Chapter 4

Uniqueness of Meromorphic Functions of class $A$
4.1 Introduction and Results.

We denote by $\mathcal{A}$ the class of meromorphic functions $f$ in $\mathbb{C}$ which satisfy the condition $N(r, f) + N\left(r, \frac{1}{f}\right) = S(r, f)$. Clearly all functions in $\mathcal{A}$ are transcendental meromorphic functions.

Many authors studied the value distribution and uniqueness of meromorphic functions like $f^n f'$, $f^n (f-1)f'$, $f^n (f-1)^2 f'$, $f^n P(f) f'$, $f^{(k)}$, $(f^n)^{(k)}$ and $(f^n (f-1))^{(k)}$. But, the research work is still continued on value distribution and uniqueness of such functions in improving the restriction on $n$ and $k$. Authors like Q.L.Xiong, L.Yang, H.C.Xie, C.C.Yang, Gopalakrishna and Bhoosnurmath, H.X.Yi, Bhoosnurmath and Veena L Pujari [56] and others studied the uniqueness of meromorphic functions in class $\mathcal{A}$. But, so far much work has not been done in this direction.

Hayman [50] proved the following theorem.

**Theorem 4.1.1** Suppose that $f$ is a transcendental meromorphic function and $n$ is a positive integer. Then $f^n f'$ assumes every finite non-zero value infinitely often when $n \geq 3$.

In 1998, Zhang and Song [36] proved the following result.

**Theorem 4.1.2** Suppose that $f$ is a transcendental meromorphic function and that $n$, $k$ are two a positive integers. Then $f(f^{(k)})^n - A(z)$, $n \geq 2$, has infinitely many zeros where $A(z) \neq 0$ is a small function such that $T(r, A) = S(r, f)$.

Theorem 4.1.2 motivated us to think for a similar result in class $\mathcal{A}$ of meromorphic functions. In this chapter, we considered the zeros of $(f^{(k)})^n$ in class...
A and also investigated the uniqueness of meromorphic functions that share 1 value in class $A$. In proving our results, we mainly use the result of Gopalakrishna and Bhoosnurmath [25] i.e., $T(r, P) \sim nT(r, f)$, where $P$ is a homogeneous differential polynomial in $f$ of degree $n$, which plays a cardinal role.

4.2 Definition and Lemmas.

**Definition 4.2.1** [20]. Any expression of the type

$$P(f) = \sum_{i=1}^{n} \alpha_i(z) f^{n_{i0}} (f')^{n_{i1}} (f'')^{n_{i2}} \ldots (f^{(m)})^{n_{im}}$$

is called a differential polynomial in $f$ of degree $d(P)$, lower degree $\bar{d}(P)$ and weight $\Gamma_P$ where for each $i = 1, 2, \ldots, n$, $n_{i0}, n_{i1}, \ldots, n_{im}$ are non-negative integers, $\alpha_i = \alpha_i(z)$ are meromorphic functions satisfying $T(r, \alpha_i) = S(r, f)$ and

$$\bar{d}(P) = \max \left\{ \sum_{j=0}^{m} n_{ij} : 1 \leq i \leq n \right\}, \quad d(P) = \min \left\{ \sum_{j=0}^{m} n_{ij} : 1 \leq i \leq n \right\}$$

and

$$\Gamma_P = \max \left\{ \sum_{j=0}^{m} (j+1) n_{ij} : 1 \leq i \leq n \right\}.$$ 

If $\bar{d}(P) = d(P) = n$ (fixed integer), then $P(f)$ is called homogeneous differential polynomial of degree $n$.

We need following lemmas to prove our results.

**Lemma 4.2.1** [25]. If $P$ is a homogeneous differential polynomial in $f$ of degree $n \geq 1$, then

$$m\left(r, \frac{P}{f^n}\right) = S(r, f).$$
The following lemmas play a cardinal role in proving our theorems which are due to Gopalakrishna and Bhoosnurmath [25]. For the sake of completeness we give the proof of the following lemma:

**Lemma 4.2.2** [25]. Let $f$ be a meromorphic function of finite order and $P$ a homogeneous differential polynomial in $f$ of degree $n$. Let

$$
\alpha = \lim_{r \to \infty} \frac{N(r, f) + \overline{N}(r, 1/f)}{T(r, f)}.
$$

Then

$$
n(1 - m\alpha) \leq \lim_{r \to \infty} \frac{T(r, P)}{T(r, f)} \leq \lim_{r \to \infty} \frac{T(r, P)}{T(r, f)} \leq n(1 + m\alpha) \quad (4.2.1)
$$

where $f^{(m)}$ is the order of highest derivative of $f$ occurring in $P$, provided that $P$ does not reduce to constant.

**Proof.** Since a zero or a pole of $f$, which is not a pole of any co-efficient $a(z)$ of $P$, is a pole of $\frac{P}{f^n}$ of degree $mn$ at most, we have

$$
N \left( r, \frac{P}{f^n} \right) \leq mn \left[ N(r, f) + \overline{N} \left( r, \frac{1}{f} \right) \right] + S(r, f) \quad (4.2.2)
$$

Now,

$$
T \left( r, \frac{P}{f^n} \right) = m \left( r, \frac{P}{f^n} \right) + N \left( r, \frac{P}{f^n} \right) \leq mn \left[ N(r, f) + \overline{N} \left( r, \frac{1}{f} \right) \right] + S(r, f) \quad (4.2.3)
$$

by (4.2.2) and Lemma 4.2.1. Therefore, by using (4.2.3), we get

$$
T(r, P) \leq T \left( r, \frac{P}{f^n} \right) + T(r, f^n) \leq mn \left[ N(r, f) + \overline{N} \left( r, \frac{1}{f} \right) \right] + nT(r, f) + S(r, f) \quad (4.2.4)
$$
The right inequality in (4.2.1) follows from (4.2.4).

On the other hand,

\[ nT(r, f) = T(r, f^n) \]
\[ \leq T\left(r, \frac{f^n}{P}\right) + T(r, P) \]
\[ = T(r, P) + T\left(r, \frac{P}{f^n}\right) + o(1) \]
\[ \leq T(r, P) + \frac{1}{n} \left[ \mathcal{N}(r, f) + \mathcal{N}\left(r, \frac{1}{f}\right) \right] + S(r, f) \]

which gives the left inequality in (4.2.1). This completes the proof of Lemma.

An immediate consequence of Lemma 4.2.2 is given below.

**Lemma 4.2.3** [25]. Let \( f \) be a meromorphic function of finite order and \( P \) a homogeneous differential polynomial in \( f \) of degree \( n \). If \( \Theta(0, f) = \Theta(\infty, f) = 1 \), then

\[ T(r, P) \sim nT(r, f). \]

**Lemma 4.2.4** [32]. Let \( f_j (j = 1, 2, 3) \) be meromorphic functions that satisfy

\[ \sum_{j=1}^{3} f_j = 1 \]

Assume that \( f_1 \) is not a constant, and

\[ \sum_{j=1}^{3} N_2 \left( r, \frac{1}{f_j} \right) + \sum_{j=1}^{3} \mathcal{N}(r, f_j) < (\lambda + o(1))T(r), \quad r \in I, \]

where \( \lambda < 1 \), \( T(r) = \max\{T(r, f_1), T(r, f_2), T(r, f_3)\} \), \( N_2 \left( r, \frac{1}{f_j} \right) \) is the counting function of zeros of \( f_j (j = 1, 2, 3) \), where a multiple zero is counted two times and a simple zero is counted once. Then \( f_2 = 1 \) or \( f_3 = 1 \).

**Lemma 4.2.5** [47]. Let \( f \) be a non-constant meromorphic function. Then

\[ N \left( r, \frac{1}{f^{(k)}} \right) \leq N \left( r, \frac{1}{f} \right) + k \mathcal{N}(r, f) + S(r, f) \]
where $k$ is a positive integer.

Lemma 4.2.6 [46]. Suppose that $f$ is a non-constant meromorphic function, $k \geq 2$ is an integer. If

$$N(r, f) + N \left( r, \frac{1}{f} \right) + N \left( r, \frac{1}{f(f^k)} \right) = S \left( r, f' \right),$$

then $f = e^{az+b}$, where $a \neq 0$, $b$ are constants.

Lemma 4.2.7 [47]. Let $f$ and $g$ be distinct non-constant meromorphic functions and $a_j (1, 2, 3, 4)$ be four distinct values. If $f$ and $g$ share $a_j (j = 1, 2, 3, 4)$ CM, then $f(z) = T(g(z))$, where $T$ is a Mobius transformation such that two of the four values are fixed points and another two (are Picard exceptional values of $f$ and $g$) exchange each other under $T$.

Lemma 4.2.8 [47]. Let $f$ and $g$ be distinct non-constant meromorphic functions and $a_j (1, 2, 3, 4)$ be four distinct values. If $f$ and $g$ share $a_1, a_2$ CM and share $a_3, a_4$ IM, then $f$ and $g$ share $a_3, a_4$ CM and thus conclusion of Lemma 4.2.7 holds.

Lemma 4.2.9 [47]. If $f$ and $g$ are distinct non-constant meromorphic functions that share four values $a_1, a_2, a_3, a_4$ CM, then $f$ is Mobius transformation of $g$; two of the shared values, say $a_1$ and $a_2$ are Picard exceptional values and the cross ratio $(a_1, a_2, a_3, a_4) = -1$.

Lemma 4.2.10 If $f, g \in \mathcal{A}$, $n \geq 3$ and $k$ be a positive integer. If $(f^{(k)})^n$ and $(g^{(k)})^n$ share 1 CM, then

$$T(r, g) \leq \left( \frac{n}{n-2} \right) T(r, f) + S(r, g)$$
Proof. Let \( G = (g^{(k)})^n \) be a differential polynomial of degree \( n \).

By Lemma 4.2.3, we have

\[
T(r, G) \sim nT(r, g)
\]

(4.2.5)

Applying Nevanlinna's second fundamental theorem to \( G \), we get

\[
T(r, G) \leq \overline{N}(r, G) + \overline{N}
\left(r, \frac{1}{G}\right) + \overline{N}
\left(r, \frac{1}{G - 1}\right) + S(r, G)
\]

(4.2.5a)

Noting that

\[
\overline{N}(r, (g^{(k)})^n) \leq \overline{N}(r, g^{(k)}) \leq N(r, g^{(k)})
\]

(4.2.5b)

and

\[
\overline{N}
\left(r, \frac{1}{(g^{(k)})^n}\right) \leq \overline{N}
\left(r, \frac{1}{g^{(k)}}\right)
\leq N
\left(r, \frac{1}{g}\right) + k \overline{N}(r, g) + S(r, g)
\]

(4.2.5c)

and by (4.2.5) we write \( S(r, G) = S(r, g) \).

Using (4.2.5), (4.2.5b)(4.2.5c) in (4.2.5a), we get

\[
nT(r, g) \leq N(r, g) + k \overline{N}(r, g) + N
\left(r, \frac{1}{g^{(k)}}\right) + k \overline{N}(r, g) + \overline{N}
\left(r, \frac{1}{(g^{(k)})^n - 1}\right) + S(r, g)
\]

Since \((f^{(k)})^n \) and \((g^{(k)})^n \) share 1 CM, it implies that \((f^{(k)})^n - 1\) and \((g^{(k)})^n - 1\) have the same zeros with the same multiplicities. Using this in the above equation, we obtain that

\[
nT(r, g) \leq N(r, g) + 2k \overline{N}(r, g) + N
\left(r, \frac{1}{g}\right) + \overline{N}
\left(r, \frac{1}{(f^{(k)})^n - 1}\right)
\]

\[
+ S(r, g)
\]

(4.2.6)

By hypothesis, we have

\[
\overline{N}(r, f) + \overline{N}
\left(r, \frac{1}{f}\right) = S(r, f) \quad , \quad \overline{N}(r, g) + \overline{N}
\left(r, \frac{1}{g}\right) = S(r, g).
\]

73
Using Nevanlinna's first fundamental theorem and Lemma 4.2.3, we have

\[
N\left( r, \frac{1}{(f^{(k)})^n - 1} \right) \leq T\left( r, \frac{1}{(f^{(k)})^n - 1} \right) = T(r, (f^{(k)})^n) + O(1) \\
\sim n \, T(r, f) + O(1)
\]

So,

\[
N\left( r, \frac{1}{(f^{(k)})^n - 1} \right) \leq n \, T(r, f) + O(1) \quad (4.2.7)
\]

Using (4.2.7), (4.2.6) becomes

\[
n \, T(r, g) \leq N(r, g) + N\left( r, \frac{1}{g} \right) + n \, T(r, f) + S(r, g)
\]

\[
\leq 2 \, T(r, g) + n \, T(r, f) + S(r, g)
\]

or,

\[
(n - 2) \, T(r, g) \leq n \, T(r, f) + S(r, g)
\]

or,

\[
T(r, g) \leq \left( \frac{n}{n - 2} \right) T(r, f) + S(r, g)
\]

This completes the proof of the Lemma.

Lemma 4.2.11 Let \( f, g \in \mathcal{A}, \ n \geq 3 \) and \( k \) be a positive integer. If \( (f^{(k)})^n \) and \( (g^{(k)})^n \) share 1 CM, then \( S(r, f) = S(r, g) \).

Proof. Proceeding as in the proof of Lemma 4.2.10, we have

\[
T(r, g) \leq \left( \frac{n}{n - 2} \right) T(r, f) + S(r, g)
\]

Similarly, we have

\[
T(r, f) \leq \left( \frac{n}{n - 2} \right) T(r, g) + S(r, f)
\]

using above two inequalities we easily obtain

\[
S(r, f) = S(r, g).
\]
Lemma 4.2.12 Let $f, g \in \mathcal{A}$, $n \geq 3$ and $k$ be a positive integer. If $(f^{(k)})^n (g^{(k)})^n = 1$, then $f = c_3 e^{p^2 z}$ and $g = c_4 e^{-p^2 z}$ where $c_3, c_4$ and $p$ are constants such that $(-1)^{nk} (c_3 c_4)^n p^{2nk} = 1$.

Proof.

Let $P = (f^{(k)})^n$ and $G = (g^{(k)})^n$.

By Lemma 4.2.3, we have

$$T(r, F) \sim n T(r, f), \quad T(r, G) \sim n T(r, g)$$

Clearly $S(r, F) = S(r, f)$ and $S(r, G) = S(r, g)$.

By Lemma 4.2.10, we have $S(r, f) = S(r, g)$.

Thus, $S(r, F) = S(r, f) = S(r, g) = S(r, G)$. (4.2.9)

By hypothesis, we have

$$(f^{(k)})^n (g^{(k)})^n = 1 \quad \text{or} \quad FG = 1 \quad (4.2.10)$$

From (4.2.10) and $f$ and $g$ are transcendental functions, it follows that

$$N \left( r, \frac{1}{f} \right) = 0 \quad \text{and} \quad N \left( r, \frac{1}{g} \right) = 0 \quad (4.2.11)$$

By hypothesis, we have

$$\overline{N}(r, f) + \overline{N}(r, \frac{1}{f}) = S(r, f) \quad , \quad \overline{N}(r, g) + \overline{N}(r, \frac{1}{g}) = S(r, g) \quad (4.2.12)$$

(4.2.10) can be expressed as,

$$(f^{(k)})^n = \frac{1}{(g^{(k)})^n} \quad (4.2.13)$$
We deduce from (4.2.13) that

\[ N(r, (f^{(k)})^n) = N\left(r, \frac{1}{(g^{(k)})^n}\right) \quad (4.2.14) \]

Using (4.2.12), we get

\[ N(r, (f^{(k)})^n) = n N(r, f^{(k)}) \]
\[ = n \left[ N(r, f) + k \overline{N}(r, f) \right] \]
\[ = n N(r, f) + S(r, f) \]

using this with Lemma 4.2.5, (4.2.9), (4.2.11) and (4.2.12), (4.2.14) can be written as

\[ n N(r, f) + S(r, f) = n N\left(r, \frac{1}{g^{(k)}}\right) \]
\[ \leq n \left[ N\left(r, \frac{1}{g}\right) + k \overline{N}(r, g) \right] + S(r, g) \]
\[ = S(r, g) = S(r, f) \]

which implies that \( N(r, f) = S(r, f) \) \quad (4.2.15)

Similarly, \( N(r, g) = S(r, g) \) \quad (4.2.16)

By (4.2.8), (4.2.11), (4.2.12) and Lemma 4.2.5, we have

\[ N\left(r, \frac{1}{f}\right) = N\left(r, \frac{1}{(f^{(k)})^n}\right) \]
\[ = n N\left(r, \frac{1}{f^{(k)}}\right) \]
\[ \leq n \left[ N\left(r, \frac{1}{f}\right) + k \overline{N}(r, f) \right] + S(r, f) \]
\[ = S(r, f) \]

Similarly

\[ N\left(r, \frac{1}{G}\right) = S(r, g) \]
Since $S(r, f) = S(r, g)$, we obtain

$$N\left(r, \frac{1}{F}\right) = N\left(r, \frac{1}{G}\right)$$  \hspace{1cm} (4.2.17)

Using (4.2.8), (4.2.12) and (4.2.15), we have

$$N(r, F) = N(r, (f^{(k)})^n)$$

$$= n N(r, f^{(k)})$$

$$= n \left[N(r, f) + kN(r, f)\right]$$

$$= S(r, f)$$

Similarly, $N(r, G) = S(r, g)$

Therefore, $N(r, F) = N(r, G)$ \hspace{1cm} (4.2.18)

From (4.2.17) and (4.2.18), it implies that $F$ and $G$ share 0 and $\infty$ CM. In view of (4.2.10), we know that $F$ and $G$ share 1 and -1 IM. Together with this and Lemma 4.2.8, it implies that $F$ and $G$ share 1, -1, 0, $\infty$ CM. Thus by Lemma 4.2.9 we get that 0 and $\infty$ are Picard values of $F$ and $G$ and the cross ratio $(1, -1, 0, \infty) = -1$. We deduce from (4.2.8) that both $f$ and $g$ are transcendental entire functions.

By (4.2.11) we have

$$f(z) := e^{\alpha(z)}, \quad g(z) = e^{\beta(z)}$$  \hspace{1cm} (4.2.19)

where $\alpha(z)$ and $\beta(z)$ are non-constant entire functions.

Then

$$T\left(r, \frac{f'}{f}\right) = T\left(r, \frac{e^{\alpha} \alpha'}{e^{\alpha'}}\right) = T(r, \alpha').$$

We claim that $\alpha(z) + \beta(z) = c$, $c$ is a constant. From (4.2.19), we know that either $\alpha$ and $\beta$ are transcendental functions or both $\alpha$ and $\beta$ polynomials.
From (4.2.10), we have

\[ N \left( r, \frac{1}{f^{(k)}} \right) = N(r, g^{(k)}) \]

\[ \leq N(r, g) + kN(r, g) \]

\[ = 0 \]

From this and (4.2.11), we get

\[ N(r, f) + N \left( r, \frac{1}{f} \right) + N \left( r, \frac{1}{f^{(k)}} \right) = 0 \]

If \( k \geq 2 \), suppose that \( \alpha \) is transcendental entire function. From Lemma 4.2.6, we have \( f = e^{\alpha(z)} = e^{az+b} \), it implies that \( \alpha(z) = az + b \), a polynomial, which is a contradiction. Thus \( \alpha \) and \( \beta \) polynomials.

We deduce from (4.2.19) that:

\[ f^{(k)} = [(\alpha')^k + P_{k-1}(\alpha')] e^\alpha, \quad g^{(k)} = [(\beta')^k + Q_{k-1}(\beta')] e^\beta \]

where \( P_{k-1}(\alpha') \) and \( Q_{k-1}(\beta') \) are differential polynomials in \( \alpha' \) and \( \beta' \) of degree at most \((k-1)\) respectively.

Thus by (4.2.10) we obtain that

\[ [(\alpha')^k + P_{k-1}(\alpha')] [(\beta')^k + Q_{k-1}(\beta')] e^{n(\alpha+\beta)} = 1 \quad (4.2.20) \]

we deduce from (4.2.20) that \( \alpha(z) + \beta(z) = c \), \( c \) is a constant.

If \( k = 1 \), from (4.2.19) we get,

\[ (f')^n(g')^n = 1 \]

\[ (e^a \alpha')^n(e^b \beta')^n = 1 \]

\[ (\alpha')^n(\beta')^n e^{n(\alpha+\beta)} = 1 \quad (4.2.21) \]
Let $\alpha + \beta = \gamma$. If $\alpha$ and $\beta$ are transcendental entire functions, then $\gamma$ is not a constant and (4.2.21) implies that

$$(\alpha')^n(\gamma' - \alpha')^n e^{an} = 1 \quad (4.2.22)$$

Since

$$T(r, \gamma') = m(r, \gamma')$$

$$= m \left( r, \frac{e^{an\gamma'}}{e^{an\gamma}} \right)$$

$$= m \left( r, \frac{(e^{an\gamma'})'}{(e^{an\gamma})'} \right) = S(r, e^{an\gamma})$$

Thus (4.2.22) implies that

$$T(r, e^{an\gamma}) = T \left( r, \frac{1}{(\alpha')^n(\gamma' - \alpha')^n} \right)$$

$$\leq T \left( r, (\alpha')^n(\gamma' - \alpha')^n \right) + O(1)$$

$$\leq T \left( r, (\alpha')^n \right) + T \left( r, (\gamma' - \alpha')^n \right) + O(1)$$

$$\leq nT(r, \alpha') + nT(r, \gamma') + nT(r, \alpha') + O(1)$$

$$\leq 2nT(r, \alpha') + S(r, e^{an\gamma})$$

which implies that

$$T(r, e^{an\gamma}) = O(T(r, \alpha'))$$

Thus, $T(r, \gamma') = S(r, \alpha')$

By the second fundamental theorem of Nevanlinna for small functions, we get

$$T(r, (\alpha')^n) \leq N(r, (\alpha')^n) + N \left( r, \frac{1}{(\alpha')^n} \right) + N \left( r, \frac{1}{(\alpha' - \gamma')^n} \right) + S(r, (\alpha')^n)$$

Where, $(\alpha' - \gamma')^n = (\alpha')^n + ((\alpha')^{n-1}\gamma' + \ldots + (\gamma')^n)$ and

$(\alpha')^n + ((\alpha')^{n-1}\gamma' + \ldots + (\gamma')^n) = o(T(r, (\alpha')^n))$. From this and (4.2.22) it is clear that

$$nT(r, \alpha') \leq S(r, \alpha') \quad (4.2.23)$$
Since $n \geq 3$, from (4.2.23) we get a contradiction. Thus $\alpha$ and $\beta$ are both polynomials and $\alpha(z) + \beta(z) = c$, for a constant $c$.

Hence from (4.2.20) we get,

$$(\alpha')^{2kn} = 1 + \tilde{P}_{2kn-1}(\alpha') \quad (4.2.24)$$

where $\tilde{P}_{2kn-1}(\alpha')$ is a differential polynomial in $\alpha'$ of degree at most $(2kn - 1)$.

From (4.2.24) we have,

$$2kn T(r, \alpha') = T(r, (\alpha')^{2kn}) = m(r, (\alpha')^{2kn})$$

$$\leq m(r, \tilde{P}_{2kn-1}(\alpha')) + O(1)$$

$$\leq m\left(r, \frac{\tilde{P}_{2kn-1}(\alpha')}{(\alpha')^{2kn-1}}(\alpha')^{2kn-1}\right) + O(1)$$

$$\leq m\left(r, \frac{\tilde{P}_{2kn-1}(\alpha')}{(\alpha')^{2kn-1}}\right) + m(r, (\alpha')^{2kn-1}) + O(1)$$

$$\leq (2kn - 1) T(r, \alpha') + S(r, \alpha')$$

Therefore $T(r, \alpha') \leq S(r, \alpha')$, which implies that $\alpha'$ is a constant.

Thus $\alpha = pz + c_1$, $\beta = -pz + c_2$. By (4.2.19), we represent $f$ and $g$ as

$$f = c_3 e^{pz}, \quad g = c_4 e^{-pz},$$

where $c_3$, $c_4$ and $p$ are constants such that $(-1)^n(c_3c_4)^n p^{nk} = 1$.

This completes the proof of Lemma.

4.3 Statement and Proofs of Main Results.

Theorem 4.3.1 If $f, g \in A$, $n \geq 3$ and $k$ be a positive integer. Then $(f^{(k)})^n = 1$ has infinitely many zeros.
Proof. By the second fundamental theorem of Nevanlinna and Lemma 4.2.4, we have

\[ n T(r, f) \sim T(r, (f^{(k)})^n) \]
\[ \leq \overline{N}(r, (f^{(k)})^n) + \overline{N}\left(r, \frac{1}{(f^{(k)})^n}\right) + \overline{N}\left(r, \frac{1}{(f^{(k)})^n - 1}\right) \]
\[ + S(r, (f^{(k)})^n) \quad (4.3.1) \]

Noting that \( \overline{N}(r, (f^{(k)})^n) \leq N(r, f^{(k)}) = N(r, f) + k N(r, f) \)

\[ \overline{N}\left(r, \frac{1}{(f^{(k)})^n}\right) \leq N\left(r, \frac{1}{f^{(k)}}\right) \leq N\left(r, \frac{1}{f}\right) + k \overline{N}(r, f) + S(r, f) \]

Substituting these in (4.3.1), we obtain

\[ n T(r, f) \leq N(r, f) + 2k \overline{N}(r, f) + N\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{(f^{(k)})^n - 1}\right) + S(r, f) \]

By hypothesis, we have

\[ N(r, f) = S(r, f) \quad , \quad \overline{N}\left(r, \frac{1}{f}\right) = S(r, f). \]

Therefore,

\[ n T(r, f) \leq N(r, f) + N\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{(f^{(k)})^n - 1}\right) + S(r, f) \]
\[ \leq 2 T(r, f) + \overline{N}\left(r, \frac{1}{(f^{(k)})^n - 1}\right) + S(r, f) \]

or,

\[ (n - 2) T(r, f) \leq \overline{N}\left(r, \frac{1}{(f^{(k)})^n - 1}\right) + S(r, f) \]

which implies that \((f^{(k)})^n - 1\) has infinitely many zeros for \(n \geq 3\).

**Theorem 4.3.2** Let \(f, g \in \mathcal{A}, \ n \geq 8\) and \(k\) be a positive integer. If \((f^{(k)})^n\) and \((g^{(k)})^n\) share \(1\) CM, then either \(f = dg\) for a constant \(d\) such that \(d^{k+1} = 1\) or \(f = c_3 e^{p^2}, \ g = c_4 e^{-p^2}\) where \(c_3, c_4\) and \(p\) are constants such that \((-1)^n(c_3c_4)^{p^{2nk}} = 1\).
Proof. By hypothesis, \((f^{(k)})^n\) and \((g^{(k)})^n\) share 1 CM.

Let

\[
H(z) = \frac{(f^{(k)})^n - 1}{(g^{(k)})^n - 1}
\]  

(4.3.2)

Then \(H(z)\) is a meromorphic function satisfying \(T(r, H) = O(T(r, f) + T(r, g))\), by the first fundamental theorem and Lemma 4.2.3.

From (4.3.2), we see that the zeros and poles of \(H(z)\) are multiple and satisfy

\[
N(r, H) \leq N_L(r, f), \quad N\left(r, \frac{1}{H}\right) \leq N_L(r, g)
\]  

(4.3.3)

Let

\[
F_1 = (f^{(k)})^n, \quad F_2 = -H(g^{(k)})^n, \quad F_3 = H
\]  

(4.3.4)

Then, we easily see that \(F_1 + F_2 + F_3 = 1\).

Assuming that \(F_1\) is non-constant and by Lemma 4.2.4, we have

\[
\sum_{j=1}^{3} N_2\left(r, \frac{1}{F_j}\right) + \sum_{j=1}^{3} N(r, F_j) = N_2\left(r, \frac{1}{F_1}\right) + N_2\left(r, \frac{1}{F_2}\right) + N_2\left(r, \frac{1}{F_3}\right)
\]

\[+
N(r, F_1) + N(r, F_2) + N(r, F_3)
\]

\[\leq N_2\left(r, \frac{1}{(f^{(k)})^n}\right) + N_2\left(r, \frac{1}{(g^{(k)})^n}\right) + N_2\left(r, \frac{1}{H}\right)
\]

\[+
N(r, (f^{(k)})^n) + N(r, (g^{(k)})^n) + N(r, H)
\]  

(4.3.5)

Noting that

\[
N(r, (f^{(k)})^n) \leq N(r, f) + k N(r, f)
\]

\[
N(r, (g^{(k)})^n) \leq N(r, g) + k N(r, g)
\]

\[
N_2\left(r, \frac{1}{(f^{(k)})^n}\right) \leq 2 N(r, \frac{1}{f}) + 2k N(r, f) + S(r, f)
\]

\[similiarly, \quad N_2\left(r, \frac{1}{(g^{(k)})^n}\right) \leq 2 N(r, \frac{1}{g}) + 2k N(r, g) + S(r, g)
\]
Using this with (4.3.3), (4.3.5) becomes

\[ \sum_{j=1}^{3} N_2 \left( r, \frac{1}{F_j} \right) + \sum_{j=1}^{3} N(r, F_j) \leq 2 N \left( r, \frac{1}{f} \right) + 2k \overline{N}(r, f) + 2 N \left( r, \frac{1}{g} \right) + 2k \overline{N}(r, g) + 2 N_L(r, g) + N(r, f) + k \overline{N}(r, f) + N(r, g) + k \overline{N}(r, g) + \overline{N}_L(r, f) + S(r, f) + S(r, g) = 2 \left( N \left( r, \frac{1}{f} \right) + N \left( r, \frac{1}{g} \right) \right) + N(r, f) + 3k \left( \overline{N}(r, f) + \overline{N}(r, g) \right) + N(r, g) + 2 \overline{N}_L(r, g) + \overline{N}_L(r, f) + S(r, f) + S(r, g) \]

Since \( f, g \in A \), we have

\[ \overline{N}(r, f) + \overline{N}(r, \frac{1}{f}) = S(r, f), \quad \overline{N}(r, g) + \overline{N}(r, \frac{1}{g}) = S(r, g). \]

Noting that

\[ 2 \overline{N}_L(r, g) + \overline{N}_L(r, f) \leq 2 \overline{N}(r, f) = S(r, f) \]

and \[ 2 \overline{N}_L(r, g) + \overline{N}_L(r, f) \leq 2 \overline{N}(r, g) = S(r, g), \]

Therefore, using these with (4.2.10) and Lemma 4.2.4, we get

\[ \sum_{j=1}^{3} N_2 \left( r, \frac{1}{F_j} \right) + \sum_{j=1}^{3} N(r, F_j) \leq 2 \left( N \left( r, \frac{1}{f} \right) + N \left( r, \frac{1}{g} \right) \right) + N(r, f) + N(r, g) + S(r, f) + S(r, g) \leq 3 \left( T(r, f) + T(r, g) \right) + S(r, f) + S(r, g) \leq 3 T(r, f) + \frac{3n}{n-2} T(r, f) + S(r, f) + S(r, g) = \frac{6n - 6}{n-2} T(r, f) + S(r, f) + S(r, g) \leq \frac{6n - 6}{(n-2)n} T(r) + S(r, f) \leq \left( \frac{6n - 6}{(n-2)n} + o(1) \right) T(r), \]
Since $n \geq 8$, $\frac{6n-8}{(n-2)n} < 1$, using Lemma 4.2.4, we get $F_2 = 1$ or $F_3 = 1$.

Next we consider two cases:

**case 1.** $F_2 = 1$ i.e, $-H(g^{(k)})^n = 1$

\[ or \quad (f^{(k)})^n (g^{(k)})^n = 1. \]

By Lemma 4.2.11, we get the conclusion of Theorem.

**case 2.** $F_3 = 1$ i.e, $H = \frac{1}{z}$

\[ or \quad f^{(k)} = g^{(k)} \quad (4.3.6) \]

it implies that, $f(z) \equiv g(z) + P(z)$, where $P(z)$ is a polynomial of degree atmost $(k-1)$ and $P(z) \neq 0$.

By the second fundamental theorem of Nevanlinna, we have

\[
T(r, f) \leq \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f - \frac{1}{g}}\right) + S(r, f)
\]

\[
= \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{g}\right) + S(r, f)
\]

\[
= S(r, f)
\]

which is impossible. Hence $f(z) \equiv g(z)$.  

84