CHAPTER 3

THERMOHALINE AND THERMO-VISCOELASTIC CONVECTION WITH

ROTATION AND MAGNETIC FIELD

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THREMOHALINE CONVECTION WITH ROTATION AND MAGNETIC FIELD

3.1 THE PHYSICAL PROBLEM

The physical configuration to be mathematically investigated in this chapter is precisely the following:
an infinite horizontal layer of viscous, incompressible fluid is statically confined between two horizontal boundaries \( z = 0 \) and \( z = d \) maintained at constant temperatures \( T_0 \) and \( T_1 \) and salinity \( S_0 \) and \( S_1 \) at the lower and upper boundaries respectively, where either \( T_0 > T_1, S_0 > S_1 \) or \( T_0 < T_1, S_0 < S_1 \), in the presence of a uniform rotation and magnetic field acting in a direction opposite to that of gravity. The object is to investigate the stability of the above configuration.

Let the origin be taken on the lower boundary \( z = 0 \) with the \( z \)-axis perpendicular to it along the vertically upward direction so that the \( xy \) plane then constitutes the horizontal plane \( z = 0 \).

3.2 THE BASIC EQUATIONS AND THE INITIAL STATIONARY STATE

For a general treatment of thermohaline convection problem as described in (3.1), the basic equations are as follows:
(i) **Equation of Momentum**

\[
\rho \frac{\partial U_i}{\partial t} + \rho U_j \frac{\partial U_i}{\partial x_j} = \rho \frac{\partial X_i}{\partial x_i} + \frac{\partial}{\partial x_j} \left[ \mu \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3} \mu \frac{\partial U_k}{\partial x_k} \delta_{ij} \right] + \frac{\mu e}{\partial x_j} \left( \frac{\partial X_i}{\partial x_j} \right)
\]

\[
\frac{\partial}{\partial x_j} \left( \frac{\partial H_i}{\partial x_j} \right) + 2 \epsilon_{ijk} U_j \Omega_k \rho + \frac{1}{2} \rho \frac{\partial}{\partial x_i} \left( \Omega \times r \right)^2.
\]  

(3.2.1)

(ii) **Equation of Continuity**

\[
\frac{\partial \rho}{\partial t} + U_j \frac{\partial \rho}{\partial x_j} = 0.
\]  

(3.2.2)

For an incompressible fluid

\[
\frac{\partial \rho}{\partial t} + U_j \frac{\partial \rho}{\partial x_j} = 0,
\]  

(3.2.3)

so that equation (3.2.2) reduces to

\[
\frac{\partial U_j}{\partial x_j} = 0,
\]  

(3.2.4)

(iii) **Equation of Heat Conduction**

\[
\frac{\partial}{\partial t} \left( \rho c \frac{T}{v} \right) + \frac{\partial}{\partial x_j} \left( \rho c \frac{T}{v} U_j \right) = \frac{\partial}{\partial x_j} \left( K \frac{T}{v} \right) - \rho \frac{\partial U_j}{\partial x_j} + \phi,
\]

(3.2.5)

where the rate of viscous dissipation

\[
\phi = 2 \mu e_{ij}^2 - \frac{2}{3} \mu (e_{ij}^2)^2,
\]  

(3.2.6)

(which gives the rate at which energy is dissipated irreversibly.
by viscosity in each element of volume of the fluid) and the rate of strain tensor

$$e_{ij} = \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right).$$

(3.2.7)

For an incompressible fluid $e_{jj} = 0$, and the corresponding expression for $\phi$ is given by

$$\phi = 2\mu e_{ij}^2.$$

Making use of (3.2.2), we can simplify (3.2.5) to the form

$$p \frac{\partial}{\partial t} (c \cdot T) + \rho U_j \frac{\partial}{\partial x_j} (c \cdot T) = \frac{\partial}{\partial x_j} (K \frac{\partial T}{\partial x_j}) - p \frac{\partial U_j}{\partial x_j} + \phi.$$

(3.2.8)

(iv) **Equation of Magnetic Induction**

$$\frac{\partial H_i}{\partial t} + \frac{\partial}{\partial x_j} (U_j H_i - U_i H_j) = \eta \nabla^2 H_i.$$

(3.2.9)

(v) **Solenoidal Character of Magnetic Field**

$$\frac{\partial H_i}{\partial x_i} = 0.$$

(3.2.10)

(vi) **Equation of State**

$$\rho = \rho_0 \left[ 1 - \alpha(T - T_0) + \alpha'(S - S_0) \right].$$

(3.2.11)

(vii) **Equation of Mass Diffusion**

$$p \frac{\partial}{\partial t} (c \cdot S) + \rho U_j \frac{\partial}{\partial x_j} (c \cdot S) = \frac{\partial}{\partial x_j} \left( \kappa \frac{\partial S}{\partial x_j} \right).$$

(3.2.12)
In the above equations $c_v$ is the specific heat of the fluid at constant volume, $K$ is the coefficient of heat conduction, $K'$ is the coefficient of salt diffusion, $\alpha$ is the thermal coefficient of expansion, $\alpha'$ is the analogous solvant coefficient and various other symbols occurring in the above equations have the same meaning as in CHAPTER 1.

**THE BOUSSINESQ APPROXIMATION**

The above fundamental equations of motion governing the system appear quite complicated and it seems rather difficult to handle them as such. Consequently, the need for introducing some mathematical approximation which simultaneously points out to the appropriate physical situation is strongly felt in order to simplify the basic equations. One of the contributions of Boussinesq (1903) in these problems of thermal instability is precisely at this point in the form of an approximation which is known after his name. This approximation has also gained a wide recognition in other problems of nonhomogeneous fluids, for example, the problems of Kelvin-Helmholtz instability type. In fact, while dealing with homogeneous fluids, one observes that density variations, mainly, have a two-fold effect on the stability of the problem:

(a) When coupled with the inertial terms in the equations of motion, namely $\frac{\partial U}{\partial x_j}$, it gives an inertial acceleration to the system and

(b) it interacts with the external forces acting on the system, which is gravity here, to produce a gravitational
acceleration. Boussinesq pointed out that there are situations in the domain of meteorology and oceanography where one is justified in neglecting the inertial effects of density variations as compared to its gravitational effects. This is as if \( \rho \) is taken as constant everywhere in the equation of motion except in the term with external force. Boussinesq approximation is precisely as above. One can easily verify that for the classical Bénard problem, the perturbation equation derived in the usual way and using Boussinesq approximation are one and the same because of the smallness of the coefficients of volume expansion of the fluid (Chandrasekhar (1970)) but for the problem, such as we are presently investigating, Boussinesq's approximation will definitely simplify the basic equations. Consequently, in the calculations that follow, we replace \( \rho_0 [1 - \alpha (T - T_0) + \alpha' (S - S_0)] \) by \( \rho_0 \) everywhere in the equation of motion except in the term representing the external body force \( X_i \).

On the basis of foregoing remarks the equation of momentum, namely (3.2.1) becomes

\[
\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = - \frac{\partial}{\partial x_i} \left[ P - \frac{1}{2} \left| \mathbf{U} \right|^2 \right] + \frac{\nu \nabla^2 U_i}{\rho_0} + \frac{\nabla \rho'}{\rho_0} X_i + \\
\nu \nabla^2 U_i + \frac{\mu}{4\pi \rho_0} \frac{\partial H_i}{\partial x_j} + 2 \epsilon_{ijk} \frac{\partial x_j}{\partial x_i} U_k \nabla \cdot \nabla \left( H_i H_j \right),
\]

where \( \nu = \frac{\mu}{\rho_0} \) represents the coefficient of kinematic viscosity,
\[ \nabla \rho = \rho_0 \alpha (T_0 - T), \quad (3.2.14) \]

and
\[ \nabla \rho = \rho_0 \alpha' (S - S_0). \quad (3.2.15) \]

Since we are assuming temperature variations to be small, we shall take \( \alpha \) constant with value corresponding to \( T_0 \). It is to be noted at this point that while simplifying equation (3.2.1) \( \mu \) has been treated as constant and hence taken outside the differentiation sign on account of the fact that for small temperature variations as mentioned above, the variations in \( \mu \) are small, in fact of the order of density variations and hence can be ignored.

Similarly, in equation of heat conduction (3.2.5), we can treat \( c_v \) and \( K \) as constants and hence take them outside the sign of differentiation. The term \(-p \frac{\partial \mathbf{U}}{\partial x_j}\) does not contribute anything because of equation (3.2.4). Further, viscous dissipation \( \phi \) can also be neglected because, according to equations (3.2.13) - (3.2.15) the prevailing velocities are of the order \( \mu \Delta T |x|d \) and hence, the term \( \phi \) is of the order \( \frac{\mu \alpha |x|d}{K} \), and hence, the term \( \phi \) is of the order relative to the term arising from the conduction of heat.

But this ratio for ordinary liquids (such as water and mercury) is \( 10^{-7} \) or \( 10^{-8} \) for \( d \sim 1 \text{ cm.} \) and \( \lambda \sim g \) (the acceleration due to gravity). In the above situations, the equation of heat conduction reduces to
\[ \frac{\partial T}{\partial t} + U_j \frac{\partial T}{\partial x_j} = K \nabla^2 T. \quad (3.2.17) \]
In a similar manner it can be shown that the equation of salt diffusion namely (3.2.12) reduces to

\[ \frac{\partial S}{\partial t} + U_j \frac{\partial S}{\partial x_j} = K' \nabla^2 S. \tag{3.2.18} \]

Equations (3.2.9) and (3.2.10) obviously remain unaltered as they donot contain the density field. Thus equations (3.2.9), (3.2.10), (3.2.13), (3.2.17) and (3.2.18) represent the basic equations under the Boussinesq's approximation.

We now proceed to obtain the initial stationary state solutions. Since the equilibrium situation under discussion is a static one, it is clearly characterized by the following equations.

\[ U_i = (u,v,w) = (0,0,0), \]
\[ H_i = (H_x, H_y, H_z) = (0,0,H), \]
\[ n_i = (n_x, n_y, n_z) = (0,0,n). \]

\[ T = T(z), \]
\[ S = S(z), \]

and \[ \rho = \rho_0 [1 - \alpha (T - T_0) + \alpha' (S - S_0)]. \tag{3.2.19} \]

When no motions are present, the magnetohydrodynamic equations required that the pressure distribution is governed by

\[ \frac{2}{\partial x_i} \left[ p - \frac{1}{2} \rho_0 \| \Omega \times \mathbf{E} \|^2 \right] = \rho \frac{\partial^2 x_i}{\partial t^2} = -g \rho. \tag{3.2.20} \]

The temperature and salinity distributions are governed by the equations
\[ \nabla^2 T = 0, \quad (3.2.21) \]

and \[ \nabla^2 S = 0. \quad (3.2.22) \]

The solution of equations (3.2.21) and (3.2.22) appropriate to the problem on hand is

\[ T = T_0 - \beta z, \quad (3.2.23) \]

and \[ S = S_0 - \beta' z, \quad (3.2.24) \]

where \( \beta \) (\( = \frac{T_0 - T}{d} \)) and \( \beta' \) (\( = \frac{S_0 - S}{d} \)) are the maintained uniform temperature and salinity gradients respectively.

Hence from equation (3.2.19) the corresponding distribution of density is given by

\[ \rho = \rho_0 \left[ 1 + \alpha \beta z - \alpha' \beta' z \right]. \quad (3.2.25) \]

Using the expression for \( \rho \) given by equation (3.2.25), equation (3.2.20) can be integrated to give

\[ p = p_0 - g \rho_0 \left[ z + \frac{\alpha \beta z^2}{2} - \frac{\alpha' \beta' z^2}{2} \right], \quad (3.2.26) \]

where \( p_0 \) represents the pressure at the lower boundary \( z = 0 \).

3.3 THE PERTURBATION EQUATIONS AND BOUNDARY CONDITIONS

Let the initial state described by equations (3.2.19) be slightly perturbed so that the perturbed state is given by

\[ (U_1', U_2', U_3') = (u, v, w), \]

\[ p' = p + \delta p, \]

\[ T' = T + \theta, \]

\[ S' = S + \phi, \]
\((H', H', H') = (h', h', h' + h),\)

and
\[\rho = \rho_0 [1 + \alpha' \{ (S + \phi) - S_0 \} - \alpha (T + \theta - T_0)],\]

(3.3.1)

where \((u, v, w), \ \delta p, \ \eta, \ \phi, (h', h', h')\) are respectively the perturbations in the initial velocity, pressure \(p\), temperature \(T\), salinity \(S\) and magnetic field.

Then the linearized perturbation equations of momentum, continuity, heat conduction, mass diffusion, magnetic induction and solenoidal character of magnetic field are respectively given by

\[
\rho \frac{\partial u}{\partial t} = - \frac{\partial}{\partial x} \delta p + \mu \nabla^2 u + 2 \rho \Omega v + \frac{\mu H}{4\pi} \left[ \frac{\partial h}{\partial x} - \frac{\partial h}{\partial z} \right],
\]

(3.3.2)

\[
\rho \frac{\partial v}{\partial t} = - \frac{\partial}{\partial y} \delta p + \mu \nabla^2 v - 2 \rho \Omega w + \frac{\mu H}{4\pi} \left[ \frac{\partial h}{\partial y} - \frac{\partial h}{\partial z} \right],
\]

(3.3.3)

\[
\rho \frac{\partial w}{\partial t} = - \frac{\partial}{\partial z} \delta p + \mu \nabla^2 w + g \rho_0 \alpha^0 - g \rho_0 \alpha' \phi',
\]

(3.3.4)

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,
\]

(3.3.5)

\[
\frac{\partial \phi}{\partial t} = \beta w + K \nabla^2 \theta,
\]

(3.3.6)

\[
\frac{\partial \phi}{\partial t} = \beta' w + K' \nabla^2 \phi',
\]

(3.3.7)

\[
\frac{\partial h_x}{\partial t} = H \frac{\partial u}{\partial z} + \eta \nabla^2 h_x,
\]

(3.3.8)

\[
\frac{\partial h_y}{\partial t} = H \frac{\partial v}{\partial z} + \eta \nabla^2 h_y,
\]

(3.3.9)
The text is about the equations and boundary conditions for a fluid system. The equations are:

\[ \frac{\partial h}{\partial t} = H \frac{\partial w}{\partial z} + \eta \nabla^2 h, \quad (3.3.10) \]

and \[ \frac{\partial h}{\partial x} + \frac{\partial h}{\partial y} + \frac{\partial h}{\partial z} = 0. \quad (3.3.11) \]

From equations (3.3.2) and (3.3.3), and (3.3.8) and (3.3.9) it follows that

\[ \frac{\partial Z}{\partial t} = \mu \nabla^2 Z + \frac{\mu e}{4\pi} \frac{\partial X}{\partial z} - 2 \rho \Omega \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right), \quad (3.3.12) \]

and \[ \frac{\partial X}{\partial t} = \eta \nabla^2 X + H \frac{\partial Z}{\partial z}, \quad (3.3.13) \]

where \( Z \) and \( X \) are respectively the \( z \)-components of vorticity and current density.

Equations (3.3.2) - (3.3.11) are the required perturbation equations.

**Boundary Conditions**

Since the fluid under consideration is confined between two horizontal planes \( z = 0 \) and \( z = d \), the field quantities must satisfy certain boundary conditions on them. Further, because the above surfaces are fixed and maintained at constant temperature and concentration, we must have \( w, \theta \) and \( \phi \) vanish on them. Thus

\[ w = 0 = \theta = \phi \text{ on } z = 0 \text{ and } z = d, \quad (3.3.14) \]

and rest of the boundary conditions are the same as in §§ 2.3 of chapter 2.
3.4 THE ANALYSIS IN TERMS OF NORMAL MODES

Analysing the perturbation in terms of normal modes by seeking solutions whose dependence on $x$, $y$ and $t$ is given by

$$\exp [i(K_x x + K_y y) + nt], \quad (3.4.1)$$

equations (3.3.2) - (3.3.13) become

$$\rho_{0u} = -iK_x \delta p + \mu \left( \frac{d^2}{dz^2} - K^2 \right) u + \frac{\mu e^H}{4\pi} \frac{dh}{dz} - iK_x h \frac{d}{dz} x + \frac{d}{dz} y + \frac{d}{dz} z + \frac{d}{dz} w, \quad (3.4.2)$$

$$\rho_{0v} = -iK_y \delta p + \mu \left( \frac{d^2}{dz^2} - K^2 \right) v + \frac{\mu e^H}{4\pi} \frac{dh}{dz} - iK_y h \frac{d}{dz} x + \frac{d}{dz} y + \frac{d}{dz} z + \frac{d}{dz} w, \quad (3.4.3)$$

$$\rho_{0w} = - \frac{d}{dz} (\delta p) + \mu \left( \frac{d^2}{dz^2} - K^2 \right) w + g \rho_{0\phi} + g \rho_{a' \phi'}, \quad (3.4.4)$$

$$iK_x u + iK_y v = - \frac{d}{dz} w, \quad (3.4.5)$$

$$n\phi = \beta' w + K \left( \frac{d^2}{dz^2} - K^2 \right) \phi, \quad (3.4.6)$$

$$n\phi = \beta' w + K' \left( \frac{d^2}{dz^2} - K^2 \right) \phi, \quad (3.4.7)$$

$$n h_x = H \frac{du}{dz} + \eta \left( \frac{d^2}{dz^2} - K^2 \right) h, \quad (3.4.8)$$

$$n h_y = H \frac{dv}{dz} + \eta \left( \frac{d^2}{dz^2} - K^2 \right) h, \quad (3.4.9)$$

$$n h_z = H \frac{dw}{dz} + \eta \left( \frac{d^2}{dz^2} - K^2 \right) h, \quad (3.4.10)$$
\[
\begin{align*}
\frac{iK_h}{x} + iK_h \frac{d}{dy} &= -\frac{dh}{dz}, \quad (3.4.11) \\
\rho n Z &= \mu \left( \frac{d^2}{dz^2} - K^2 \right) Z + \frac{\mu H}{4\pi} \frac{dX}{dz} - 2 \rho_0 \Omega [iK_x u + iK_y v], \\
\rho n Z &= \eta \left( \frac{d^2}{dz^2} - K^2 \right) X + H \frac{dZ}{dz}, \quad (3.4.12)
\end{align*}
\]

and
\[
\frac{\partial n X}{\partial z} = \eta \left( \frac{d^2}{dz^2} - K^2 \right) X + H \frac{dZ}{dz}, \quad (3.4.13)
\]

where \( K = \sqrt{\frac{K_x^2 + K_y^2}{y}} \) is the wave number of the perturbation, \( n \) is a constant which can be complex and \( u, v, w, \delta p, h_x, h_y, h_z, \theta, Z \) and \( X \) are now functions of \( z \) only.

Multiplying equation (3.4.2) by \( iK_x \) and equation (3.4.3) by \( iK_y \), adding the resulting equation and making use of equations (3.4.5) and (3.4.11), we have

\[
\begin{align*}
\frac{\partial n w}{\partial z} &= K^2 \delta p - \mu \left( \frac{d^2}{dz^2} - K^2 \right) \frac{dw}{dz} + 2 \rho_0 \Omega Z - \\
&\quad \frac{\mu H}{4\pi} \frac{d^2}{dz^2} - K^2 \frac{dz}{dz} \frac{dw}{dz} \\
&\quad \frac{\mu H}{4\pi} \frac{d^2}{dz^2} - K^2 \frac{dz}{dz} \frac{dw}{dz}.
\end{align*}
\]

Eliminating \( \delta p \) between equations (3.4.14) and (3.4.4), we get

\[
\begin{align*}
\left( \frac{d^2}{dz^2} - K^2 \right) \left[ \mu \left( \frac{d^2}{dz^2} - K^2 \right) - \rho n \right] w - 2 \rho_0 \Omega \frac{dZ}{dz} + \\
&\quad \frac{\mu H}{4\pi} \frac{d}{dz} \left( \frac{d^2}{dz^2} - K^2 \right) h - g \rho \sigma \alpha k^2 + g \rho \sigma \alpha' K^2 = 0.
\end{align*}
\]

Using the non-dimensional quantities defined by

\[
\begin{align*}
\Lambda &= \frac{\gamma}{K}, \quad \nu = \frac{\nu}{K}, \quad \Lambda = \frac{d}{dz}, \quad \Lambda = \frac{nd}{K}, \\
\Lambda &= \frac{\omega d}{u}, \quad \alpha = \frac{\alpha K}{\beta d}, \quad \phi = \frac{\phi K}{\beta d}, \quad \Lambda = \frac{\gamma}{z}, \quad \Lambda = \frac{h n}{H}, \\
\Lambda &= \frac{q K}{2udH} \text{ and } \sigma = \frac{\nu}{n}, \quad (3.4.16)
\end{align*}
\]
and dropping the cap for convenience in writing, the system of equations (3.4.15), (3.4.10), (3.4.12), (3.4.13), (3.4.6) and (3.4.7) yield the following linearized nondimensional perturbation equations:

\[ (D^2 - a^2)(D^2 - a^2 - \frac{p}{\nu})W + Q D(D^2 - a^2)h_z = R a^2 \theta - R_s a^2 \phi + T DZ, \]

\[ (D^2 - a^2 - \frac{p}{\nu}) \theta_z = - DW, \]  

\[ (D^2 - a^2 - \frac{p}{\nu}) Z = - DW - Q DX, \]

\[ (D^2 - a^2 - \frac{p}{\nu}) X = - DZ, \]

\[ (D^2 - a^2 - p) \phi = - W, \]

and

\[ (D^2 - a^2 - \frac{p}{\nu}) \phi = - \frac{W}{T}, \]

where \( R = \frac{g a d^4}{\nu} \) is the thermal Rayleigh number,

\[ R_s = \frac{g a' \beta d^4}{\nu} \] is the salinity Rayleigh number,

\[ Q = \frac{\mu_e H^2 d^2}{4 \pi f_0 v_f} \] is the Chandrasekhar number,

and \( T = \frac{4 Q d^2}{\nu^2} \) is the Taylor number.

Solution of equations (3.4.17) - (3.4.22) must be sought which satisfy the following boundary conditions.

\[ W = 0 = \phi = \theta \] at \( z = 0 \) and \( z = 1 \),

and either \( D^2 W = 0 = DZ = X = Dh_z \) at \( z = 0 \) and \( z = 1 \),

(Free insulating boundaries) (FIB)

or \( DW = 0 = Z = DX = h_z \) at \( z = 0 \) and \( z = 1 \),

(Rigid perfectly conducting boundaries) (RPCB)
or \( DW = 0 = Z = X = D_{h z} + a_{h z} \) at \( z = 0 \) and \( z = 1 \), (Rigid insulating boundaries)(RIB)

or any combination of FIB, RPCB and RIB. \( (3.4.23) \)

Equations (3.4.17) - (3.4.22) together with the boundary conditions (3.4.23) present an eigen value problem for \( p(=p_r + i p_i) \)
for given values of the other parameters and a given state of the system is stable, marginal or unstable provided that the real part \( p_r \) of \( p \) is negative, zero or positive respectively. Further, if \( p_r = 0 \) implies \( p_i = 0 \) for every wave number 'a', than the principle of exchange of stabilities is valid, otherwise, we will have overstability at least when instability sets in as certain modes.

3.5 MATHEMATICAL ANALYSIS

We prove the following theorems:

**THEOREM I:** If \((p, w, \phi, h_z, Z, X), p = p_r + i p_i, p_r \geq 0 \) is a solution of equations (3.4.17) - (3.4.23) with \( R > 0 \), \( R_s > 0 \) and

\[
\frac{R}{2 \tau^2} \frac{\sigma}{H^4} + \frac{T}{H^4} + \frac{Q \sigma}{H^2} \leq 1, \text{ then}
\]

\( p_i = 0. \)

In particular, if \( \frac{R}{2 \tau^2} \frac{\sigma}{H^4} + \frac{T}{H^4} + \frac{Q \sigma}{H^2} \leq 1, \) then

\( p_r = 0 \Rightarrow p_i = 0, \forall a \) and

\( p_i \neq 0 \Rightarrow p_r < 0. \)
PROOF: Multiplying both sides of equation (3.4.17) by \( W^* \) (the complex conjugate of \( W \)) and integrating the resulting equation over the vertical range of \( z \), we have

\[
\int_{0}^{1} W^* (D^2-a^2) (D^2-a^2 - \frac{\partial}{\partial z}) W \, dz = T \int_{0}^{1} W^* DZ \, dz + Ra^2 \int_{0}^{1} W^* \theta \, dz - R_S a^2 \int_{0}^{1} W^* \phi \, dz - Q \int_{0}^{1} W^* D(D^2-a^2) h \, dz. \tag{3.5.1}
\]

Now, taking the complex conjugate of both sides of equations (3.4.21), (3.4.22), (3.4.18) and (3.4.19), multiplying the resulting equations by \( \theta, \phi, (D^2-a^2) h \), and \( Z \), respectively throughout and integrating over the vertical range of \( z \), we get

\[
\int_{0}^{1} W^* \theta \, dz = - \int_{0}^{1} \eta (D^2-a^2) \theta^* \, dz, \tag{3.5.2}
\]

\[
\int_{0}^{1} W^* \phi \, dz = - T \int_{0}^{1} \phi (D^2-a^2 - \frac{\partial}{\partial z}) \phi^* \, dz, \tag{3.5.3}
\]

\[
\int_{0}^{1} D W^* (D^2-a^2) h \, dz = - \int_{0}^{1} (D^2-a^2) h \left[ D^2-a^2 - \frac{\partial}{\partial z} \right] h^* \, dz, \tag{3.5.4}
\]

and

\[
\int_{0}^{1} Z D W^* \, dz = - \int_{0}^{1} Z \left[ (D^2-a^2 - \frac{\partial}{\partial z}) Z^* + Q DX^* \right] dz. \tag{3.5.5}
\]

Integrating equations (3.5.2) - (3.5.5) by parts a suitable number of times, and utilizing the boundary conditions (3.4.23), we get

\[
\int_{0}^{1} W^* \theta \, dz = \int_{0}^{1} \left[ |D\theta|^2 + a^2 |\theta|^2 + p^* |\theta|^2 \right] \, dz. \tag{3.5.6}
\]
\[
\int_0^1 W^* \phi \, dz = \tau \int_0^1 \left[ |D\phi|^2 + a^2 |\phi|^2 + \frac{p^*}{\tau} |\phi|^2 \right] dz, \quad (3.5.7)
\]

\[
\int_0^1 DW^*(D^2 - a^2)h \, dz = -\int_0^1 \left| (D^2 - a^2)h \right|^2 dz -
\frac{p^*}{\sigma} \left[ \gamma + \int_0^1 \left( |Dh_z|^2 + a^2 |h_z|^2 \right) dz \right], \quad (3.5.8)
\]

and
\[
\int_0^1 Z DW^* dz = \int_0^1 \left( |DZ|^2 + a^2 |Z|^2 + \frac{p^*}{\sigma} |Z|^2 \right) dz + Q \int_0^1 X^* DZ \, dz, \quad (3.5.9)
\]

where \( \gamma = a \left( |h_z|^2 \right)_0 + (|h_z|^2)_1 \geq 0 \).

Substituting the value of \( DZ \) from equation (3.4.20) in equation 3.5.9, integrating the resulting equation by parts, and using the boundary conditions (3.4.23), we have
\[
\int_0^1 Z DW^* dz = \int_0^1 \left( |DZ|^2 + a^2 |Z|^2 + \frac{p^*}{\sigma} |Z|^2 \right) dz +
Q \int_0^1 (|DX|^2 + a^2 |X|^2 + \frac{p^*}{\sigma} |X|^2) \, dz. \quad (3.5.10)
\]

Now, integrating both sides of equation (3.5.1) by parts, a suitable number of times and using boundary conditions (3.4.23), we have
\[
\int_0^1 \left( (D^2 - a^2)W \right) \, dz + \int_0^1 \left( |DW|^2 + a^2 |W|^2 \right) dz = R \int_0^1 W^* \psi \, dz -
R_s a^2 \int_0^1 \phi W^* \, dz + Q \int_0^1 DW^*(D^2 - a^2)h \, dz - T \int_0^1 Z DW^* \, dz. \quad (3.5.11)
\]

Combining equations (3.5.6) - (3.5.8), (3.5.10) and (3.5.11), we have
Comparing the imaginary parts of both sides of equations (3.5.12) and cancelling $p_i \neq 0$ throughout, we have

$$\frac{1}{\sigma} \int_0^1 (|D\phi|^2 + a^2 |\phi|^2) \, dz + Ra^2 \int_0^1 |\theta|^2 \, dz + \frac{QT\sigma}{\sigma} \int_0^1 |X|^2 \, dz = 0$$

$$\frac{Q\sigma}{\sigma} \left[ \gamma + \int_0^1 (|Dh_z|^2 + a^2 |h_z|^2) \, dz \right] + \frac{T}{\sigma} \int_0^1 |Z|^2 \, dz +$$

$$R_S a^2 \int_0^1 |\phi|^2 \, dz.$$  \hspace{1cm} (3.5.13)

Multiplying equation (3.4.22) by its complex conjugate and integrating the resulting equation over the vertical range of $z$ utilizing the boundary conditions (3.4.23), we get

$$\int_0^1 |D^2 \phi|^2 \, dz + |(a^2 + \frac{P}{E}) \phi|^2 \int_0^1 |\phi|^2 \, dz + 2(a^2 + \frac{P}{E}) \int_0^1 |D\phi|^2 \, dz = 0.$$  \hspace{1cm} (3.5.14)
Since $p_{\ell} > 0$, it follows from equation (3.5.14) that
\[ 2a^2 \int_0^1 |D\phi|^2 \, dz < \frac{1}{\ell} \int_0^1 |W|^2 \, dz, \]
which upon using Poincare inequality, viz,
\[ \int_0^1 |Df|^2 \, dz \geq \pi^2 \int_0^1 |f|^2 \, dz, \tag{3.5.15} \]
where $f(0) = 0 = f(1)$, gives
\[ 2a^2 \pi^2 \int_0^1 |\phi|^2 \, dz < \frac{1}{\ell \pi^2} \int_0^1 |D\phi|^2 \, dz. \tag{3.5.16} \]

Multiplying equation (3.4.18) by $h_z^*$, integrating over the vertical range of $z$ and equating the real parts of the resulting equation, we get
\[ \gamma + \oint \frac{1}{0} (|Dh_z|^2 + a^2 |h_z|^2) \, dz + \frac{p_{\ell} \sigma}{\sigma} \oint \frac{1}{0} |h_z|^2 \, dz = \]
\[ \text{Re} \left[ - \oint \frac{1}{0} Dh_z^* \, W \, dz \right], \tag{3.5.17} \]
where Re denotes the real part.

Since $p_{\ell} > 0$, it follows from equation (3.5.17) that
\[ \oint \frac{1}{0} |Dh_z|^2 \, dz < \oint \frac{1}{0} |Dh_z| \, |W| \, dz \]
\[ < \oint \frac{1}{2} \oint \frac{1}{0} (|Dh_z|^2 + |W|^2) \, dz. \tag{3.5.18} \]

Inequality (3.5.18) implies that
\[ \oint \frac{1}{0} |Dh_z|^2 \, dz < \oint \frac{1}{0} |W|^2 \, dz. \tag{3.5.19} \]
Combining equation (3.5.17) and inequality (3.5.19), we get
\[ \gamma + \frac{1}{2} \int_0^1 \left( |Dh_{z_2}|^2 + a^2 |h_{z_2}|^2 \right) dz \leq \frac{1}{2} \int_0^1 \left( |Dh_z|^2 + |w|^2 \right) dz \]
\[ < \int_0^1 |w|^2 dz, \]
which upon using Poincare inequality (3.5.15), with \( f = W \) gives
\[ \gamma + \frac{1}{2} \int_0^1 \left( |Dh_{z_2}|^2 + a^2 |h_{z_2}|^2 \right) dz \leq \frac{1}{2\pi} \int_0^1 |DW|^2 dz. \]
(3.5.20)

Multiplying equation (3.4.19) by \( Z^* \), integrating the resulting equation by parts and using the boundary conditions (3.4.23), we get
\[ \int_0^1 \left( |DZ|^2 + a^2 |Z|^2 \right) dz + \frac{p}{q} \int_0^1 |Z|^2 dz = \int_0^1 Z^* DW dz + \]
\[ Q \int_0^1 Z^* DX dz. \]
(3.5.21)

Equating the real parts of both sides of equation (3.5.21), we have
\[ \int_0^1 \left( |DZ|^2 + a^2 |Z|^2 \right) dz + \frac{p}{q} \int_0^1 |Z|^2 dz = \Re \left( \int_0^1 Z^* DW dz + \right. \]
\[ Q \int_0^1 Z^* DX dz \right]. \]
(3.5.22)

Now
\[ \int_0^1 Z^* DX dz = - \int_0^1 X DZ^* dz. \]
(3.5.23)

Substituting the value of \( DZ^* \) from equation (3.4.20) in equation (3.5.23) and integrating resulting equation by parts, we get
\[
\int_0^1 Z^* DX \, dz = -\int_0^1 (|DX|^2 + a^2 |X|^2 + \frac{p^* \sigma}{\sigma} |X|^2) \, dz,
\]
so that
\[
\text{Re} \left( \int_0^1 Z^* DX \, dz \right) = -\int_0^1 (|DX|^2 + a^2 |X|^2 + \frac{p^* \sigma}{\sigma} |X|^2) \, dz.
\]

(3.5.24)

Since \( p \geq 0 \), it follows from equation (3.5.24) that
\[
\text{Re} \left( \int_0^1 Z^* DX \, dz \right) < 0.
\]

(3.5.25)

Combining equation (3.5.22) and inequality (3.5.25), we have
\[
\int_0^1 (|DZ|^2 + a^2 |Z|^2 + \frac{p^*}{\sigma} |Z|^2) \, dz < \text{Re} \left[ \int_0^1 Z^* DW \, dz \right]
\]
\[
= -\text{Re} \left[ \int_0^1 W DZ^* \, dz \right]
\]
\[
\leq \int_0^1 |W| |DZ| \, dz
\]
\[
\leq \left[ \int_0^1 |W|^2 \, dz \right]^{1/2} \left[ \int_0^1 |DZ|^2 \, dz \right]^{1/2}.
\]

(3.5.26)

Since \( p \geq 0 \), inequality (3.5.26) implies that
\[
\int_0^1 |DZ|^2 \, dz < \int_0^1 |W|^2 \, dz.
\]

(3.5.27)

Inequality (3.5.27) upon using Poincare inequality (3.5.15) with \( f=Z \) and \( f=W \) respectively, gives
\[
\int_0^1 |Z|^2 \, dz < \frac{1}{\beta} \int_0^1 |DW|^2 \, dz.
\]

(3.5.28)
Combining equation (3.5.13) with the inequalities (3.5.16), (3.1.20) and (3.5.28), we get

\[
\frac{1}{\alpha}[1 - \left(\frac{R}{2} \sigma + \frac{Q}{\pi^2} \tau + \frac{T}{\pi^4}\right)] \left(\int_0^1 |\delta W|^2 \, dz + \frac{\alpha^2}{\sigma} \int_0^1 |\delta W|^2 \, dz + \int_0^1 |\delta W|^2 \, dz\right)
\]

\[
\frac{1}{R} \int_0^1 |\theta|^2 \, dz + \frac{1}{\sigma} \int_0^1 |X|^2 \, dz < 0. \quad (3.5.29)
\]

Inequality (3.5.29) clearly implies that

\[
\frac{R}{2} \sigma + \frac{Q}{\pi^2} \tau + \frac{T}{\pi^4} > 1.
\]

Hence \( p_1 \neq 0, \quad p_r > 0 \Rightarrow \frac{R}{2} \sigma + \frac{Q}{\pi^2} \tau + \frac{T}{\pi^4} > 1. \)

In other words,

\[
\frac{R}{2} \sigma + \frac{Q}{\pi^2} \tau + \frac{T}{\pi^4} \leq 1, \quad p_r > 0 \Rightarrow p_1 = 0.
\]

This proves the theorem.

The essential content of THEOREM I, from the point of view of hydrodynamic instability is that an arbitrary neutral \((p_r = 0)\) or unstable \((p_r > 0)\) mode of the system is definitely non-oscillatory \((p_1 = 0)\) in character if \(\frac{R}{2} \sigma \frac{Q}{\pi^2} \tau \frac{T}{\pi^4} \leq 1.\)

In particular it follows that the principle of exchange of stabilities is valid for the problem under consideration and an arbitrary non-oscillatory mode of the system is stable if \(\frac{R}{2} \sigma \frac{Q}{\pi^2} \tau \frac{T}{\pi^4} \leq 1.\) It also shows that for the classical Rayleigh-Bénard convection problem \((R = 0 = T = Q)\), the principle of exchange of stabilities is valid.
SPECIAL CASES: It follows from THEOREM I, that an arbitrary neutral or unstable mode of the system is definitely non-oscillatory in character and in particular the principle of exchange of stabilities is valid for

(i) Rayleigh-Bénard convection (RBC) \((T = 0 = Q = R_S)\), (Pellew and Southwell, (1940)).

(ii) Rotatory RBC \((Q = 0 = R_S)\) if \(\frac{T}{\pi^4} \leq 1\).

(iii) Hydromagnetic RBC \((T = 0 = R_S)\) if \(\frac{Q}{\pi^4} \leq 1\) (Banerjee, et al, (1985)).

(iv) Rotatory Hydromagnetic RBC \((R_S = 0)\) if \(\frac{Q}{\pi^4} + \frac{T}{\pi^4} \leq 1\).

(v) Veronis thermohaline configuration (VTC) \((T = 0 = Q)\) if \(\frac{R_S}{2 \pi^2 \pi^4} \leq 1\).

(vi) Rotatory VTC \((Q = 0)\) if \(\frac{R_S}{2 \pi^2 \pi^4} + \frac{T}{\pi^4} \leq 1\).

(vii) Hydromagnetic VTC \((T = 0)\) if \(\frac{R_S}{2 \pi^2 \pi^4} + \frac{Q}{\pi^4} \leq 1\).

All the above results are uniformly valid for all combinations of free insulating, rigid perfectly conducting and rigid insulating boundaries.

THEOREM II: If \((p, W, \eta, \phi, h, z, X)\), \(p = p_r + i p_i\), \(p_r \geq 0\), \(p_i \neq 0\), is a solution of equations (3.4.17)-(3.4.23) with \(R > 0\) and \(R_S > 0\), then

\[ |p| \leq \max \left[ \sqrt{R_S}, \frac{B + \sqrt{B^2 + 4C}}{2} \right], \]

where \(B = Q \sigma\)
and \(C = T \sigma^2 (Q + 1) \left( \frac{Q}{\pi^4} + 1 \right).\)
PROOF: For an arbitrary oscillatory perturbation, neutral or unstable, we have $p_\perp \neq 0$ and $p_r \geq 0$.

Multiplying equations (3.4.22), (3.4.18) and (3.4.19) respectively by their complex conjugates, integrating the resulting equations over the vertical range of $z$ and using boundary conditions (3.4.23), we got

\[
\frac{1}{2} \int \left| (D^2 - a^2) \phi \right|^2 dz + \frac{2}{\sigma} \int \left| \frac{p_r}{\tau} \phi \right|^2 \frac{1}{2} \left( \left| D\phi \right|^2 + a^2 \left| \phi \right|^2 \right) dz + \frac{1}{\tau} \int |p|^2 \phi \phi^* dz = 0.
\]

\[
\frac{1}{2} \int |\phi|^2 dz = \frac{1}{\tau} \int |W|^2 dz, \tag{3.5.30}
\]

\[
\frac{1}{2} \int \left| (D^2 - a^2) h_z \right|^2 dz + \frac{2}{\sigma} \int \left| \frac{p_r}{\tau} \right| h_z \frac{1}{2} \left( \left| Dh_z \right|^2 + a^2 \left| h_z \right|^2 \right) dz + \frac{1}{\sigma^2} \int |p|^2 h_z h_z^* dz = 0.
\]

\[
\frac{1}{2} \int |h_z|^2 dz = \frac{1}{\sigma} \int |DW|^2 dz, \tag{3.5.31}
\]

and

\[
\frac{1}{2} \int \left| (D^2 - a^2) Z \right|^2 dz + \frac{2}{\sigma} \int \left| \frac{p_r}{\tau} \right| Z \frac{1}{2} \left( \left| DZ \right|^2 + a^2 \left| Z \right|^2 \right) dz + \frac{1}{\sigma^2} \int |p|^2 Z Z^* dz = 0.
\]

\[
\frac{1}{2} \int |Z|^2 dz = \frac{1}{\sigma} \int |DW|^2 dz + Q \int |DX|^2 dz + 2Q \text{ Re} \left[ \frac{1}{\sigma} \int DX DW^* dz \right]. \tag{3.5.32}
\]

Equations (3.5.30) - (3.5.32) under the conditions of the theorem implies that

\[
\frac{1}{2} \int |\phi|^2 dz < \frac{1}{|p|^2} \int |W|^2 dz, \tag{3.5.33}
\]

\[
\frac{1}{2} \int \left| (D^2 - a^2) h_z \right|^2 dz < \frac{1}{|p|} \int |DW|^2, \tag{3.5.34}
\]
\[
\begin{align*}
\int_0^1 |h_z|^2 dz & \leq \frac{\sigma^2}{\sigma_1^2 |p|^2} \int_0^1 |\mathbf{D}W|^2 dz, & (3.5.35)\\
and \\
\int_0^1 |Z|^2 dz & < \frac{\sigma^2}{|p|^2} \left[ \int_0^1 |\mathbf{D}W|^2 dz + Q^2 \int_0^1 |\mathbf{D}X|^2 dz + 2Q \Re \left( \int_0^1 \mathbf{D}X \mathbf{D}W^* dz \right) \right]. & (3.5.36)
\end{align*}
\]

Now
\[
\gamma + \frac{1}{2} \left( |Dh_z|^2 + a^2 |h_z|^2 \right) dz = - \int_0^1 h^*(D^2 - a^2)h_z dz
\]
\[
= | - \int_0^1 h^*(D^2 - a^2)h_z dz| = \int_0^1 |h_z|^2 |(D^2 - a^2)h_z| dz.
\]

Using Schwartz inequality it follows from inequality (3.5.37), that
\[
\gamma + \frac{1}{2} \left( |Dh_z|^2 + a^2 |h_z|^2 \right) dz \leq \left\{ \int_0^1 |hz|^2 dz \right\}^{\frac{1}{2}} \left\{ \int_0^1 |(D^2 - a^2)hz|^2 dz \right\}^{\frac{1}{2}}
\]
which upon using inequalities (3.5.34) and (3.5.35) gives
\[
\gamma + \frac{1}{2} \left( |Dh_z|^2 + a^2 |h_z|^2 \right) dz < \frac{\sigma^2}{\sigma_1^2 |p|^2} \int_0^1 |\mathbf{D}W|^2 dz. & (3.5.38)
\]
Multiplying equation (3.4.20) by \(X^*\), integrating the resulting equation by parts and using boundary conditions (3.4.23), we have
\[
\int_0^1 (|\mathbf{D}X|^2 + a^2 |X|^2 + \frac{p \sigma}{|\sigma|} |X|^2) dz = \int_0^1 X^* \mathbf{D}Z dz. & (3.5.39)
\]
Equating the real part of both sides of equation (3.5.39) and using \(p_r \gg 0\), we have
\[ \frac{1}{\pi} \int_0^1 (|DX|^2 + a^2|X|^2 + \frac{\rho \sigma}{\sigma - \frac{1}{\rho}} |X|^2) dz = \]

\[ \text{Re} \left[ \int_0^1 X^* DZ \, dz \right] = - \text{Re} \left[ \int_0^1 DX^* \cdot Z \, dz \right], \]

which implies that

\[ \frac{1}{\pi} \int_0^1 |DX|^2 \, dz < \frac{1}{\pi} \int_0^1 |DX| \cdot |Z| \, dz \leq \left[ \frac{1}{\pi} \int_0^1 |DX|^2 \, dz \right]^{\frac{1}{2}} \left[ \frac{1}{\pi} \int_0^1 |Z|^2 \, dz \right]^{\frac{1}{2}}. \]

The above inequality upon using inequality (3.5.15) with \( f = Z \), gives

\[ \frac{1}{\pi} \int_0^1 |DX|^2 \, dz < \frac{1}{\pi^2} \int_0^1 |DZ|^2 \, dz. \quad (3.5.40) \]

Inequalities (3.5.27), (3.5.40) and (3.5.15) with \( f = W \), gives

\[ \frac{1}{\pi} \int_0^1 |DX|^2 \, dz < \frac{1}{\pi^4} \int_0^1 |DW|^2 \, dz. \quad (3.5.41) \]

Further, \( \text{Re} \left[ \int_0^1 DX \cdot DW^* \, dz \right] \leq \int_0^1 |DX| \cdot |DW| \, dz \leq \int_0^1 \left( \frac{|DX|^2 + |DW|^2}{2} \right) \, dz. \quad (3.5.42) \]

Inequality (3.5.36) upon using inequalities (3.5.41) and (3.5.42), gives

\[ \frac{1}{\pi} \int_0^1 |Z|^2 \, dz < \frac{\sigma^2(Q+1)(Q/\pi^4 + 1)}{|p|^2} \int_0^1 |DW|^2 \, dz. \quad (3.5.43) \]
Combining equation (3.5.13) and inequalities (3.5.33), (3.5.38) and (3.5.43), we get

\[
\left( \frac{1}{\sigma} - \frac{Q}{|p|} - \frac{T \sigma (Q+1)(Q/\pi^4 + 1)}{|p|^2} \right) \int_0^1 |Dw|^2 \, dz + a^2 \left( \frac{1}{\sigma} - \frac{RS}{|p|^2} \right) \int_0^1 |W|^2 \, dz < 0.
\]

Hence either \( \frac{1}{\sigma} - \frac{RS}{|p|^2} < 0 \) or \( \frac{1}{\sigma} - \frac{Q}{|p|} - \frac{T \sigma (Q+1)(Q/\pi^4 + 1)}{|p|^2} < 0 \)

or \( |p| < \max \left[ \sqrt{\frac{RS}{\sigma}}, \frac{B + \sqrt{B^2 + 4C}}{2} \right] \).

This proves the theorem.

The essential content of **THEOREM II**, from the point of view of hydrodynamic instability is that it provides us with an upper bound for the modulus of the complex growth rate of an arbitrary oscillatory perturbation, neutral or unstable, for the problem of thermohaline convection in the presence of a uniform vertical rotation and magnetic field. The theorem states that the complex growth rate \( \rho = \rho_r + i \rho_i \) of an arbitrary oscillatory perturbation, neutral or unstable, for the problem under consideration with \( R > 0 \) and \( R_S > 0 \) must lie inside a semicircle whose centre is at the origin and radius \( \max[\sqrt{\frac{RS}{\sigma}}, \frac{B + \sqrt{B^2 + 4C}}{2}] \), in the right half of the \( \rho_r \rho_i \)-plane, where

\[
B = Q\sigma
\]

and

\[
C = T \sigma^2 (Q+1) \left( \frac{Q}{\pi^4} + 1 \right).
\]

This result is uniformly valid for all combinations of free insulating, rigid perfectly conducting and rigid insulating boundaries.
SPECIAL CASES: It follows from THEOREM II, that the complex growth rate of an arbitrary oscillatory perturbation, neutral or unstable, in each of the linear stability problems \( p_i(i=1,2,\ldots,6) \) given below, must lie in a semicircle each of which has its centre at the origin and \((\text{radius})^2 = r_i^2(i=1,2,\ldots,6)\), as shown in front of each problem:

\[
\begin{align*}
P_1 &= \text{Rotatory Rayleigh-Bénard convection (RBC)} \\
&\quad (Q = 0 = R_S), \quad r_1^2 = T \sigma^2 (\text{Banerjee et al. (1981)}).
\end{align*}
\]

\[
\begin{align*}
P_2 &= \text{Hydromagnetic RBC} (T = 0 = R_S) \quad r_2^2 = Q^2 \sigma^2 \\
&\quad (\text{Gupta et al. (1982)}).
\end{align*}
\]

\[
\begin{align*}
P_3 &= \text{Rotatory hydromagnetic RBC} (R_S = 0), \\
&\quad r_3^2 = \frac{B + \sqrt{B^2 + 4C}}{2}.
\end{align*}
\]

\[
\begin{align*}
P_4 &= \text{VTC} (T = 0 = Q), \quad r_4^2 = R_S \sigma (\text{Banerjee et al. (1981)}).
\end{align*}
\]

\[
\begin{align*}
P_5 &= \text{Rotatory VTC} (Q = 0), \quad r_5^2 = \max \{ R_S \sigma, T \sigma^2 \} \\
&\quad (\text{Banerjee et al. (1981)}).
\end{align*}
\]

\[
\begin{align*}
P_6 &= \text{Hydromagnetic VTC} (T=0), \quad r_6^2 = \max \{ R_S \sigma, Q^2 \sigma^2 \} \\
&\quad (\text{Gupta et al. (1982)}).
\end{align*}
\]

The above results are uniformly valid for all combinations of free insulating, rigid perfectly conducting and rigid insulating boundaries.

THEOREM III: If \((p,W,\omega,\phi, h, Z,X)\), \(p = p_r + ip_i, p_r \geq 0\), is a solution of equations (3.4.17) - (3.4.23), with \(R < 0, R_S < 0\)

\[
\begin{align*}
\frac{|R|}{2\pi} \frac{\sigma}{T} + \frac{Q}{\pi^2} \leq 1, \text{ then}
\end{align*}
\]

\[p_i = 0.\]
In particular, if \( \frac{|R|}{2\pi^4} + \frac{T}{\pi^4} + \frac{Q}{P} \leq 1 \), then

\[
p_r = 0 \Rightarrow p_i = 0, \forall a \text{ and } p_i \neq 0 \Rightarrow p_r < 0.
\]

**PROOF:** The governing equations and the boundary conditions in the present case are given by equations (3.4.17) - (3.4.23) with

\[
R = -|R| \text{ and } R_S = -|R_S|.
\]

(3.4.44)

Using (3.5.44) in equations (3.4.17) - (3.4.23) and proceeding exactly as in THEOREM I, we have for \( p_i \neq 0 \),

\[
\frac{1}{\sigma^2} \left( |DW|^2 + a^2 |W|^2 \right) dz + |R_S| \frac{1}{\sigma^2} \left| \phi \right|^2 dz +
\]

\[
\frac{Q}{\sigma^2} \left[ y + \frac{1}{\sigma^2} (|Dh|^2 + a^2 |h|^2) dz \right].
\]

(3.5.45)

Multiplying equation (3.4.21) by its complex conjugate and integrating the resulting equation over the vertical range of \( z \) using boundary conditions (3.4.23), we get

\[
\frac{1}{\sigma^2} \int_0^1 |D^2 \varphi|^2 dz + \frac{1}{\sigma^2} \int_0^1 |\varphi|^2 dz + 2 (a^2 + p_r) \frac{1}{\sigma^2} \int_0^1 |D\varphi|^2 dz =
\]

\[
\frac{1}{\sigma^2} \int_0^1 |W|^2 dz.
\]

(3.5.46)

Since \( p_r \geq 0 \), it follows from equation (3.5.46) that

\[
2a^2 \int_0^1 |D\varphi|^2 dz < \int_0^1 |W|^2 dz,
\]
which upon using inequality (3.5.15) with \( f = \phi \) and \( f = W \) respectively gives

\[
2a^2 \int_0^1 |\phi|^2 \, dz < \frac{1}{\pi^2} \int_0^1 |DW|^2 \, dz. \quad (3.5.47)
\]

Combining equation (3.5.45) and inequalities (3.5.47), (3.5.20) and (3.5.28), we get

\[
\frac{1}{\sigma} \left[ 1 - \left( \frac{|R| \sigma}{2 \pi^4} + \frac{Q \sigma}{\pi^2} + \frac{T}{\pi^4} \right) \right] \int_0^1 |DW|^2 \, dz + \frac{a^2}{\sigma} \int_0^1 |W|^2 \, dz + |R_s| a^2 \int_0^1 |\phi|^2 \, dz + \frac{TQ \sigma}{\sigma} \int_0^1 |X|^2 \, dz < 0. \quad (3.5.48)
\]

Inequality (3.5.48) clearly implies that

\[
\frac{|R| \sigma}{2 \pi^4} + \frac{Q \sigma}{\pi^2} + \frac{T}{\pi^4} > 1.
\]

Hence \( p_1 \neq 0 \), \( p_r = 0 \) \( \Rightarrow \) \( \frac{|R| \sigma}{2 \pi^4} + \frac{Q \sigma}{\pi^2} + \frac{T}{\pi^4} > 1 \).

In other words,

\[
\frac{|R| \sigma}{2 \pi^4} + \frac{Q \sigma}{\pi^2} + \frac{T}{\pi^4} \leq 1, \quad p_r = 0 \Rightarrow p_1 = 0.
\]

This proves the theorem.

The essential content of THEOREM III, from the point of view of hydrodynamic instability is similar to that of THEOREM I.

SPECIAL CASES: It follows from THEOREM III, that an arbitrary neutral or unstable mode of the system is definitely non-oscillatory in character and in particular the principle of exchange of stabilities is valid for

(i) Stern's thermohaline configuration (STC) \((T = 0 = Q)\)

if \( \frac{|R| \sigma}{2 \pi^4} \leq 1 \).
(ii) Rotatory STC \((Q = 0)\) if \(\frac{|R|\sigma}{2\pi^4} + \frac{1}{\sigma^2} \leq 1\).

(iii) Hydromagnetic STC \((T = 0)\) if \(\frac{|R|\sigma}{2\pi^4} + \frac{Q}{\sigma^2} \leq 1\).

All the above results are uniformly valid for all combinations of free insulating, rigid perfectly conducting and rigid insulating boundaries.

**THEOREM IV:** If \((p, W, \theta, \phi, h, Z, X)\), \(p = p^r + i p^i\), \(p^r \geq 0\), \(p^i \neq 0\), is a solution of equations (3.4.17) - (3.4.23) with \(R < 0\) and \(R_S < 0\), then

\[
|p| < \max \left\{ \sqrt{|R|\sigma}, \frac{B + A\sqrt{B^2 + 4C}}{2} \right\},
\]

where \(B = Q\sigma\),

and \(C = T\sigma^2 (Q + 1) (Q/\pi^4 + 1)\).

**PROOF:** Multiplying equation (3.4.21) by its complex conjugate, integrating the resulting equation over the vertical range of \(z\) and using boundary conditions (3.4.23), we have

\[
\frac{1}{0} |(D^2 - a^2)\theta|^2 dz + 2 p^r \frac{1}{0} [|D\theta|^2 + a^2 |\theta|^2] dz + |p|^2 \frac{1}{0} |\theta|^2 dz = \frac{1}{0} |W|^2 dz.
\]

Since \(p^r \geq 0\), equation (3.5.48) implies that

\[
\frac{1}{0} |\theta|^2 dz \leq \frac{1}{|p|^2} \frac{1}{0} |W|^2 dz.
\]

Combining equation (3.5.45) with the inequalities (3.5.38), (3.5.43) and (3.5.50), we get
\[ \left( \frac{1}{\sigma} - \frac{Q}{|p|} \right) - \frac{T \sigma(Q+1)(Q/\pi^4+1)}{|p|^2} \int_{0}^{1} |W|^2 \, dz + a^2 \left( \frac{1}{\sigma} - \frac{|R|}{|p|^2} \right) \int_{0}^{1} |W|^2 \, dz < 0. \]  

(3.5.51)

Hence

either \( \frac{1}{\sigma} - \frac{|R|}{|p|^2} < 0 \) or \( \frac{1}{\sigma} - \frac{Q}{|p|} - \frac{T \sigma(Q+1)(Q/\pi^4+1)}{|p|^2} < 0 \).

or \( |p| < \max \left[ \sqrt{|R|/\sigma}, \frac{B+\sqrt{R^2 + 4C}}{2} \right] \).

This proves the theorem.

The essential content of **THEOREM IV**, from the point of view of hydrodynamic instability is similar to that of **THEOREM II**.

**SPECIAL CASES:** It follows from **THEOREM IV**, that the complex that the complex growth rate of an arbitrary oscillatory perturbation, neutral or unstable, in each of the linear stability problems \( S_i \) \( (i = 1, 2, \ldots, 3) \) given below, must lie in a semicircle each of which has its centre at the origin and \((\text{radius})^2 = r_i^2 \) \((i = 1, 2, 3)\) as shown in front of each problem.

(i) \( S_1 \) = Stern's thermohaline convection (STC) \( (T=0=Q) \),

\[ r_1^2 = |R|/\sigma \text{ (Banerjee, et al. (1981)).} \]

(ii) \( S_2 \) = Rotatory STC \( (Q = 0) \), \( r_2^2 = \max \left[ |R|/\sigma, T \sigma^2 \right] \text{ (Banerjee, et al. (1981)).} \)

(iii) \( S_3 \) = Hydromagnetic STC \( (T = 0) \), \( r_3^2 = \max \left[ |R|/\sigma, Q^2 \sigma^2 \right] \text{ (Gupta, et al. (1982)).} \)

All the above results are uniformly valid for all combinations of free insulating rigid perfectly conducting and rigid insulating boundaries.
3.6 THERMO-VISCOELASTIC CONVECTION WITH ROTATION AND MAGNETIC FIELD

THE PHYSICAL PROBLEM

The physical configuration to be mathematically investigated is precisely the following: a visco-elastic fluid is statically confined between two infinite horizontal boundaries \( z = 0 \) and \( z = d \) maintained at constant temperature \( T_0 \) and \( T_1 \) (\( T_0 > T_1 \)) at the lower and upper boundary respectively in the presence of a uniform rotation and magnetic field acting in a direction opposite to gravity. The object is to examine the stability of this configuration.

Let the origin be taken on the lower boundary \( z = 0 \) with \( z \) - axis perpendicular to it along the vertically upward direction so that \( xy \)-plane then constitute the horizontal plane \( z = 0 \). We assume that the visco-elastic fluid under consideration is described by Maxwell constitutive relation,

\[
p_{ij} + t \frac{d}{dt} p_{ij} = \mu \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right)
\]

(3.6.1)

where \( t_0 \) is the Maxwell's relaxation time, \( U_i (= u,v,w) \) is the velocity vector, \( \frac{d}{dt} \) \( (= \frac{\partial}{\partial t} + U_k \frac{\partial}{\partial x_k} \) is mobile operator, \( \mu \) is the coefficient of viscosity, \( p_{ij} \) is a viscous stress tensor and \( t \) is time. The total stress tensor \( p_{ij} \) is related to \( p_{ij} \), through

\[
p_{ij} = -p \delta_{ij} + p_{ij}'
\]

(3.6.2)

where \( p \) is the (scalar) pressure and \( \delta_{ij} \) is Kronecker delta.
3.7 THE LINEARIZED PERTURBATION EQUATIONS

Proceeding exactly as in §§ 3.4, one obtains the following nondimensional linearized perturbation equations for the problem under consideration:

\[
(D^2 - a^2)(D^2 - a^2 - p(1 + \Gamma p) / \sigma) \theta = R a^2 (1 + \Gamma p) \theta + T(1 + \Gamma p) DZ - Q(1 + \Gamma p) D(D^2 - a^2) h, \quad (3.7.1)
\]

\[
(D^2 - a^2 - p) \theta = - W, \quad (3.7.2)
\]

\[
(D^2 - a^2 - p \frac{\tau}{\sigma}) h = - D W, \quad (3.7.3)
\]

\[
(D^2 - a^2 - p(1 + \Gamma p) / \sigma) Z = - Q(1 + \Gamma p) DX - (1 + \Gamma p) DW, \quad (3.7.4)
\]

and

\[
(D^2 - a^2 - p \frac{\sigma}{\sigma}) X = - D Z, \quad (3.7.5)
\]

where \( \Gamma = \frac{t_0 d^2}{\kappa} \) is an elastic parameter and various other symbols have the same meaning as before.

Equations (3.7.1)-(3.7.5) together with boundary conditions (3.4.23) present an eigenvalue problem for \( p \) for given value of the other parameters and a given state of the system is stable, marginal or unstable provided that the real part \( p_\tau \) of \( p \) is negative \( (p_\tau < 0) \), zero \( (p_\tau = 0) \) or positive \( (p_\tau > 0) \) respectively.

3.8 MATHEMATICAL ANALYSIS

We prove the following theorem:

**Theorem I:** If \( (p, \theta, \phi, \psi, Z, X), p = p_\tau + i p_\sigma, p_\tau \geq 0, p_\sigma \neq 0, \) is a solution of equations (3.7.1)-(3.7.5) and (3.4.3) with \( R > 0 \), then
\[ |p| \leq \max \left( R, \sqrt{(Q+1)\sigma}, \frac{\sigma}{2} \left[ Q + \sqrt{Q^2 + 4T(Q+1)} \right] \right). \]

**Proof:** Multiplying both sides of equation (3.7.1) by \( W^* \) (the complex conjugate of \( W \)) and integrating the resulting equation over the vertical range of \( z \), we have
\[
\int_0^1 W^*(D^2-a^2)(D^2-a^2 - \frac{p(1+\Gamma_p)}{\sigma}) W \, dz = \frac{Ra^2}{(1+\Gamma_p)} \int_0^1 W^* \, dz + T(1+\Gamma_p) \int_0^1 W^* \, dz - Q(1+\Gamma_p \int_0^1 W^* \, dz. \tag{3.8.1}
\]

Multiplying the complex conjugate of both the sides of equation (3.7.4) by \( Z \) and integrating the resulting equation over the vertical range of \( z \) a suitable number of times using boundary conditions (3.4.23), we have
\[
\int_0^1 Z \frac{DZ}{(1+\Gamma_p)} \, dz = \frac{1}{(1+\Gamma_p)} \int_0^1 (|DZ|^2 \frac{a^2}{|Z|^2}) \, dz + \frac{1}{\sigma} \int_0^1 \frac{Z}{\sigma} \, dz.
\tag{3.8.2}
\]

Substituting the value of \( DZ \) from equation (3.7.5) in equation (3.8.2) and integrating the resulting equation by parts using boundary conditions (3.4.23), we have
\[
\int_0^1 Z \frac{DZ}{(1+\Gamma_p)} \, dz = \frac{1}{(1+\Gamma_p)} \int_0^1 (|DZ|^2 \frac{a^2}{|Z|^2}) \, dz + \frac{1}{\sigma} \int_0^1 \frac{Z}{\sigma} \, dz + \frac{1}{\sigma} \int_0^1 \frac{X}{|X|^2} \, dz. \tag{3.8.3}
\]

Integrating both sides of equation (3.8.1) by parts a suitable number of times using boundary conditions (3.4.23), we get
Combining equations (3.8.4), (3.8.3) and integral relations (3.5.8) and (3.5.6), we get

\[ \int_0^1 (D^2 - a^2)W^2 \, dz + \frac{(1 + rp)p}{\sigma} \int_0^1 (|D^2 + a^2|W^2) \, dz = \]

\[ Ra^2(1+rp) \int_0^1 W^* \theta \, dz - T(1+rp) \int_0^1 Z \, DW^* \, dz + \]

\[ Q(1+rp) \int_0^1 DW^* (D^2 - a^2)h \, dz. \quad (3.8.4) \]

Where \( \gamma = a \left[ (|h^2|)_{z=0} + (|h^2|)_{z=1} \right] \geq 0. \)

Comparing the imaginary parts of both sides of equation (3.8.5)
and cancelling \( p \neq 0, \) throughout we have
Multiplying equation (3.8.6) by its complex conjugate and integrating the resulting equation over the vertical range of \( z \) utilizing boundary conditions (3.4.23), we get

\[
\begin{align*}
\frac{(1 + 2 \Re \rho r)}{\sigma} \int_0^1 \left[ |D W|^2 + a^2 |\bar{W}|^2 \right] d z + Ra^2 \int_0^1 |\theta|^2 d z + \frac{2 \Gamma(1 + \rho p r)}{|1 + \rho p|^2} x \\
\int_0^1 \left[ |D Z|^2 + a^2 |Z|^2 \right] d z + T Q \Gamma \int_0^1 |X|^2 d z + T Q \Gamma \int_0^1 \left[ |D X|^2 + a^2 |X|^2 \right] d z +
\end{align*}
\]

Since \( p_r \geq 0 \), \( p_r^2 \leq 1 \) and

\[
2 \Re \int_0^1 D X D W^* d z \leq 2 \int_0^1 |D X| |D W| d z
\]

\[
\leq \int_0^1 \left( |D X|^2 + |D W|^2 \right) d z
\]

equation (3.8.7) imply that
Multiplying equation (3.7.2) by its complex conjugate, integrating the resulting equation over the vertical range of \( z \) and using boundary conditions (3.4.23), we get

\[
\frac{1}{|1+\pi r|^2} \int_0^1 (|Dz|^2 + a^2|z|^2)dz.
\]

Since \( p_r \geq 0 \), equation (3.8.9) implies that

\[
\int_0^1 (D^2-a^2)\theta^2 dz \leq \int_0^1 |W|^2 dz,
\]

\[
\int_0^1 |\theta|^2 dz \leq \frac{1}{|p|^2} \int_0^1 |W|^2 dz
\]

Now

\[
\int_0^1 (|D\theta|^2 + a^2|\theta|^2)dz = -\int_0^1 \theta^*(D^2-a^2) \theta dz
\]

\[
= -\int_0^1 \theta^*(D^2-a^2) \theta dz
\]

\[
\leq \int_0^1 |\theta|(D^2-a^2)|\theta|dz,
\]

which upon using Schwartz in equality gives

\[
\int_0^1 (|D\theta|^2 + a^2|\theta|^2)dz \leq \int_0^1 |\theta|^2 dz \leq \frac{1}{(D^2-a^2)^2} \int_0^1 |\theta|^2 dz.
\]
Inequality (3.8.13) together with inequalities (3.8.10) and (3.8.11) gives

\[ \frac{1}{\sigma} \int_0^1 \left( |D\theta|^2 + a^2 |\theta|^2 \right) dz < \frac{1}{|p|} \int_0^1 |W|^2 dz. \]  

(3.8.14)

Combining equation (3.8.6) with the inequalities (3.8.8), (3.5.20) and (3.8.14), we get

\[ \frac{1 + 2\sigma}{\sigma} \int_0^1 \left( |D W|^2 + a^2 |W|^2 \right) dz + Ra^2 \int_0^1 |\theta|^2 dz + 
\]
\[ 2T \sigma \int_0^1 \left( |D Z|^2 + a^2 |Z|^2 \right) dz + \frac{TQ}{\sigma} \int_0^1 |X|^2 dz + 
\]
\[ \frac{TQ}{\sigma} \int_0^1 \left( |D X|^2 + a^2 |X|^2 \right) dz + \frac{2TQ}{\sigma} \int_0^1 \left( |D Z|^2 + a^2 |Z|^2 \right) dz. \]

(3.8.15)

It follows from inequality (3.8.15) that

\[ \left( \frac{1}{\sigma} - \frac{R \Gamma}{|p|} \right) a^2 \int_0^1 |W|^2 dz + TQ \left( \frac{(Q+1)\sigma}{|p|^2} \right) \int_0^1 |D X|^2 + 
\]
\[ \left( \frac{1}{\sigma} - \frac{Q}{|p|} - \frac{T \sigma (Q+1)}{|p|^2} \right) \int_0^1 |D W|^2 dz < 0. \]

(3.8.16)
Inequality (3.8.6) clearly implies that

\[ |p| < \max \left[ R \sigma, \sqrt{\frac{Q+1}{\pi}}, \frac{\sigma}{2} \left( Q + \sqrt{Q^2 + 4T(Q+1)} \right) \right]. \]

This proves the theorem.

The essential content of THEOREM I, from the point of view of hydrodynamic instability is similar to that of THEOREM II of §§ 3.5.

3.9 CONCLUSIONS

In this section we summarize the principal results established in this chapter. It is shown that

1. If \( R > 0, R_S > 0 \) and \( \frac{R_S \sigma}{2T^2 \pi^4} + \frac{T}{\pi^4} + \frac{Q}{\pi^2} \leq 1 \), then an arbitrary oscillatory mode of the system is stable and in particular the principle of exchange of stabilities is valid. This result also implies that an arbitrary oscillatory mode of the system is stable and in particular the principle of exchange of stabilities is valid for:

(a) RBC \((T = 0 = Q = R_S)\), (Pellew and Southwell, (1940)).

(b) Rotatory RBC \((Q = 0 = R_S)\) if \( \frac{T}{\pi^4} \leq 1 \).

(c) Hydromagnetic RBC \((T = 0 = R_S)\) if \( \frac{Q}{\pi^2} + \frac{1}{\pi^2} \leq 1 \) (Banerjee, et al. (1985)).

(d) Rotatory Hydromagnetic RBC \((R_S = 0)\) if \( \frac{Q}{\pi^2} + \frac{T}{\pi^4} \leq 1 \).
(e) VTC \( T = 0 = Q \) if \( \frac{R_S \sigma}{2 \pi^2} \leq 1 \).

(f) Rotatory VTC \( Q = 0 \) if \( \frac{R_S \sigma}{2 \pi^2} + \frac{T}{\pi} \leq 1 \).

(g) Hydromagnetic VTC \( T = 0 \) if \( \frac{R_S \sigma}{2 \pi^2} + \frac{Q}{\pi} \leq 1 \).

2. If \( R > 0 \) and \( R_S > 0 \), then the complex growth rate \( \sigma \) of and arbitrary oscillatory perturbation, neutral or unstable, must lie inside a semicircle whose center is at the origin and radius \( \text{Max}\left( \sqrt{R_S \sigma}, \frac{B + \sqrt{B^2 + 4C}}{2} \right) \), in the right half of the complex \( \sigma \)-plane, where

\[
B = Q \sigma \quad \text{and} \quad C = T \sigma^2 (Q+1)(Q/\pi^2+1). \quad (3.9.1)
\]

This result also implies that an arbitrary oscillatory perturbation, neutral or unstable, for the problems \( P_i \) \( (i = 1, 2, \ldots, 6) \) given below, must lie in a semicircle each of which has its centre at the origin and \( \text{(radius)}^2 = r_i^2 \) \( (i = 1, 2, \ldots, 6) \) as shown in front of each problem:

\[
P_1 = \text{Rotatory RBC} \quad (Q = 0 = R_S), \quad r_1^2 = T \sigma^2
\]

(Banerjee, et al. (1981)).

\[
P_2 = \text{Hydromagnetic RBC} \quad (T = 0 = R_S), \quad r_2^2 = Q^2 \sigma^2
\]

(Gupta, et al. (1982)).

\[
P_3 = \text{Rotatory hydromagnetic RBC} \quad (R_S = 0), \quad r_3^2 = \left[ \frac{B + \sqrt{B^2 + 4C}}{2} \right]^2.
\]

\[
P_4 = \text{VTC} \quad (T = 0 = Q), \quad r_4^2 = R_S \sigma \text{Banerjee, et al. (1981)).}
\]
If $R < 0$ and $R_s < 0$ and
\[
\frac{|R|}{2\pi^4} + \frac{T}{\pi^4} + \frac{Q}{\pi^4} \leq 1,
\]
then an arbitrary oscillatory mode of the system is stable and in particular the principle of exchange of stabilities is valid. This result also implies that an arbitrary oscillatory mode of the system is stable and in particular the principle of exchange of stabilities is valid for:

(a) STC ($T = 0 = Q$) if
\[
\frac{|R|}{2\pi^4} \leq 1.
\]

(b) Rotatory STC ($Q = 0$) if
\[
\frac{|R|}{2\pi^4} + \frac{T}{\pi^4} \leq 1.
\]

(c) Hydromagnetic STC ($T = 0$) if
\[
\frac{|R|}{2\pi^4} + \frac{Q}{\pi^4} \leq 1.
\]

If $R < 0$ and $R_s < 0$ then the complex growth rate of an arbitrary oscillatory perturbation, neutral or unstable, must lie inside a semicircle whose centre is at the origin and radius
\[
\max \left[ \sqrt{|R|}, \frac{B + \sqrt{B^2 + 4C}}{2} \right],
\]
in the right half of the complex $p$-plane, where $B$ and $C$ are given by (3.9.1). This result also imply that the complex growth rate of an arbitrary oscillatory perturbation, neutral or unstable, for the problems $S_i = i = 1, 2, 3$) given below, must lie in a semicircle each of which has its centre at the origin and $(radius)^2 = r_i^2(i = 1, 2, 3)$ as shown in front of each problem:
5. If $R > 0$, $R_S = 0$ and the fluid is described by Maxwell constitutive relation then the complex growth rate $p$ of an arbitrary oscillatory perturbation, neutral or unstable, must lie inside a semicircle whose centre is at the origin and radius $= \max \left[ R \Re \sigma, \sqrt{\frac{(Q+1)\sigma}{\Gamma}}, \frac{\sigma}{2} \left[ Q + \sqrt{A^2 + 4T(Q+1)} \right] \right]$, in the right half of the complex $p$-plane.

All the above results are uniformly valid for all combinations of free insulating, rigid perfectly conducting and rigid insulating boundaries.