CHAPTER 1

INTRODUCTION: A PARTIAL SURVEY OF THE LITERATURE
1.1 RAYLEIGH-TAYLOR INSTABILITY PROBLEM

A static state, in which an incompressible fluid of variable density is arranged in horizontal strata and the pressure $p$ and the density $\rho$ are functions of the vertical co-ordinate $z$ only, is clearly kinematically realizable one. The character of equilibrium of this initial static state can be determined as usual, by supposing that the system is slightly disturbed and then by following its further evolution. For a comprehensive account of the mathematical and physical aspects of the problem of characterizing the equilibrium of such a static state as mentioned above together with its possible extensions in various other domains of interest one may be referred to Hide (1955,56), Chandrasekhar (1961), Spiegal and Zahn (1977) and Yih (1980). However, we shall summarize some of the fundamental results of the above problem in brief which will be helpful for the understanding and appreciation of the present work.

The Rayleigh-Taylor instability occurs when a fluid supports a denser fluid against gravity whereupon the two tend to interchange positions and is encountered frequently in nature and in the laboratory. Lord Rayleigh (1883) investigated the character of equilibrium of an inviscid incompressible fluid of variable density stratified in horizontal planes. Rayleigh showed that the necessary and sufficient condition
that a stratified heterogeneous fluid be stable is that \( \frac{dp}{dz} \) should be negative everywhere in the flow domain. This implies that if \( \frac{dp}{dz} \) is positive anywhere inside the flow domain, the configuration is not stable. Rayleigh's treatment was extended by Chandrasekhar (1955) to include the effect of viscosity. Chandrasekhar showed that if \( D^2 \mu \) is positive everywhere inside the flow domain then the oscillatory modes are stable, where \( D = \frac{d}{dz} \) and \( z \) is the vertical coordinate. It follows that the principle of exchange of stabilities is valid if \( D^2 \mu \) is positive everywhere inside the flow domain. Chandrasekhar also established a variational principle for solving the underlying characteristic value problem. Salig (1964) developed an alternative scheme of calculations to tackle the same problem and established that the restriction on \( D^2 \mu \) for the validity of principle of exchange of stabilities as imposed by Chandrasekhar is not necessary. Fan (1955) investigated the case of an exponentially varying density stratification increasing vertically upwards and evaluated the rate of growth of various modes for given values of the wave number and two other nondimensional number which occur in his analysis. However, the treatment of Fan is severely restricted by the fact that he assumes identical mathematical forms for \( f(Z) \) and \( \mu(Z) \) which make the governing differential equations having constant coefficients.

Banerjee and Kalthia (1971) reconsidered the problem especially with a view to fix up bounds for the rate of growth of an arbitrary mode and derive a sufficient condition for the
stability of the system for general continuous \( \rho(z) \) and \( \mu(z) \).
The eigen value problem as considered by them is essentially as follows:

\[
D[n \rho Dw - \mu D(D^2-K^2)w - D\mu(D^2+K^2)w] = K^2 \left[ -\frac{g}{n} \frac{DP}{n} w + \rho nw \right]
- \mu(D^2-K^2)w - 2 D\mu Dw],
\]

for \( 0 < z < d \),

with

\( w = 0 = Dw \) at \( z = 0 \) and \( d \) (at a rigid boundary),

where \( w(z) \) is the single Fourier component of vertical perturbation velocity; \( z \) is the vertical coordinate; \( D \) stand for \( \frac{d}{dz} \); \( n = n_r + i n_i \) is the complex growth rate of the wave mode of the wave number \( K \); \( \rho(z) \) and \( \mu(z) \) respectively represent the density and viscosity of the fluid; \( g \) is the acceleration due to gravity; \( d \) is the depth of the fluid layer.

Equation (1.1.1) together with boundary condition (1.1.2) defines an eigen value problem for \( n \), given \( K \) and two of the fundamental problems associated with the above system of equations and relevant to the recognition of a certain state of the physical system as well, is to characterize whether \( n_i = 0 \) for every admissible \( K \) or \( n_i \neq 0 \), when \( n_r = 0 \), i.e.
whether marginal state is stationary or oscillatory (if at all it exists). In other words whether 'principle of exchange of stabilities' is valid or we have overstability; and if overstability is satisfied then what are the possible eigen values for $n_1$. Banerjee and Kalthia established a circle theorem which states that the complex growth rate of an arbitrary oscillatory perturbation which may be stable, neutral or unstable must lie inside a circle with centre at the origin and $(radius)^2 = \max \text{ flow domain } [- \frac{g |D_p|}{\rho}].$ Further, a sufficient condition for the stability of the system under certain restrictions on $\rho$ and $\mu$ was obtained. It is to be noted that unlike the results of Fan which depend upon the special forms of $\rho$ and $\mu$, the results of Banerjee and Kalthia hold good for general density and viscosity distributions and are also independent of the wave number. However, the restriction on $\mu$, in their sufficient condition appears to be little curious as a physical explanation of it is by no means clear.

Hide (1955) studied the initial manner of development of an infinitesimal disturbance of a plane horizontal layer of a heavy, viscous, incompressible fluid of density $\rho$, which rotates uniformly with an angular velocity $\omega$ about an axis making an angle $\theta$ with the vertical and established that if $D^2\mu$ is positive everywhere inside the flow domain then oscillatory modes are stable. It follows that the principle of exchange of stabilities is valid if $D^2\mu$ is positive everywhere inside the flow domain. Further, it was shown that the solution can be characterized by a variational principle.
To account for the essential new features produced by rotation Chandrasekhar (1961) studied the Rayleigh-Taylor problem for an inviscid fluid when the axis of rotation is along the vertical. He considered the case of an exponentially varying density and showed that if $D\rho$ is negative everywhere inside the flow domain then the configuration is stable for disturbances of all wave numbers, whereas if $D\rho$ is positive everywhere inside the flow domain the configuration is unstable for disturbances of all wave numbers exceeding a critical and stable otherwise.

Kruskal and Schwarzschild (1954) studied Rayleigh-Taylor problem with a uniform horizontal magnetic field and established that for disturbances which are independent of the co-ordinate along the direction of the uniform magnetic field vector $H$, the presence of the magnetic field does not in any way effect the development of Rayleigh-Taylor instability. Hide (1955) studied Rayleigh-Taylor problem in the presence of a vertical magnetic field and established that the underlying characteristic value problem can also be solved as a variational problem. Further it was shown that if $D^2\mu$ is positive everywhere in the flow domain then oscillatory modes are stable. It follows that the principle of exchange of stabilities is valid if $D^2\mu$ is positive everywhere inside the flow domain.

Chakraborty (1982) investigated Rayleigh-Taylor instability of an infinitely conducting, rotating, stratified fluid in the presence of a horizontal magnetic field, in which the fluid density and the magnetic field strength were taken
to be arbitrary functions of the vertical coordinate. He obtained an upper bound for the growth rate of an arbitrary unstable mode of the system of a given wave length and a sufficient condition for the stability of a disturbance.

The investigation of the Rayleigh-Taylor instability problem with continuous variation of $\rho$ and $\mu$ in the presence of a uniform rotation and magnetic field does not appear to have been studied on account of the extremely complicated nature of the growing differential equations for the problem. The investigation of this problem forms a chapter of the present work.

For latest developments concerning the Rayleigh-Taylor problem and some of its extensions, one is referred to Plesset et al. (1974); Holiday et al. (1981); Menikoff et al. (1978); Krusin et al. (1979); Andrews (1981); Pullins (1982); Baker (1983); Donald (1983) and Ira et al. (1983).

1.2 RAYLEIGH-BÉNARD CONVECTION PROBLEM

The study of the onset of convection in a fluid layer contained between two infinite horizontal boundaries heated uniformly from below, known as Bénard convection problem has received considerable attention both experimentally and theoretically from successive generations of workers in the field of hydrodynamic and hydromagnetic stability. The whole subject started from the historical observations of J. Thompson (1882), who on the occasion of a
countryside excursion of the Belfast Naturalists Field Club noted some tessellations in a tub of warm evaporating wash-water. However, the first carefully controlled laboratory experiments were carried by the French Scientist H. Bénard in 1901. Bénard's discovery of convection cells was a rather fortuitous one and was made in early experiments on wireless reception. He observed that when the temperature gradient is small the fluid remains at rest and heat is transported through the fluid only by conduction. However, when the temperature gradient is increased beyond a certain value the fluid undergoes a time independent motion called convection current. Heat is now transported through the fluid by convection as well as conduction. In actual experiments the fluid arranged itself in a rectangular cellular pattern in the shape of hexagonal cells and motion takes place within the cell.

This work of Bénard served as a stimulus to theoretical investigations and led Lord Rayleigh to propose his theory of buoyancy driven convection. Lord Rayleigh (1916) for the first time setforth the criterion for the Bénard cells for the idealized case of free boundaries with a linear temperature gradient. He showed that a top heavy fluid layer was stable under the joint influence of viscosity and heat diffusion until the vertical temperature drop was large enough to overcome these two dissipative and stabilizing mechanisms. He observed that convective flow sets in when the rate at which free energy is liberated by the uprising of the hot, less dense fluid near
the base exceeds the rate at which energy is dissipated by thermal conduction and viscous damping. This argument was later made use of in the development of a variational principle that governs the linear mathematical stability problem.

Beginning with Rayleigh's (1916) analysis of the phenomenon a great deal of efforts have been made treating cellular convection as a buoyancy driven flow. Despite generally impressive qualitative agreement between theory and various observations this approach has not been satisfactory in describing Bénard's classic experiments which motivated the original theoretical investigations. The inappropriateness of Rayleigh's model to Bénard's experiments was adequately explained by the experimental and analytical studies by Block (1956), Pearson (1958) and Nield (1964). A detailed study of the problem showed that the sole parameter determining stability was the temperature difference, made dimensionless by a combination of parameters that corresponds to what is now known as the Rayleigh number, which is also the product of Prandtl number times Grashof number.

Further contributions to the Bénard problem mainly in the nature of verification of Rayleigh's analysis from a somewhat more general point of view and methods of solution were made by Jeffrey (1926, 28) and Low (1929). The mathematical analysis of Rayleigh is fundamentally based in terms of the vertical component of velocity while the analysis of Jeffreys (1926) proceeds primarily in terms of temperature. Jeffrey restricted himself to rectangular cells, but added the more realistic feature
of rigid boundaries. Low (1929) considered the exact solution of the basic characteristic value problem for the first time.

Pellew and Southwell (1940) reformulated the theoretical analysis. The eigen value problem as derived by them is essentially as follows:

\[(D^2-a^2) (D^2-a^2-p)w = \frac{\rho \alpha}{\gamma} d^2 a^2 \theta,\]  
\[\text{and} \quad (D^2-a^2-p)\theta = -\frac{\beta d^2}{K} w.\]  

for \(0 < z < 1,\)

with \(w = 0 = \theta\) for \(z = 0\) and \(z = 1,\)

and either \(Dw = 0\) for \(z = 0\) and \(z = 1\) (rigid boundaries),
or \(D^2w = 0\) for \(z = 0\) and \(z = 1\) (free boundaries)(1.2.3)

where \(D\) stands for \(\frac{d}{dz}\), 'a' is the wave number, \(p ( = p_r + i p_i)\) is the complex growth rate of the wave mode of wave number 'a'
\(w(z)\) and \(\theta(z)\) are respectively the single Fourier components of the vertical perturbation velocity and temperature, \(g\) is the acceleration due to gravity, \(\alpha\) is the coefficient of volume expansion of the fluid, \(d\) is the depth of the fluid layer, \(\sigma\) is the thermal prandtl number \(\beta\) is the uniform adverse temperature gradient maintained between the two bounding surfaces, and \(K\) is the coefficient of thermal diffusivity of the fluid.
Equations (1.2.1) and (1.2.2) together with boundary conditions (1.2.3) define an eigen value problem for $p$ given 'a' and two of the fundamental problems associated with the above system of equations, and relevant to the recognition of a certain state of the physical system as well, is to characterize whether $p_1 = 0$ for every admissible 'a' or $p_1 \neq 0$, when $p_r = 0$, i.e. whether marginal state is stationary or oscillatory (if at all it exists). In other words whether 'principle of exchange of stabilities' is valid or we have overstability; and if overstability is satisfied then what are the possible eigen values for $p_1$. Pellew and Southwell demonstrated in a rigorous mathematical manner that $p_1 = 0$ for every admissible 'a' in the more general case wherein the boundaries could be either both free or both rigid or any one of them free and the other rigid. In otherwords the 'principle of exchange of stabilities' is valid for the problem in the general case. Pellew and Southwell also established the criterion for Bénard motions in the general case.

On account of the importance of the Bénard configuration in the problems of meteorology, oceanography and various other fields of practical interest, its meaningful extensions in the frame work of various external force fields has been the main centre of activity in the recent past of many research workers in the field of hydrodynamic and hydromagnetic stability. Chandrasekhar (1953, 55) and others investigated
the Bénard problem under the influence of a uniform rotation acting parallel to gravity, and showed that rotation inhibit the onset of instability and elongate the cells which appear at the marginal stability for certain ranges of the concerned parameters. Chandrasekhar traces the origin of this similar behaviour in the Taylor-Proudman theorem in the case of rotation. It was also shown that the marginal state could either be stationary or oscillatory in character for which sufficient conditions may be obtained and were in good agreement with the experimental results of Nakagawa (1959), Fultz-Nakagawa and Frenzen (1954). To put the matter more explicitly it is concluded that the principle of exchange of stabilities is valid so long \( \sigma \) exceeds a certain critical value \( \varpi \), where \( \varpi \) depends on the nature of bounding surfaces, while the onset of instability will be as overstable oscillations if \( \sigma < \varpi \) and the Taylor number \( T \) exceeds a certain critical value \( T^* \) depending on \( \sigma \), and for \( T < T^* \) the onset of instability will be as stationary convection. Banerjee et al. (1981) investigated the same problem and showed that for all combinations of dynamically free or rigid boundaries the complex growth rate \( \rho \) of an arbitrary oscillatory perturbation, neutral or unstable must lie inside a semicircle whose centre is at origin and \((\text{radius})^2 = T \sigma^2 \), in the right half of the complex \( \rho \)-plane, where \( T \) is the Taylor number and \( \sigma \) is the thermal Prandtl number.
Thompson (1951) investigated simple Bénard problem in the presence of a magnetic field for inviscid fluids and derived a sufficient criterion, in terms of the parameters of the system alone, for the validity of the principle of exchange of stabilities, but his investigation is somewhat limited on account of the neglect of the effects of viscosity. Chandrasekhar (1952) analysed the same problem and solved the governing equations which include the viscous effects in a case which he states is appropriate to two free boundaries and concluded the validity of the principle of exchange of stabilities for $\kappa \leq \eta$, which is essentially Thompson's criterion, where $\kappa$ is the fluid thermal diffusivity and $\eta$ is the fluid electrical resistivity but unfortunately, his boundary conditions are not correct as they do not include the magnetic boundary conditions. An attempt to correct this error in Chandrasekhar's analysis was made by Gibson (1966) who discussed the problem in the asymptotic case $\alpha \to -\infty$ and recovered the above Thompson - Chandrasekhar criterion for the validity of this exchange principle, but Gibson's analysis too is somewhat limited since it applies only in the limit $\alpha \to -\infty$. Sherman and Ostrach (1966) examined a more general problem when the fluid is completely confined in an arbitrary region and the uniform magnetic field is applied in an arbitrary direction and derived sufficient criterion for the validity of this exchange principle for the present problem, a result
conjectured earlier by Chandrasekhar, but their analysis too is of limited value since one cannot apriori be certain when this criterion will be satisfied. In the asymptotic case $Q \to \infty$ Sherman and Ostrach also recovered the Thompson-Chandrasekhar criterion for the validity of this exchange principle but this again is somewhat limited for reasons stated earlier.

The search for a clear cut result corresponding to Pellew and Southwell's (1940), in the present magnetohydrodynamic case, has continued till the recent times and it is relevant to note in this connection that Chandrasekhar has predicted the existence of a $Q < \sigma$ such that for $\sigma > \sigma$ (i.e. $K > \eta$) and $Q < Q_{\sigma}$ the principle of exchange of stabilities will be valid, $\sigma$ being the thermal Prandtl number, though as mentioned earlier his analysis is not free from errors.

Banerjee et al. (1985) established that if $Q < \sigma \leq \sigma^2$ then the principle of exchange of stabilities is valid, which is essentially Chandrasekhar's prediction but without the condition $\sigma > \sigma$ and thus provides the natural extension of Pellew and Southwell's result which was long sought after. This result is uniformly valid for all combinations of rigid or free boundaries which may be conducting or insulating. Gupta et al. (1982) investigated the same problem and showed that for rigid or free conducting boundaries the complex growth rate of an arbitrary oscillatory perturbation, neutral or unstable, must lie inside a semicircle whose centre is
the origin and \((\text{radius})^2 = \mathcal{Q} \sigma^2\), in the right half of the complex \(p\)-plane, where \(\mathcal{Q}\) is the Chandrasekhar number and \(\sigma\) is the thermal Prandtl number.

Chandrasekhar (1954, 56) examined the problem where \(\mathcal{Q}\) and \(\mathcal{H}\) are both uniform and in the vertical direction and the boundaries are free and non-conducting. He has shown that the manner of the onset of instability, must in general, depend in an extremely complicated manner on the relevant parameters involved. Assuming the principle of exchange of stabilities to be valid, the numerical results indicated that, if \(T\) was kept fixed and \(\mathcal{Q}\) increased from zero, the critical Rayleigh number remained roughly constant until \(\mathcal{Q}\) reached a certain critical value when it started to decrease. When \(\mathcal{Q}\) was increased further, the critical Rayleigh number continued to decrease, reaching a minimum before starting to increasing again.

The work of Eltayeb (1972, 75) is also concerned with the combined effect of rotation and magnetic field on the simple Bénard problem. When both \(T\) and \(\mathcal{Q}\) are large i.e. in the double limit \(T \to \infty, \mathcal{Q} \to \infty\). The analysis is carried for three different configurations classified by the orientations of the magnetic field and rotation axes under a variety of different surface condition. It is shown that irrespective of the nature of the boundaries the asymptotic dependence of the critical Rayleigh number on \(T\) and \(\mathcal{Q}\) is the same, apart from constants of proportionally of order unity. More specifically,
assuming the principle of exchange of stabilities to be valid, \( R_c = 0 \left( T^\frac{1}{2} \right) \) when \( T = 0 \left( Q^2 \right) \) in all the configuration which, of course, implies that, for large values of \( T \), the presence of magnetic field facilitates convection (Eltayeb and Roberts, 1970).

Gupta et al. (1983) investigated the problem under the joint influence of rotation and magnetic field for the case of rigid conducting boundaries especially with a view to derive bounds for the complex growth rate \( p \) of an arbitrary oscillatory perturbation which may be neutral or unstable. It is shown that for all wave numbers \( a^2 > \frac{Q}{2\sigma} \), the complex growth rate of such a perturbation must lie inside a semi-circle whose centre is at the origin and (radius)\(^2\) = \( \frac{\sigma^2}{4} [Q + \sqrt{Q^2 + 4T}] \), in the right half of the complex \( p \)-plane, where \( \sigma \) is the magnetic Prandtl number and various other symbols have the same meanings as before.

The above result of Gupta et al. have the weakness that it is dependent on the wave number \( a^2 \) of the perturbation and holds good only for the case of rigid conducting boundaries. Further no criterion analogous to that of Banerjee et al. (1985) for the validity of the principle of exchange of stabilities exists in this case. In the present work we derive such a criterion and remedy the weakness in the result of Gupta et al. (1983).

In recent years attention has also been paid to the stability characteristics of Bénard problem for non-Newtonian
fluids due to their increasing use in many biophysical and engineering fields. Vest and Arpaci (1969) investigated Bénard problem for a Maxwellian fluid. Critical Rayleigh number, wave numbers and frequencies for overstability are determined for both free and rigid boundaries. Elasticity is found to destabilize the fluid, and the presence of rigid boundaries is found to be slightly stabilizing. Further contributions to Bénard problem for non-Newtonian fluids which are relevant to the present work are by Takashima (1970), Bhatia and Steiner (1972,73), Eltayeb (1975,76) and Gupta et al. (1983).

For latest developments concerning the problem of simple Bénard problem and some of its extensions, one is referred to Roberts (1978); Spiegel (1971); Haken (1979); Joseph (1976,80); Palm (1975); Tritton (1977); Busse (1978,81) Gershuni et al. (1976); Drazin et al. (1981); Normand et al. (1977); Koschmider (1974,81); Roppo et al. (1984) and Edwards (1984).

1.3 THERMOHALINE CONVECTION PROBLEM

Comparatively recently there has been interest in a class of hydrodynamic instabilities which arise solely as a result of differential diffusion of two or more components of a multi-component fluid. The well known example of this type of instability occurs in fluid stratified jointly by heat and salt. Instability is able to develop as a result of the disparity in diffusion rates of heat and salt even when the
overall density distribution is gravitationally stable. When the two components contribute in an opposite sense to the local density gradient of the fluid things happen which are not present in ordinary convection. For example convection may occur even in situations where the fluid is statically stable i.e. the density of the fluid decreases upwards. Also, in contrast to ordinary thermal convection the onset of convection may arise in an oscillatory manner. Diffusion, which is generally stabilizing in a fluid containing a single solute, can now act so as to allow the release of the potential energy in the component that is heavy at the top. Much of this work was initiated with an application to the ocean in mind and because heat and salt (or some other dissolved substances) are then important the process has been called thermohaline or thermosolutal convection. Related effects have now been observed in other contexts and the name Double-Diffusive convection has been used to encompass this wider range of phenomena. This type of convection deals with buoyancy driven motion which results from the diffusion of two components with different molecular diffusivities making opposing contributions to the vertical density gradient. It is the differential diffusion that produces the density differences required to drive the motion. The nature and form of thermohaline instability is determined by which of the component is stabilizing and which is destabilizing. In either case the instability is driven by the release of the potential energy associated with the unstable gradient of the destabilizing component.
The whole subject developed from the thought experiments of Stommel, Arons and Blanchard (1956), in which they considered the consequences of displacing a fluid particle vertically in such a system. When the temperature field is destabilizing (cold fresh water above warm salty water) a fluid particle displaced upwards will adjust its temperature to the local value relatively quickly. However, because of the disparity in diffusion coefficient between heat and salt the particle remains locally heavy by virtue of its excess salinity and it is forced to sink to its original level and overshoot. In its new position it experiences an upward force because of its local salinity deficiency so that it begins to oscillate about its original position, experiencing buoyancy restoring force at the extreme points in its motion, because of lag in temperature between the particle and its surrounding fluid. The particle always leaves either of the limiting positions faster than it arrives. This results in an increase in the amplitude of the oscillations with time with the instability being maintained by the potential energy in the unstable temperature field and this gives rise to distinct homogeneous layers separated by relative sharp density interfaces. In this configuration convective motions occur in a manner more similar to thermal convection and is referred to as "diffusive convection".

Such a motion is not possible in the case where the salinity profile is destabilizing. Here by the same differential diffusing argument as above, if a fluid particle is displaced
vertically, buoyancy forces always act to keep the particle moving in the direction of the displacement, i.e. any displacement will grow directly with time, the instability being driven on this occasion by the potential energy in the salinity gradient. By an analogous argument if a particle of fluid is displaced downwards it will continue to fall. In this configuration convection typically takes the form of long vertical cells and is referred to as "Salt fingure convection". It must be mentioned here that for sufficiently large temperature gradients, steady motion can occur because a large temperature field can overcome the restoring tendency of the salinity field.

The first theoretical discussion of salt fingers was given by Stern (1960). He considered a horizontal layer of fluid which is heated from above and in which the mass concentration of a chemical dissolved is maintained at $C_0$ at the lower boundary and $C_1$ at the upper boundary ($C_1 > C_0$). The temperature at the two boundaries are maintained at $T_0$ and $T_1$ respectively, with $T_1 > T_0$. He showed that even if the fluid in the undisturbed condition is lighter at the top than at the bottom, the configuration still might be unstable provided the thermal diffusivity of the solute exceeds its mass diffusivity. The above investigation of Stern is restricted by the assumption that the boundaries are free and the principle of exchange of stabilities is valid.

Veronis (1965) considered a situation in which $C_0 > C_1$. 
and \( T_0 > T_1 \) and showed that instability might still occur as overstability provided the destabilizing temperature gradient is sufficiently large but compatible with the condition that the total density field is gravitationally stable. He also obtained the critical Rayleigh number and corresponding wave number and frequency for overstable oscillations as well as for stationary cellular pattern at the marginal state. The above investigation of Veronis is restricted by the assumption that the boundaries are free.

Banerjee et al. (1981) proposed a new scheme for combining the governing differential equations for the double-diffusive convection problem with or without rotation, that leads to the following bounds on the modulus of the complex growth rate \( p \) of an arbitrary oscillatory perturbation, neutral or unstable:

\[
|p|^2 < R_S \sigma \quad \text{(Veronis' configuration)},
\]

\[
|p|^2 < -R \sigma \quad \text{(Stern's configuration)},
\]

\[
|p|^2 < \text{greater of } (R_S, T \sigma^2) \quad \text{(rotatory Veronis' configuration)},
\]

and

\[
|p|^2 < \text{greater of } (-R \sigma, T \sigma^2) \quad \text{(rotatory Stern's configuration)},
\]

where \( R, R_S, T \) and \( \sigma \) are respectively the thermal Rayleigh number, the concentration Rayleigh number, the Taylor number and the Prandtl number. The above results are uniformly valid for all combinations of dynamically free and rigid boundaries and are important since the exact solutions of the problems (especially in situations when one or both boundaries are rigid) are not obtainable in a closed form.
Gupta et al. (1982, 83) extended the above results of Banerjee et al. for hydromagnetic Veronis' and Stern's configurations and also for non-Newtonian fluids with or without rotation/magnetic field. Further under the joint influence of a uniform vertical rotation and magnetic field Gupta et al. (1983) extended the results of Banerjee et al. These results of Gupta et al. have the same weaknesses as for the Rayleigh-Bénard convection problem with rotation and magnetic field as pointed out earlier. The present work remedies these weaknesses.

For latest developments concerning the problem of thermohaline convection and some of its extensions, one may be referred to Turner (1973, 74); Huppert (1976); Nield (1967); Griffiths (1979); Huppert et al. (1981); Finalayson (1972); Stern (1975); Suzukawa et al. (1982); Narayanan (1984); Mc-Dougall (1983); Worthem et al. (1983); Rudraiah (1984) and Chen et al. (1984).

1.4 GENERALIZED BÉNARD CONVECTION PROBLEM

In the investigation of the stability of Rayleigh-Bénard convection problem the fluid is assumed to be originally homogeneous, while on the other hand for the Rayleigh-Taylor configuration it is assumed to be non-heat conducting. Consequently, it seems desirable to present a unified treatment of these two classes of problems by including a basic non-homogeneity in the Rayleigh-Bénard problem or imposing a uniform temperature
gradient upon the Rayleigh-Taylor problem when the fluid under discussion is heat conducting. Such a unified theory of these two classical problems of hydrodynamic instability was presented by Banerjee (1969, 1971, 1972). Taking the initial fluid non-homogeneity of the form \( \rho = \rho_0 e^{-\delta z} \), where \( \delta \) is a constant which can be positive or negative, in the Rayleigh-Bénard model, he sets up what we shall subsequently refer to as a generalized Rayleigh-Bénard convection or generalized Bénard convection problem. It is shown that the system becomes unstable to only overstable perturbations when the disturbance is infinitesimal. The critical Rayleigh number and the corresponding wave length and frequency for these overstable oscillations at the marginal state have also been evaluated. The above investigation is valid for dynamically free and rigid boundaries.

It is interesting to note here that the governing equations of the problem of infinitesimal amplitude instability in thermohaline convection wherein one investigates the instability of an infinite horizontal layer of fluid subjected to a destabilizing salt gradient and to a stabilizing temperature gradient (Stern' 1960) or the gravitationally opposite configuration (Veronis' 1965) coincide exactly, when the mass diffusivity of the dissolved solute is neglected, with the governing equations of the generalized Benard problem with \( \delta < 0 \) or \( \delta > 0 \) which are derived independently and with a basically different outlook namely, to present a unified treatment of
the well known Bénard and Rayleigh-Taylor instability problems.

Banerjee, Gupta, Dube and Gupta (1976), Banerjee, Gupta and Gupta (1976) and Banerjee, Gupta and Dube (1976) investigated the above problem in the cases when the system is acted upon by a uniform vertical rotation and/or magnetic field respectively. It is shown that in the case of non-conducting boundaries, which may be rigid or free, the principle of exchange of stabilities is not satisfied irrespective of whether the magnetic field and rotation are acting individually or jointly. The overstable solutions at the marginal state when the boundaries are free and non-conducting and magnetic field alone is acting show that the onset of instability is postponed further and further as $Q$ increases whenever $\zeta < \sigma$ and $R_z > 0$. The stabilizing character of the magnetic field is thus established in the above situation. Further, the above solutions at the marginal state when applied to liquid metals show that for increasing values of $R_z > 0$ i.e. the initial distribution of density becoming more and more bottom heavy the system becomes more and more stable. On the other hand the system becomes unstable through non-oscillatory modes for $\zeta < \sigma$ and $R_z < 0$. The analytical solutions at the marginal state when the boundaries are free and rotation alone is acting establish the stabilizing character of rotation as $T$ increases, for all admissible values of $\sigma$ and $R_z > 0$, while for $\sigma > 1$ and $R_z < 0$ the system becomes unstable through non-oscillatory modes.
as in the case with the magnetic field. It is noted here that while for $R_2 = 0$ the marginal state, in both the problems wherein magnetic field or rotation are individually included, could either be stationary or oscillatory in character depending on certain values of the parameters involved (Chandrasekhar, 1961), it is definitely oscillatory in character for $R_2 > 0$. The investigation of the situation when magnetic field and rotation are simultaneously present and the boundaries are free and non-conducting with $R_2 > 0$ leads to an interesting result. The overstable solutions at the marginal state when applied to liquid metals show an important effect. Thus, while in the presence of a uniform magnetic field acting separately, an increasing stable initial density stratification i.e. increasing values of $R_2 > 0$ stabilizes the system more and more, it plays a dual role, i.e. it first destabilizes the system and then stabilizes it when magnetic field and rotation are simultaneously present.

1.5 MOTIVATIONS, AIMS AND CONTRIBUTIONS OF THE PRESENT WORK

The work embodied in the present thesis represents our attempt to investigate theoretically some hydrodynamic and hydromagnetic stability problems of relevance in oceanography, astrophysics and non-Newtonian fluid mechanics in the framework of linear theory. The thesis is divided into five chapters with the following contents:
CHAPTER I: This chapter is introductory in nature and gives a partial survey of the literature on various hydrodynamic and hydromagnetic stability problems considered in the thesis.

CHAPTER II: This chapter is primarily based on the works of Chandrasekhar (1955), Hide (1955,56), Salig (1964) and Banerjee and Kalthia (1971) on the Rayleigh-Taylor instability problem and is motivated by the following results proved by them respectively:

(i) If $D'\mu > 0$ everywhere inside the flow domain, then oscillatory modes are stable, where $\mu$ is the viscosity of the fluid and $D = \frac{d}{dz}$, $z$ being the vertical coordinate.

(ii) The above result holds under the individual effect of a uniform vertical rotation or magnetic field on the problem.

(iii) The variational principle which describes the stability problem of susperposed fluids can be written in a form which is free of any ambiguities and allows a very simple physical interpretation.

(iv) If $D\rho < 0$ and $D'\mu > 0$ everywhere inside the flow domain, then instability cannot occur, where $\rho$ is the density of the fluid.

The dependence of the above results on $D^2\rho$ appears to be curious as the physical interpretation to it is by no means clear.

This chapter of the thesis mathematically investigates the Rayleigh-Taylor instability problem with continuously
varying density $\rho(z)$ and viscosity $\mu(z)$ under the combined effect of a uniform vertical rotation and magnetic field. It is shown that

(a) An arbitrary neutral or unstable mode of the system is non-oscillatory in character. In particular oscillatory modes of the system are stable.

(b) If $\frac{d\rho}{dz} < 0$ everywhere inside the flow domain, then the system is stable.

(c) If $\frac{d\rho}{dz} > 0$ everywhere inside the flow domain, then the growth rate $n_r (= n_r + i n_i)$ of an arbitrary unstable mode ($n_r > 0$) (which is non-oscillatory ($n_i = 0$) in character in view of result (a)) lies inside an open interval on the right of $n_i$-axis.

(d) A variational principle free from any ambiguities can also be enunciated for the determination of the growth rate of an arbitrary unstable mode of the system.

Result (a) in particular implies that the restrictions on $\rho$ which are used by Chandrasekhar (1955) and Hide (1955, 56) to derive this result are no longer required. Further, result (b) shows that Banerjee and Kalthia's condition on $D^2\mu$ to derive this result is also not required. In particular it also shows that under the individual effect of a uniform vertical rotation or magnetic field the system is stable if $D\rho < 0$ everywhere inside the flow domain. All these results are new and do not appear to have been reported earlier in the literature with the same degree of generality to the best of our knowledge.
CHAPTER III: This chapter is based on a recent new way of combining the governing differential equations for the thermohaline convection problem (THCP) proposed by Banerjee et al. (1981) and its subsequent extensions by Gupta et al. (1982, 83) that lead to bounds for the modulus of the complex growth rate of an arbitrary oscillatory perturbation neutral or unstable in THCP, hydromagnetic THCP, thermo-viscoelastic convection problem (TVCP), rotatory TVCP and hydromagnetic TVCP. These results are important since the exact solutions of these problems, especially in situations when one or both boundaries are rigid, are not obtainable in closed form. However, when we consider the above problems under the combined effect of a uniform vertical rotation and magnetic field, the governing differential equations are not amenable to the analysis of Banerjee et al. (1981) and Gupta et al. (1982, 83) on account of the coupling between current density and vorticity and therefore require a special consideration.

This chapter of the thesis mathematically investigates the following problem:

\[ P_1 : \text{Rayleigh-Bénard convection problem}, \]

\[ P_2 : \text{Thermohaline convection problem (Veronis' (1965) configuration)}, \]

\[ P_3 : \text{Thermohaline convection problem (Stern's (1960) configuration)}, \]

and \[ P_4 : \text{Thermo-viscoelastic convection problem (Vest and Arpaci's (1969) configuration)}, \]
under the combined effect of a uniform vertical rotation and magnetic field especially with a view to determine sufficient conditions for the validity of the 'principle of exchange of stabilities' and deriving bounds for the modulus of the complex growth rate of an arbitrary oscillatory perturbation, neutral or unstable. It is shown that for each of the problems $P_i$ ($i = 1, 2, 3, 4$) the complex growth rate of an arbitrary oscillatory perturbation, neutral or unstable, must lie inside a semicircle each of which has its centre at the origin and radius $r_i$ ($i = 1, 2, 3, 4$) given by

$$ r_1 = \frac{A + \sqrt{A^2 + 4B}}{2}, $$

$$ r_2 = \max \left[ \frac{\sqrt{R_S}}{\sigma}, \frac{A + \sqrt{A^2 + 4B}}{2} \right], $$

$$ r_3 = \max \left[ \frac{\sqrt{R_T}}{\sigma}, \frac{A + \sqrt{A^2 + 4B}}{2} \right], $$

and

$$ r_4 = \max \left[ R_T \sigma', \frac{\sqrt{\sigma(Q+1)}}{\tau'}, \frac{\sigma}{2} \left( \frac{Q+\sqrt{Q^2+4T(Q+1)}}{1} \right) \right]. $$

Here $A = Q\sigma$, $B = T\sigma^2 (Q+1) (Q+D)$, $Q$ is the Chandrasekhar number, $R_S$ is the salinity Rayleigh number, $R_T$ is the thermal Rayleigh number, $T$ is the Taylor number, $\sigma$ is the thermal Prandtl number and $\sigma'$ is an elastic parameter.

Further sufficient conditions are obtained for the validity of the principle of exchange of stabilities in these problems. All these results are new and are uniformly valid for all combinations of free insulating or rigid perfectly conducting boundaries.
CHAPTER IV: This chapter is also based on the work of Banerjee et al. (1981) and is principally motivated due to the reason that their analysis is restricted to the horizontal layer configurations of Veronis (1965) and Stern (1960).

This chapter of the thesis mathematically investigates the viscoelastic thermohaline convection problem for fluids completely confined in an arbitrary region with rigid bounding surfaces. The viscoelastic fluid is assumed to be described by Oldroyd's constitutive relation. It is shown that

(a) If $R_T > 0$, $R_S > 0$ the complex growth rate $p$ of an arbitrary oscillatory perturbation, neutral or unstable, must lie inside a semicircle whose centre is at the origin and

$$\text{radius} = \frac{R_T (\Gamma_1 - \Gamma_2) \sigma + \sqrt{R_T^2 (\Gamma_1 - \Gamma_2)^2 \sigma^2 + 4 R_S \sigma}}{2},$$

in the right half of the complex $p$-plane.

Further, if $\sigma < \frac{\Lambda}{R_T (\Gamma_1 - \Gamma_2)^2}$, then this region for the complex growth rate can further be reduced.

(b) If $R_T < 0$, $R_S < 0$ the complex growth rate $p$ of an arbitrary oscillatory perturbation, neutral or unstable, must lie inside a semicircle whose centre is at the origin and

$$\text{radius} = \frac{|R_S| (\Gamma_1 - \Gamma_2) \sigma + \sqrt{|R_S|^2 (\Gamma_1 - \Gamma_2)^2 \sigma^2 + 4 |R_T| \sigma}}{2},$$

in the right half of the complex $p$-plane.

Further, if $\sigma < \frac{\Lambda}{|R_S| (\Gamma_1 - \Gamma_2)^2}$, then this region for the complex growth rate can further be reduced.
(c) If $R_T > 0$, $R_S > 0$ and $\frac{R_S}{\tau} \leq \frac{1}{\Lambda} + \frac{R_T (\tau_1 - \tau_2)}{\tau_2} \leq 1$, then an arbitrary neutral or unstable mode of the system is non-oscillatory in character.

(d) If $R_T < 0$, $R_S < 0$ and $\frac{|R_T|}{\Lambda} - \frac{|R_S| (\tau_1 - \tau_2)}{\tau_2} \leq 1$, then an arbitrary neutral or unstable mode of the system is non-oscillatory in character.

Results for Vest and Arpaci's configuration (1969), Green's configuration (1968), and Banerjee at al's configurations (1981) in the present frame work follow as a consequence.

Here $\tau_1$ and $\tau_2$ are elastic parameters with $0 \leq \tau_2 \leq \tau_1$, $\tau$ is the smallest distance between two parallel planes that just contain the region within which the fluid is confined, $\Lambda (> 2)$ is a constant, $\tau$ is the ratio of mass diffusivity to heat diffusivity and various other symbols have the same meaning as before.

CHAPTER V: This chapter is based on the work of Banerjee (1971, 72, 73), Banerjee et al. (1974, 75, 76) and Banerjee et al. (1983) on thermal and thermohaline instability problem and is principally motivated due to the following reasons:

First, a fluid is in general initially non-homogeneous and even a small amount of this initial non-homogeneity might effect the onset of instability in Bénard convection problem (Banerjee, 1971, 72, 73) and its various extensions (Banerjee et al., 1974, 75, 76).

Second, Rayleigh's theory shows that the gravity dominated thermal instability problem in a liquid layer heated underside (Bénard convection problem) depends only upon the
Rayleigh number which is proportional to the uniform temperature difference maintained across the layer and thus does not distinguish between a hotter and cooler layer of the same fluid under almost identical conditions otherwise. This inadequacy of Rayleigh's theory has recently been modified by Banerjee et al. (1983). It is found that Rayleigh's utilization of Boussinesq approximation overlooks a term in the equation of heat conduction, which is on account of the variations in specific heat at constant volume due to variations in temperature and which is such that in the usual circumstances it cannot be ignored if the Boussinesq approximation were to be consistently and relatively more accurately applied throughout the analysis. The essential argument on which this term finds a place in their analysis is that it is the temperature differences which are of moderate amounts but not necessarily the temperature itself and an incorporation of this term into the calculations adequately completes the qualitative and quantitative gaps in Rayleigh's theory as pointed out earlier. Subsequently the Boussinesq approximation together with the incorporation of the above term will be referred to as the modified Boussinesq approximation.

This chapter of the thesis mathematically investigates the generalized thermo-viscoelastic convection problem (i.e. Bénard convection problem of an initially non-homogeneous viscoelastic fluid) in the frame work of modified Boussinesq approximation. The viscoelastic fluid is assumed to be described by Maxwell's constitutive relation and the
initial non-homogeneity of the fluid is taken to be of the exponential type, namely, \( \rho = \rho_0 e^{-\beta z} \), where \( z \) is the vertical coordinate having the dimensions of length and satisfying \( 0 \leq z \leq d \), \( d \) being the depth of the fluid layer while \( \beta \), the relative rate of change of density with altitude, is a dimensional constant (having the dimensions of inverse of length) which can be positive or negative or zero. The small-\( d \delta \) approximation (Banerjee, 1972) is adopted throughout the analysis and the problem thus formulated on the basis of normal mode technique is shown to depend on the nondimensional thermal Rayleigh number \( R_I \), the Rayleigh number corresponding to initial non-homogeneity \( R_N \), the thermal Prandtl number \( \sigma \), the wave number \( a \) and an elastic parameter \( \psi \).

In the framework of the theory mentioned above it is shown that the principle of exchange of stabilities (PES) is not valid.

The overstable solutions at the marginal state in the case of dynamically free boundaries show that the initial stable non-homogeneity can stabilize or destabilize the configuration depending on certain conditions on the parameters of the problem. Further the modified theory distinguishes between a hotter and a cooler layer of the same fluid under almost identical conditions otherwise which is lacking in the earlier theories of the problem. It is to be noted that in the absence of initial non-homogeneity, marginal state could be stationary or oscillatory in character but it is
definitely oscillatory in character for $R_N > 0$. Further, in the absence of viscoelasticity an initial stable non-homogeneity stabilizes the configuration whereas it can stabilize or destabilize the configuration when the viscoelastic nature of the fluid is taken into consideration.

The complex growth rate of an arbitrary oscillatory perturbation, neutral or unstable is shown to lie inside a semicircle with centre as origin. This result is uniformly valid for all combinations of dynamically free or rigid boundaries.

A sufficient conditions is derived for the stability of the non-oscillatory modes of the system.

The first result concerning the PES mentioned above is independent of the signs of $R_T$ and $R_N$ whereas the succeeding results have been derived for the case $R_T > 0$ and $R_N > 0$.

If $R_T > 0$ and $R_N < 0$, a sufficient condition is derived for the stability of oscillatory modes of the system. Further bounds are given for the growth rate of an arbitrary non-oscillatory unstable mode of the system. Similar results are proved for the case $R_T < 0$ and $R_N < 0$.

All these results are valid for all combinations of dynamically free or rigid boundaries unless stated otherwise.