_R} \exp(-i_1 \omega_1 t) \hat{f}_1(\omega_1) d\omega_1 = 0. \quad (6.12)$$
Finally as,
\[
\lim_{R\to\infty} \int_{\Gamma_R} \exp(-i_1\omega_1 t) \hat{f}_1(\omega_1) d\omega_1 = \int_{C_R} \exp(-i_1\omega_1 t) \hat{f}_1(\omega_1) d\omega_1 + \int_{-R+i_1x_2}^{R+i_1x_2} \exp(-i_1\omega_1 t) \hat{f}_1(\omega_1) d\omega_1
\]
then for \( R \to \infty \), on using (6.11) and (6.12) we obtain that
\[
\int_{-\infty+i_1x_2}^{\infty+i_1x_2} \exp(-i_1\omega_1 t) \hat{f}_1(\omega_1) d\omega_1 = 2\pi i_1 \sum_{\text{Im}({\omega}_1^{(k)}) > 0} \text{Res} \{ \exp(-i_1\omega_1 t) \hat{f}_1(\omega_1) : \omega_1 = \omega_1^{(k)} \} \text{ for } t < 0
\]
and so
\[
f(t) = i_1 \sum_{\text{Im}({\omega}_1^{(k)}) > 0} \text{Res} \{ \exp(-i_1\omega_1 t) \hat{f}_1(\omega_1) : \omega_1 = \omega_1^{(k)} \} \text{ for } t < 0.
\]

**Case II:** Suppose \( \hat{f}_1(\omega_1) \) has infinitely many poles \( \omega_1^{(k)} \) for \( k = 1, 2, ..., n \) in \( x_2 > \beta \) and we arrange them in such a way that \( |\omega_1^{(1)}| \leq |\omega_1^{(2)}| \leq |\omega_1^{(3)}| \).....where \( |\omega_1^{(k)}| \to \infty \) as \( k \to \infty \). We then consider a sequence of contours \( \Gamma_k \) consisting of the segments \([- x_1^{(k)} + i_1x_2, x_1^{(k)} + i_1x_2]\) and the semicircles \( C_k \) of radii \( r_k = |\omega_1^{(k)}| > \beta \) enclosing the first \( k \) poles \( \omega_1^{(1)}, \omega_1^{(2)}, \omega_1^{(3)}, ..., \omega_1^{(k)} \) (See Figure 3 in Appendix). Then by Cauchy’s residue theorem we get that
\[
2\pi i_1 \sum_{\text{Im}({\omega}_1^{(k)}) > 0} \text{Res} \{ \exp(-i_1\omega_1 t) \hat{f}_1(\omega_1) : \omega_1 = \omega_1^{(k)} \}
= \int_{\Gamma_R} \exp(-i_1\omega_1 t) \hat{f}_1(\omega_1) d\omega_1
= \int_{C_R} \exp(-i_1\omega_1 t) \hat{f}_1(\omega_1) d\omega_1
\]
\[ x_1^{(k)} + i x_2 + \int_{-x_1^{(k)} + i x_2}^{x_1^{(k)} + i x_2} \exp(-i \omega_1 t) \hat{f}_1(\omega_1) d\omega_1. \]  
(6.13)

Now for \( t < 0 \), in the procedure similar to Case I, employing Jordan lemma here also we may deduce that

\[
\lim_{|\omega_1^{(k)}| \to \infty} \int_{C_R} \exp(-i \omega_1 t) \hat{f}_1(\omega_1) d\omega_1 = 0.
\]

Hence in the limit \( |\omega_1^{(k)}| \to \infty \) (which implies that \( |x_1^{(k)}| \to \infty \)), (6.13) leads to

\[
\int_{-\infty + i x_2}^{\infty + i x_2} \exp(-i \omega_1 t) \hat{f}_1(\omega_1) d\omega_1 = 2\pi i \sum_{\text{Res}\{\exp(-i \omega_1 t) \hat{f}_1(\omega_1) : \omega_1 = \omega_1^{(k)}\} \text{ for } t < 0}
\]

and as its consequence

\[
f(t) = i_1 \sum_{\text{Res}\{\exp(-i \omega_1 t) \hat{f}_1(\omega_1) : \omega_1 = \omega_1^{(k)}\} \text{ for } t < 0}
\]

Thus for \( x_2 \geq 0 \), whatever the number of poles is finite or infinite, from the above two cases we obtain the complex version of Fourier inverse transform as

\[
f(t) = i_1 \sum_{\text{Res}\{\exp(-i \omega_1 t) \hat{f}_1(\omega_1) : \omega_1 = \omega_1^{(k)}\} \text{ for } t < 0}. \quad (6.14)
\]

We now consider the Case \( \text{Im} (\omega_1) = x_2 \leq 0 \). The complex valued function \( \hat{f}_1(\omega_1) \) is continuous for \( x_2 \leq 0 \) and holomorphic in \( -\alpha < x_2 \leq 0 \). Introducing a contour \( \Gamma'_{R'} \) consisting of the segment \([-R', R']\) and a semi-circle \( C'_{R'} \) of radius \( |\omega_1| = R' > \alpha \) with centre at the origin, we see that all possible singularities (if exists) of \( \hat{f}_1(\omega_1) \) must lie in the region below the horizontal line \( x_2 = -\alpha \). If \( \omega_1^{(k)} \) for \( k = 1, 2, \ldots \) are the poles in \( x_2 < \alpha \), whatever the number of poles are finite or not for \( R' \to \infty \), in similar to
the previous consideration for $x_2 \geq 0$ we see that for $t > 0$ the conditions for Jordan lemma are met and so

$$f(t) = -i_1 \sum_{Im(\omega_1^{(k)}) < 0} \text{Res}\{\exp(-i_1 \omega_1 t) \hat{f}_1(\omega_1) : \omega_1 = \bar{\omega}_1^{(k)}\} \text{ for } t > 0.$$  

(6.15)

We then assign the value of $f(t)$ at $t = 0$ fulfilling the requirement of continuity of it in $-\infty < t < 0$. This completes our procedure in complex $\omega_1$ plane.

Similarly in $\omega_2 = (y_1 + i_1 y_2)$ plane the complex version of Fourier inverse transform of $\hat{f}_2(\omega_2)$ will be

$$f(t) = \frac{1}{2\pi} \lim_{y_1 \to -\infty} \int_{-y_1 + i_1 y_2}^{y_1 + i_1 y_2} \exp(-i_1 \omega_2 t) \hat{f}_2(\omega_2) d\omega_2$$  

(6.16)

where the integration is to be performed along the horizontal line in $\omega_2$ plane. Employing the contour integration method, we can obtain that

$$f(t) = i_1 \sum_{Im(\omega_2^{(k)}) > 0} \text{Res}\{\exp(-i_1 \omega_2 t) \hat{f}_2(\omega_2) : \omega_2 = \omega_2^{(k)}\} \text{ for } t < 0$$

$$= -i_1 \sum_{Im(\omega_2^{(k)}) < 0} \text{Res}\{\exp(-i_1 \omega_2 t) \hat{f}_2(\omega_2) : \omega_2 = \omega_2^{(k)}\} \text{ for } t > 0$$  

(6.17)

and the value of $f(t)$ at $t = 0$ can be assigned fulfilling the requirement of continuity of it in $-\infty < t < \infty$.

### 6.2.9 Bicomplex version of Fourier inverse transform.

Suppose $\hat{f}(\omega)$ is the bicomplex Fourier transform of the real valued continuous function $f(t)$ for $-\infty < t < \infty$ where $\omega = \omega_1 e_1 + \omega_2 e_2$ and $\hat{f}(\omega) = \hat{f}_1(\omega_1)e_1 + \hat{f}_2(\omega_2)e_2$ in their idempotent representations. Here the symbols $\omega_1, \omega_2, \hat{f}_1$ and $\hat{f}_2$ have their same representation as defined in Subsection 6.4.1. Then $\hat{f}(\omega)$ is holomorphic in

$$\Omega = \{\omega = (x_1 + i_1 x_2)e_1 + (y_1 + i_1 y_2)e_2 \in \mathbb{C}_2 : -\alpha < x_2, y_2 < \beta, -\infty < x_1, y_1 < \infty\}.$$  

(6.18)
Now using complex inversions (6.10) and (6.16), we obtain that

\[ f(t) = f(t)e_1 + f(t)e_2 \]

\[ = \left[ \frac{1}{2\pi i} \int_{D_1} \exp(-i\omega_1 t)\hat{f}_1(\omega_1) d\omega_1 \right] e_1 + \left[ \frac{1}{2\pi i} \int_{D_2} \exp(-i\omega_2 t)\hat{f}_2(\omega_2) d\omega_2 \right] e_2 \]

\[ = \frac{1}{2\pi i} \int_{D} \exp\{-i_1(\omega_1 e_1 + \omega_2 e_2)t\}\{\hat{f}_1(\omega_1)e_1 + \hat{f}_2(\omega_2)e_2\} d(\omega_1 e_1 + \omega_2 e_2) \]

\[ = \frac{1}{2\pi i} \int_{D} \exp\{-i_1(\omega t)\hat{f}(\omega) d\omega \} \quad (6.19) \]

where

\[ D_1 = \{ \omega = x_1 + i_1 x_2 \in \mathbb{C}(i_1) : -\infty < x_1 < \infty, -\alpha < x_2 < \beta \}, \]

\[ D_2 = \{ \omega = y_1 + i_1 y_2 \in \mathbb{C}(i_1) : -\infty < y_1 < \infty, -\alpha < y_2 < \beta \} \]

and D be such that \( D_1 = P_1(D) \), \( D_2 = P_2(D) \). The integration in \( D_1 \) and \( D_2 \) are to be performed along the lines parallel to \( x_1 \)-axis in \( \omega_1 \) plane and \( y_1 \)-axis in \( \omega_2 \) plane respectively inside the respective strips \(-\alpha < x_2 < \beta\) and \(-\alpha < y_2 < \beta\). As a result,

\[ D = \{ \omega \in \mathbb{C}_2 : \omega = \omega_1 e_1 + \omega_2 e_2 = (x_1 + i_1 x_2)e_1 + (y_1 + i_1 y_2)e_2 \} \quad (6.20) \]

where \(-\infty < x_1, y_1 < \infty, -\alpha < x_2, y_2 < \beta\). In four-component form D can be alternatively expressed as

\[ D = \{ \omega \in \mathbb{C}_2 : \frac{x_1 + y_1}{2} + i_1 \frac{x_2 + y_2}{2} + i_2 \frac{y_2 - x_2}{2} + i_1 i_2 \frac{x_1 - y_1}{2}, \]

\[ -\infty < x_1, y_1 < \infty, -\alpha < x_2, y_2 < \beta \}. \]

Conversely, if the integration in D is performed then the integrations in mutually complementary projections of D namely \( D_1 \) and \( D_2 \) are to be performed along the lines parallel to \( x_1 \)-axis in \( \omega_1 \) plane and \( y_1 \)-axis in \( \omega_2 \) plane respectively inside the strips \(-\alpha < x_2, y_2 < \beta\) by using the contour integration technique. So using (6.10) and (6.16), we obtain that
\[
\frac{1}{2\pi} \int_D \exp\{-i_1(\omega t)\hat{f}(\omega)d\omega \\
= \frac{1}{2\pi} \int_D \exp\{-i_1(\omega_1e_1 + \omega_2e_2)t\}\{\hat{f}_1(\omega_1)e_1 + \hat{f}_2(\omega_2)e_2\}d(\omega_1e_1 + \omega_2e_2) \\
= [\frac{1}{2\pi} \int_{D_1} \exp(-i_1\omega_1t)\hat{f}_1(\omega_1)d\omega_1]e_1 + [\frac{1}{2\pi} \int_{D_2} \exp(-i_1\omega_2t)\hat{f}_2(\omega_2)d\omega_2]e_2 \\
= \frac{1}{2\pi} \int_{D_1} \exp(-i_1\omega_1t)\hat{f}_1(\omega_1)d\omega_1]e_1 + [\frac{1}{2\pi} \int_{D_2} \exp(-i_1\omega_2t)\hat{f}_2(\omega_2)d\omega_2]e_2 \\
= f(t)e_1 + f(t)e_2 = f(t)
\]

which guarantees the existence of Fourier inverse transform for bicomplex-valued functions.

In the following, we define the bicomplex version of Fourier inverse transform method.

**Definition 6.2.1** Let \( \hat{f}(\omega) \) be the bicomplex Fourier transform of a real valued continuous function \( f(t) \) for \(-\infty < t < \infty \) which is holomorphic in \([6.18]\). The Fourier inverse transform of \( \hat{f}(\omega) \) is defined as

\[
f(t) = \frac{1}{2\pi} \int_D \exp\{-i_1(\omega t)\hat{f}(\omega)d\omega \\
\]

where \( D \) is given by \([6.20]\). On using \([6.14]\), \([6.15]\) and \([6.17]\) this inversion method amounts to

\[
f(t) = i_1e_1 \sum_{\text{Im}(\omega^{(k)}_1)>0} \text{Res}\{\exp(-i_1\omega_1t)\hat{f}_1(\omega_1) : \omega_1 = \omega^{(k)}_1\} \\
+ i_1e_2 \sum_{\text{Im}(\omega^{(k)}_2)>0} \text{Res}\{\exp(-i_1\omega_2t)\hat{f}_2(\omega_2) : \omega_2 = \omega^{(k)}_2\} \text{ for } t < 0 \quad (6.21)
\]

and

\[
f(t) = -i_1e_1 \sum_{\text{Im}(\omega^{(k)}_1)<0} \text{Res}\{\exp(-i_1\omega_1t)\hat{f}_1(\omega_1) : \omega_1 = \overline{\omega^{(k)}_1}\} \\
\]
\[-i_1 e_2 \sum_{\text{Im}(\omega_2^{(k)}) < 0} \text{Res}\{\exp(-i_1 \omega_2 t) \hat{f}_2(\omega_2) : \omega_2 = \omega_2^{(k)}\} \text{ for } t > 0. \tag{6.22}\]

We assign the value of $f(t)$ at $t = 0$ fulfilling the requirement of continuity of it in the whole real line ($-\infty < t < \infty$).

The following examples will make our notion clear:

**Example 6.2.6 1.** If $\hat{f}(\omega) = \frac{2a}{a^2 + \omega^2}$ for $a > 0$ then

\[
\hat{f}_1(\omega_1) = \frac{2a}{a^2 + \omega_1^2},
\]

\[
\hat{f}_2(\omega_2) = \frac{2a}{a^2 + \omega_2^2}
\]

and in each of $\omega_1$ and $\omega_2$ planes the poles are simple at $i_1 a$ and $i_1 a$.

Now employing \([6.21]\) and \([6.22]\), for $t < 0$ we obtain that

\[
f(t) = i_1 e_1 \text{Re}\{\exp(-i_1 \omega_1 t) \frac{2a}{a^2 + \omega_1^2} : \omega_1 = i_1 a\}
\]

\[
+i_1 e_2 \text{Re}\{\exp(-i_1 \omega_2 t) \frac{2a}{a^2 + \omega_2^2} : \omega_2 = i_1 a\}
\]

\[
= i_1 e_1 \{-i_1 \exp(at)\} + i_1 e_2 \{-i_1 \exp(at)\}
\]

\[
= \exp(-a|t|)
\]

and for $t > 0$,

\[
f(t) = -i_1 e_1 \text{Re}\{\exp(-i_1 \omega_1 t) \frac{2a}{a^2 + \omega_1^2} : \omega_1 = i_1 a\}
\]

\[
- i_1 e_2 \text{Re}\{\exp(-i_1 \omega_2 t) \frac{2a}{a^2 + \omega_2^2} : \omega_2 = i_1 a\}
\]

\[
= -i_1 e_1 \{i_1 \exp(-at)\} - i_1 e_2 \{i_1 \exp(at)\}
\]

\[
= \exp(-a|t|).
\]

Now for the continuity of $t$ in the real line, we find $f(0) = 1$. Thus the Fourier inverse transform of $\hat{f}(\omega)$ is $f(t) = \exp(-a|t|)$. 


Example 6.2.7 2. If
\[
\hat{f}(\omega) = \frac{1}{2} \left\{ \frac{1}{\omega + \omega_0 + \frac{i}{T}} - \frac{1}{\omega - \omega_0 + \frac{i}{T}} \right\} \quad \text{for } T, \omega_0 > 0
\]

then in each of \( \omega_1 \) and \( \omega_2 \) plane the poles are at \((\omega_0 - \frac{i}{T})\) and \((-\omega_0 - \frac{i}{T})\).
For both the poles the imaginary components are negative and so the poles are in lower half of both the planes. In other words, no poles exist in upper half of \( \omega_1 \) or \( \omega_2 \) planes and as its consequence \( f(t) = 0 \) for \( t < 0 \). Now at \( t > 0 \),
\[
f(t) = -i_1 e_1 \text{Res} \{ \exp(-i_1 \omega_1 t) \hat{f}_1(\omega_1) : \omega_1 = -\omega_0 - \frac{i_1}{T} \}
\]
\[
- i_1 e_1 \text{Res} \{ \exp(-i_1 \omega_1 t) \hat{f}_1(\omega_1) : \omega_1 = \omega_0 - \frac{i_1}{T} \}
\]
\[
- i_1 e_2 \text{Res} \{ \exp(-i_1 \omega_2 t) \hat{f}_2(\omega_2) : \omega_2 = -\omega_0 - \frac{i_1}{T} \}
\]
\[
- i_1 e_2 \text{Res} \{ \exp(-i_1 \omega_2 t) \hat{f}_2(\omega_2) : \omega_2 = \omega_0 - \frac{i_1}{T} \}
\]
\[
= -i_1 e_1 \frac{1}{2} \exp\left(-\frac{t}{T}\right) \exp(i_1 \omega_1 t)
+ i_1 e_1 \frac{1}{2} \exp\left(-\frac{t}{T}\right) \exp(-i_1 \omega_1 t)
- i_1 e_2 \frac{1}{2} \exp\left(-\frac{t}{T}\right) \exp(i_1 \omega_1 t)
+ i_1 e_2 \frac{1}{2} \exp\left(-\frac{t}{T}\right) \exp(-i_1 \omega_1 t)
= -i_1 \frac{1}{2} \exp\left(-\frac{t}{T}\right) \exp(i_1 \omega_1 t)
+ i_1 \frac{1}{2} \exp\left(-\frac{t}{T}\right) \exp(-i_1 \omega_1 t)
= \exp\left(-\frac{t}{T}\right) \sin(\omega_0 t).
\]

Finally, the continuity of \( f(t) \) in the whole real line implies that \( f(0) = 0 \).