Chapter 3

Weber-Ramanujan’s Class Invariants

3.1 Introduction

We set

\[ (a;q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n), \quad |q| < 1, \]

and, recall from (2.1.26) that

\[ \chi(q) = (-q;q^2)_\infty. \tag{3.1.1} \]

If \( q = \exp(-\pi \sqrt{n}) \), where \( n \) is any positive rational number, then Weber-Ramanujan’s class invariants \( G_n \) and \( g_n \) are defined by

\[ G_n := 2^{-1/4} q^{-1/24} \chi(q), \quad g_n := 2^{-1/4} q^{-1/24} \chi(-q). \tag{3.1.2} \]

Note: This chapter is identical to our paper [6], which has been accepted for publication in the Journal of Indian Mathematical Society.
In his book, H. Weber [61] calculated 105 class invariants, or monic, irreducible polynomials satisfied by them. He was motivated to calculate class invariants so that he could construct Hilbert class fields. At the scattered places in his first notebook [48], Ramanujan recorded 107 class invariants, or monic, irreducible polynomials satisfied by them. On pages 294-299 in his second notebook [48], he recorded a table of 77 class invariants, three of which are not found in the first notebook. By the time Ramanujan wrote his paper [47], he came to know about Weber's work, and therefore his table of 46 class invariants in [47] does not contain any that are found in Weber's book [61]. Except for \( g_{325} \) and \( G_{363} \), all of the remaining values are found in his notebooks [48]. G.N. Watson [54]-[60] established 28 of these 46 class invariants. Ten of the class invariants had been proved by using Ramanujan's modular equations and the rest had been proved by using his unrigorous "empirical process". So, after Watson's work, 18 invariants of Ramanujan from his paper [47] and notebooks [48] remained to be verified. These 18 class invariants are: \( G_{65}, G_{69}, G_{77}, G_{117}, G_{141}, G_{145}, G_{153}, G_{205}, G_{213}, G_{217}, G_{265}, G_{301}, G_{441}, G_{445}, G_{509}, G_{553}, g_{90}, \) and \( g_{198} \). These invariants are proved by B.C. Berndt, H.H. Chan, L.C. Zhang [24], [26]. In [24], five of the invariants, viz., \( G_{117}, G_{153}, G_{441}, g_{90}, \) and \( g_{198} \), are proved by employing two new theorems that relate \( G_{9n} \) with \( G_{n} \), and \( g_{9n} \) with \( g_{n} \), respectively. In [26], they used modular equations to prove six of the remaining thirteen invariants. To prove the other seven invariants via modular equations, one needs modular equations of degrees 31, 41, 43, 53, 79, 89, and, 101. But, only for degree 31 Ramanujan recorded modular equations, for he recorded no modular equations for the other degrees. They could not utilize those modular equations of degree 31 to effect a proof for \( G_{217} \). They [26] proved all the remaining invariants, including \( G_{217} \), by using Kronecker's limit formula, an idea completely unknown to Ramanujan, and Watson's "empirical process." For a detail discussion on their evaluations see Berndt's book [18].
3.1. **INTRODUCTION**

In Section 3.3, we shall establish the class invariant \( G_{217} \) by using Ramanujan's modular equations of degrees 7 and 31. In Section 3.4, we employ, for the first time, some of the Schläfli-type "mixed" modular equations discussed in Chapter 2, along with some other Schläfli-type modular equations of prime degrees to evaluate Ramanujan's class invariants \( G_{15}, G_{21}, G_{33}, G_{39}, G_{55}, \) and \( G_{65} \). It is worthwhile to note that our evaluation of \( G_{65} \) is much more easier than that of Berndt, Chan, and Zhang [18], [26]. Most important feature of our method is that we can also simultaneously get the values of \( G_{5/3}, G_{7/3}, G_{11/3}, G_{13/3}, G_{11/5}, \) and \( G_{13/5} \). Previously, these values were found by verifications. We also note that, these class invariants can be utilised to find some of the explicit, values of the famous Rogers-Ramanujan continued fraction, \( R(q) \), defined by

\[
R(q) := \frac{q^{1/5}}{1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \ldots}}}}, \quad |q| < 1, \tag{3.1.3}
\]

some of the values of Ramanujan's product of theta-functions \( a_{m,n} \) (\( m, n \) are positive integers), recorded on pages 338-339 of his first notebook [48], and defined by

\[
a_{m,n} := ne^{-(\pi/4)(n-1)} \sqrt{m/n} \frac{\psi(e^{-\pi\sqrt{mn}})\phi(-e^{-2\pi\sqrt{mn}})}{\psi(e^{\pi\sqrt{m/n}})\phi(-e^{2\pi\sqrt{m/n}})}
\]

and, the values of the quotient of eta-functions, \( \lambda_n \), recorded by Ramanujan on page 212 of his lost notebook [49], and defined by

\[
\lambda_n := \frac{e^{\pi/2\sqrt{3}}}{3\sqrt{3}} \left\{ (1 + e^{-\pi\sqrt{3}})(1 - e^{-2\pi\sqrt{3}})(1 - e^{-4\pi\sqrt{3}}) \right\}^6.
\]

For details of the above evaluations see [25], [27], and [30].
We complete this introduction by noting that, since from (2.1.37),

\[
\chi(q) = 2^{1/6} \{\alpha(1 - \alpha)/q\}^{-1/24},
\]

it follows from (3.1.2) that

\[
G_n = \{4\alpha(1 - \alpha)/q\}^{-1/24} \quad \text{and} \quad G_{r^n} = \{4\beta(1 - \beta)/q\}^{-1/24}, \tag{3.1.4}
\]

where \(\beta\) has degree \(r\) over \(\alpha\) and \(q = \exp(-\pi \sqrt{n})\).

### 3.2 Preliminary Lemmas

In this section we state some lemmas which will be used in our evaluation.

**Lemma 3.2.1** ([18, p. 247]; [26]) \(\beta\) has degree \(r\) over \(\alpha\), then \(\beta\) has degree \(p\) over \(1 - \alpha\), where \(p\) and \(r\) are two coprime positive integers.

In the next three lemmas we state three Schl"{a}fli-type modular equations of Ramanujan [15, pp. 231, 282, 315] for prime degrees.

**Lemma 3.2.2** Let

\[
P = \{16\alpha\beta(1 - \alpha)(1 - \beta)\}^{1/8} \quad \text{and} \quad Q = \left(\frac{\beta(1 - \beta)}{\alpha(1 - \alpha)}\right)^{1/4}
\]

Then

\[
Q + \frac{1}{Q} + 2\sqrt{2} \left(\frac{P}{Q} - \frac{1}{P}\right) = 0,
\]

where \(\beta\) has degree 3 over \(\alpha\).
3.2. PRELIMINARY LEMMAS

Lemma 3.2.3 Let

\[ P = \{16\alpha\beta(1 - \alpha)(1 - \beta)\}^{1/12} \quad \text{and} \quad Q = \left(\frac{\beta(1 - \beta)}{\alpha(1 - \alpha)}\right)^{1/8}. \]

Then

\[ Q + \frac{1}{Q} + 2\left(P - \frac{1}{P}\right) = 0, \]

where \( \beta \) has degree 5 over \( \alpha \).

Lemma 3.2.4 Let

\[ P = \{16\alpha\beta(1 - \alpha)(1 - \beta)\}^{1/8} \quad \text{and} \quad Q = \left(\frac{\beta(1 - \beta)}{\alpha(1 - \alpha)}\right)^{1/6}. \]

Then

\[ Q + \frac{1}{Q} + 7 = 2\sqrt{2}\left(P + \frac{1}{P}\right), \]

where \( \beta \) has degree 7 over \( \alpha \).

In the following three lemmas, we state three of Ramanujan’s Schlüffli-type modular equations for composite degrees.

Lemma 3.2.5 ([48, Vol. I, p. 86], [15, p. 324]) If \( \alpha, \beta, \gamma, \) and \( \delta \) have degrees 1, 3, 5, and 15, respectively, then

\[ Q^3 + \frac{1}{Q^3} = \sqrt{2}\left(P + \frac{1}{P}\right). \quad (3.2.1) \]

Lemma 3.2.6 ([48, Vol. I, p. 88], [18, p. 381]) If \( \alpha, \beta, \gamma, \) and \( \delta \) have degrees 1, 3, 13, and 39, respectively, then

\[ Q^4 + \frac{1}{Q^4} - 3\left(Q^2 + \frac{1}{Q^2}\right) - \left(R^2 + \frac{1}{R^2}\right) + 3 = 0. \quad (3.2.2) \]
Lemma 3.2.7 ([48, Vol. I, p. 88], [18, p. 381]) If $\alpha$, $\beta$, $\gamma$, and $\delta$ have degrees 1, 5, 13, and 65, respectively, then

$$Q^6 + \frac{1}{Q^6} - 5 \left( Q + \frac{1}{Q} \right)^2 \left( R + \frac{1}{R} \right)^2 - \left( R^4 + \frac{1}{R^4} \right) = 0. \quad (3.2.3)$$

The next lemma due to Landau [40, p. 53] will be very useful in simplifying some of our radicals.

Lemma 3.2.8 If $a^2 - qb^2 = d^2$, a perfect square, then

$$\sqrt{a + b\sqrt{q}} = \sqrt{\frac{a + d}{2}} + (\text{sgn}b)\sqrt{\frac{a - d}{2}}. \quad (3.2.4)$$

Our last lemma is original due to Bruce Reznick. For a proof via Chebyshev polynomials one may see [28, p. 150].

Lemma 3.2.9 If $a, b \geq 1/2$, then

$$\{ (8a^2 - 1) + \sqrt{(8a^2 - 1)^2 - 1} \}^{1/4} = \sqrt{a + \frac{1}{2}} + \sqrt{a - \frac{1}{2}} \quad (3.2.5)$$

and

$$\{ (32b^3 - 6b) + \sqrt{(32b^3 - 6b)^2 - 1} \}^{1/6} = \sqrt{b + \frac{1}{2}} + \sqrt{b - \frac{1}{2}}. \quad (3.2.6)$$
3.3. CLASS INVARIANT $G_{217}$

3.3 Class invariant $G_{217}$

Theorem 3.3.1

$$G_{217} = \left(\sqrt{\frac{11 + 4\sqrt{7}}{2}} + \sqrt{\frac{9 + 4\sqrt{7}}{2}}\right)^{1/2} \left(\sqrt{\frac{12 + 5\sqrt{7}}{4}} + \sqrt{\frac{16 + 5\sqrt{7}}{4}}\right)^{1/2}.$$ 

Proof: From Entries 19(i) and 19(iii) of Berndt's book [15, p. 314], we note that

$$\left(\frac{(1 - \beta)^7}{(1 - \alpha)}\right)^{1/8} - \left(\frac{\beta^7}{\alpha}\right)^{1/8} = m \left(1 - (\alpha\beta(1 - \alpha)(1 - \beta)^{1/8}\right), \tag{3.3.1}$$

and

$$\left(\frac{\alpha^7}{\beta}\right)^{1/8} - \left(\frac{(1 - \alpha)^7}{(1 - \beta)}\right)^{1/8} = \frac{7}{m} \left(1 - (\alpha\beta(1 - \alpha)(1 - \beta)^{1/8}\right), \tag{3.3.2}$$

where $\beta$ has degree 7 over $\alpha$, and $m$ is the multiplier connecting $\alpha$ and $\beta$.

Multiplying (3.3.1) and (3.3.2), we find that

$$\alpha(1 - \beta) + \beta(1 - \alpha) = A \left[7(1 - A)^2 + (\alpha\beta)^{3/4} + ((1 - \alpha)(1 - \beta))^{3/4}\right], \tag{3.3.3}$$

where $A = (\alpha\beta(1 - \alpha)(1 - \beta))^{1/8}$.

Now, by the first equality of Entry 19(i) of Berndt's book [15, p. 314], we obtain

$$(\alpha\beta)^{3/4} + ((1 - \alpha)(1 - \beta))^{3/4} = 1 - 6A + 9A^2 - 2A^3. \tag{3.3.4}$$

From (3.3.3) and (3.3.4), we deduce that

$$\alpha(1 - \beta) + \beta(1 - \alpha) = 2A \left(4 - 10A + 8A^2 - A^3\right). \tag{3.3.5}$$

Now, suppose, $G_{31/7} = (4\alpha(1 - \alpha))^{-1/24}$. If $\beta$ has degree 7 over $\alpha$, then, by (3.1.4), we find that

$G_{217} = (4\beta(1 - \beta))^{-1/24}$. 

Method due to Berndt, Chan, Shapiro
Thus,

\[ \frac{1}{A} = \sqrt{2}(G_{217}G_{31/7})^3. \]  

We now recall the following two modular equations of degree 31 from Entries 22(ii) and (iii) of Berndt’s book [15, p. 439].

\[
1 + (\alpha \beta)^{1/4} + (\alpha(1 - \beta))^{1/4} - 2((\alpha \beta)^{1/8} + ((1 - \alpha)(1 - \beta))^{1/8} + A) \\
= 2A^{1/2}(1 + (\alpha \beta)^{1/8} + ((1 - \alpha)(1 - \beta))^{1/8})^{1/2}, \tag{3.3.7}
\]

and

\[
1 + (\alpha \beta)^{1/4} + (\alpha(1 - \beta))^{1/4} - \left(\frac{1}{2} \{1 + (\alpha \beta)^{1/2} + ((1 - \alpha)(1 - \beta))^{1/2}\}\right)^{1/2} \\
= (\alpha \beta)^{1/8} + ((1 - \alpha)(1 - \beta))^{1/8} + A, \tag{3.3.8}
\]

where \( \beta \) has degree 31 over \( \alpha \).

Replacing \( \alpha \) by \( 1 - \alpha \) in (3.3.7) and (3.3.8), and employing Lemma 3.2.1, we obtain

\[
1 + \{(1 - \alpha)\beta\}^{1/4} + \{(1 - \alpha)(1 - \beta)\}^{1/4} - 2\{(\alpha(1 - \beta))^{1/8} + \{(1 - \alpha)\beta\}^{1/8} + A\} \\
= 2A^{1/2}[1 + \{(1 - \alpha)\beta\}^{1/8} + \{(1 - \alpha)(1 - \beta)\}^{1/8}]^{1/2}, \tag{3.3.9}
\]

and

\[
1 + \{(1 - \alpha)\beta\}^{1/4} + \{(1 - \alpha)\beta\}^{1/4} - \left[\frac{1}{2} \{1 + \sqrt{(1 - \alpha)\beta} + \sqrt{(1 - \alpha)(1 - \beta)}\}\right]^{1/2} \\
= \{(1 - \alpha)\beta\}^{1/8} + \{(1 - \alpha)(1 - \beta)\}^{1/8} + A, \tag{3.3.10}
\]

where, now, \( \beta \) has degree 7 over \( \alpha \).
3.3. CLASS INVARIANT $G_{217}$

From (3.3.5), (3.3.9) and (3.3.10), we arrive at (simplification is done by using Mathematica)

\[
1 - 160A - 66848A^2 - 4978240A^3 + 88485264A^4 - 657312128A^5 + 2752494208A^6
- 7235315456A^7 + 12522323040A^8 - 14470630912A^9 + 11009976832A^{10} - 5258497024A^{11}
+ 1415764224A^{12} - 159303680A^{13} - 4278272A^{14} - 20480A^{15} + 256A^{16} = 0. \tag{3.3.11}
\]

Factoring the left side of (3.3.11) by using Mathematica, we find that

\[
(1 - 376A + 1048A^2 - 752A^3 + 4A^4)(1 - 8A + 24A^2 - 16A^3 + 4A^4)(1 + 224A
+ 15088A^2 - 80192A^3 + 166728A^4 - 160384A^5 + 60352A^6 + 1792A^7 + 16A^8) = 0. \tag{3.3.12}
\]

Thus,

\[
1 - 376A + 1048A^2 - 752A^3 + 4A^4 = 0 \tag{3.3.13}
\]

Since from the other two factors we will not get positive real values of $A$.

We can rewrite (3.3.13) as

\[
A^2 \left(4A^2 + \frac{1}{A^2} - 376 \left(2A + \frac{1}{A}\right) + 1048\right) = 0. \tag{3.3.14}
\]

Since $A^2 \neq 0$, we find that

\[
\left(2A + \frac{1}{A}\right)^2 - 376 \left(2A + \frac{1}{A}\right) + 1044 = 0. \tag{3.3.15}
\]

Solving (3.3.15), we find that

\[
2A + \frac{1}{A} = 188 + 70\sqrt{7}. \tag{3.3.16}
\]

Hence,

\[
\frac{1}{A} - 2A = 2\sqrt{17409 + 6580\sqrt{7}}. \tag{3.3.17}
\]
From (3.3.16) and (3.3.17), we obtain

\[
\frac{1}{A} = 94 + 35\sqrt{7} + \sqrt{17409 + 6580\sqrt{7}}. \tag{3.3.18}
\]

Now, by Lemma 2.2.4, we note that

\[
Q + \frac{1}{Q} = 2 \left(2A + \frac{1}{A}\right) - 7, \tag{3.3.19}
\]

where \(Q = (G_{217}/G_{317})^4\). From (3.3.16) and (3.3.19), we find that

\[
Q + \frac{1}{Q} = 369 + 140\sqrt{7}. \tag{3.3.20}
\]

Solving (3.3.20) for \(Q\), we obtain

\[
Q = \frac{1}{2}(369 + 140\sqrt{7} + \sqrt{273357 + 103320\sqrt{7}}). \tag{3.3.21}
\]

From (3.3.6), (3.3.18) and (3.3.21), we deduce that

\[
G_{217} = \left(\frac{94 + 35\sqrt{7}}{\sqrt{2}} + \sqrt{\frac{17409 + 6580\sqrt{7}}{2}}\right)^{1/6} \left(\frac{369 + 140\sqrt{7}}{2} + \sqrt{\frac{273357 + 103320\sqrt{7}}{4}}\right)^{1/8}. \tag{3.3.22}
\]

Now, substituting \(a = (14 + 5\sqrt{7})/4\) in Lemma 2.2.9, we find that

\[
\left(\frac{369 + 140\sqrt{7}}{2} + \sqrt{\frac{273357 + 103320\sqrt{7}}{4}}\right)^{1/4} = \sqrt{\frac{12 + 5\sqrt{7}}{4}} + \sqrt{\frac{16 + 5\sqrt{7}}{4}}.
\]

Hence, it remains to show that

\[
\frac{94 + 35\sqrt{7}}{\sqrt{2}} + \sqrt{\frac{17409 + 6580\sqrt{7}}{2}} = \left(\sqrt{\frac{11 + 4\sqrt{7}}{2}} + \sqrt{\frac{9 + 4\sqrt{7}}{2}}\right)^3,
\]

which is a routine work. This completes the theorem.
3.4 Class invariants from "mixed" modular equations

Theorem 3.4.1

\[ G_{15} = 2^{1/4} \left( \frac{1 + \sqrt{5}}{2} \right)^{1/3} \text{ and } G_{5/3} = 2^{1/4} \left( \frac{\sqrt{5} - 1}{2} \right)^{1/3}. \]

Proof: If

\[ G_n = (4\alpha (1 - \alpha))^{-1/24} \]

and \( \beta, \gamma, \) and \( \delta \) have degrees 3, 5, and 15, respectively, over \( \alpha \), then by (3.1.4), we obtain

\[ G_{9n} = (4\beta (1 - \beta))^{-1/24}, \quad G_{25n} = (4\gamma (1 - \gamma))^{-1/24} \quad \text{and} \quad G_{225n} = (4\delta (1 - \delta))^{-1/24}. \quad (3.4.1) \]

Employing Lemma 3.2.3, we find that

\[ \left( \frac{G_n}{G_{25n}} \right)^3 + \left( \frac{G_{25n}}{G_n} \right)^3 = 2 \left[ (G_n G_{25n})^2 - \frac{1}{(G_n G_{25n})^2} \right]. \quad (3.4.2) \]

Putting \( n = 1/15 \) in (3.4.2), we obtain

\[ \left( \frac{G_{15}}{G_{5/3}} \right)^3 + \left( \frac{G_{5/3}}{G_{15}} \right)^3 = 2 \left[ (G_{15} G_{5/3})^2 - \frac{1}{(G_{15} G_{5/3})^2} \right], \quad (3.4.3) \]

where we have used the fact that, \( G_n = G_{1/n} \).

Now, by Lemma 3.2.5, we obtain that

\[ \left( \frac{G_n G_{225n}}{G_{9n} G_{25n}} \right)^{3/2} + \left( \frac{G_{9n} G_{25n}}{G_n G_{225n}} \right)^{3/2} = \sqrt{2} \left[ (G_n G_{9n} G_{25n} G_{225n})^{1/2} + \frac{1}{(G_n G_{9n} G_{25n} G_{225n})^{1/2}} \right]. \quad (3.4.4) \]

Putting \( n = 1/15 \) in (3.4.4), we find that

\[ \left( \frac{G_{15}}{G_{5/3}} \right)^3 + \left( \frac{G_{5/3}}{G_{15}} \right)^3 = \sqrt{2} \left[ (G_{15} G_{5/3})^2 - \frac{1}{(G_{15} G_{5/3})^2} \right]. \quad (3.4.5) \]
Setting $x := G_{15}G_{5/3}$ in (3.4.3) and (3.4.5), we deduce that

$$2(x^2 - \frac{1}{x^2}) = \sqrt{2}(x + \frac{1}{x}). \quad (3.4.6)$$

As $x + \frac{1}{x} > 0$, from (3.4.6), we conclude that

$$\sqrt{2}(x - \frac{1}{x}) = 1. \quad (3.4.7)$$

Solving (3.4.7) for $x$, we find that

$$x = G_{15}G_{5/3} = \sqrt{2}. \quad (3.4.8)$$

Using this value of $x$ in (3.4.5), we deduce that

$$y^3 + \frac{1}{y^3} = 3, \quad (3.4.9)$$

where, $y = G_{15}/G_{5/3}$.

Solving (3.4.9) for $y^3$, we find that

$$y^3 = \left(\frac{G_{15}}{G_{5/3}}\right)^3 = \frac{3 + \sqrt{5}}{2}. \quad (3.4.10)$$

From (3.4.8) and (3.4.10), we obtain

$$G_{15}^6 = y^3x^3 = 2^{3/2}\frac{3 + \sqrt{5}}{2} = \sqrt{2}(3 + \sqrt{5}), \quad (3.4.11)$$

and

$$G_{5/3}^6 = y^{-3}x^3 = 2^{3/2}\frac{2}{3 + \sqrt{5}} = \sqrt{2}(3 - \sqrt{5}), \quad (3.4.12)$$

Now, from Lemma 3.2.8, we see that

$$\sqrt{3 \pm \sqrt{5}} = \sqrt{5/2} \pm \sqrt{1/2}.$$ 

Thus, from (3.4.11) and (3.4.12), we can arrive at the required values of $G_{15}$ and $G_{5/3}$. 
Theorem 3.4.2

\[ G_{21} = \left( \frac{\sqrt{3} + \sqrt{7}}{2} \right)^{1/4} \left( \frac{3 + \sqrt{7}}{2} \right)^{1/6} \quad \text{and} \quad G_{7/3} = \left( \frac{\sqrt{7} - \sqrt{3}}{2} \right)^{1/4} \left( \frac{3 + \sqrt{7}}{2} \right)^{1/6} \]

**Proof:** As in the proof of Theorem 3.4.1, if

\[ G_n = (4\alpha(1 - \alpha))^{-1/24} \]

\( \leftarrow \) and \( \beta, \gamma, \) and \( \delta \) have degrees 3, 7, and 21, respectively, over \( \alpha, \) then by (3.1.4),

\[ G_{3n} = (4\beta(1 - \beta))^{-1/24}, \quad G_{49n} = (4\gamma(1 - \gamma))^{-1/24} \quad \text{and} \quad G_{441n} = (4\delta(1 - \delta))^{-1/24}. \] (3.4.13)

Therefore, by Lemma 3.2.2, we find that

\[ \left( \frac{G_n}{G_{9n}} \right)^6 + \left( \frac{G_{9n}}{G_n} \right)^6 = 2\sqrt{2} \left[ (G_n G_{9n})^3 - \frac{1}{(G_n G_{9n})^3} \right]. \] (3.4.14)

Putting \( n = 1/21 \) in (3.4.14), we deduce that

\[ \left( \frac{G_{21}}{G_{7/3}} \right)^6 + \left( \frac{G_{7/3}}{G_{21}} \right)^6 = 2\sqrt{2} \left[ (G_{21} G_{7/3})^3 - \frac{1}{(G_{21} G_{7/3})^3} \right], \] (3.4.15)

where we have again used the fact that, \( G_n = G_{1/n}. \)

Now, by Theorem 2.1.4, we deduce that

\[ R^4 + \frac{1}{R^4} + 7 \left( R^3 + \frac{1}{R^3} \right) + 14 \left( R^2 + \frac{1}{R^2} \right) + 21 \left( R + \frac{1}{R} \right) - 8 \left( P^6 + \frac{1}{P^6} \right) + 42 = 0, \] (3.4.16)

where, now,

\[ R = \left( \frac{G_n G_{9n}}{G_{49n} G_{441n}} \right) \quad \text{and} \quad P^2 = 1/ (G_n G_{9n} G_{49n} G_{441n}). \]
Putting \( n = 1/21 \) in (3.4.16), we find that

\[
\left( G_{21} G_{7/3} \right)^6 + \frac{1}{\left( G_{21} G_{7/3} \right)^6} = 16 \tag{3.4.17}
\]

Solving (3.4.17) for \( \left( G_{21} G_{7/3} \right)^6 \), we obtain

\[
\left( G_{21} G_{7/3} \right)^6 = 8 + 3\sqrt{7} = \left( \frac{3 + \sqrt{7}}{\sqrt{2}} \right)^2 \tag{3.4.18}
\]

Employing (3.4.18) in (3.4.15), we find that

\[
\left( \frac{G_{21}}{G_{7/3}} \right)^6 + \left( \frac{G_{7/3}}{G_{21}} \right)^6 = 2\sqrt{2} \left( \frac{3 + \sqrt{7}}{\sqrt{2}} - \frac{\sqrt{2}}{3 + \sqrt{7}} \right) = 4\sqrt{7} \tag{3.4.19}
\]

Solving (3.4.19) for \( \left( G_{21}/G_{7/3} \right)^6 \), we obtain

\[
\left( \frac{G_{21}}{G_{7/3}} \right)^6 = 3\sqrt{3} + 2\sqrt{7} = \left( \frac{\sqrt{3} + \sqrt{7}}{2} \right)^3 \tag{3.4.20}
\]

From (3.4.18) and (3.4.20), we obtain

\[
G_{21}^{12} = \left( \frac{\sqrt{3} + \sqrt{7}}{2} \right)^3 \left( \frac{3 + \sqrt{7}}{\sqrt{2}} \right)^2 \tag{3.4.21}
\]

and

\[
G_{7/3}^{12} = \left( \frac{\sqrt{7} - \sqrt{3}}{2} \right)^3 \left( \frac{3 + \sqrt{7}}{\sqrt{2}} \right)^2 \tag{3.4.22}
\]

From (3.4.21) and (3.4.22), we get the required values of \( G_{21} \) and \( G_{7/3} \) as given in the theorem.

Theorem 3.4.3

\[
G_{33} = \left( \frac{3 + \sqrt{11}}{\sqrt{2}} \right)^{1/6} \left( \frac{1 + \sqrt{3}}{\sqrt{2}} \right)^{1/2} \quad \text{and} \quad G_{11/3} = \left( \frac{\sqrt{11} - 3}{\sqrt{2}} \right)^{1/6} \left( \frac{1 + \sqrt{3}}{\sqrt{2}} \right)^{1/2}.
\]
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Proof: In this case also, if

\[ G_n = (4\alpha(1-\alpha))^{-1/24} \]

and \( \beta, \gamma, \) and \( \delta \) have degrees 3, 11, and 33, respectively, over \( \alpha \), then by (3.1.4),

\[ G_{9n} = (4\beta(1-\beta))^{-1/24}, \quad G_{121n} = (4\gamma(1-\gamma))^{-1/24} \quad \text{and} \quad G_{1089n} = (4\delta(1-\delta))^{-1/24}. \]  

By putting \( n = 1/33 \) in (3.4.14), we deduce that

\[ \left( \frac{G_{33}}{G_{11/3}} \right)^6 + \left( \frac{G_{11/3}}{G_{33}} \right)^6 = 2\sqrt{2} \left[ (G_{33}G_{11/3})^3 - \frac{1}{G_{33}G_{11/3}} \right]. \]  

Now, by Theorem 2.1.2, we find that

\[ T^2 + \frac{1}{T^2} + 3 \left( T + \frac{1}{T} \right) - 2 \left( \frac{P^2 + 1}{P^2} \right) = 0, \]  

where, now,

\[ T = \frac{G_n G_{121n}}{G_{9n} G_{1089n}} \quad \text{and} \quad P^2 = 1/(G_n G_{9n} G_{121n} G_{1089n}). \]

Putting \( n = 1/33 \) in (3.4.25), we find that

\[ \left( G_{33}G_{11/3} \right)^2 + \frac{1}{G_{33}G_{11/3}} = 4. \]  

Solving (3.4.26) for \( (G_{33}G_{11/3})^2 \), we obtain

\[ (G_{33}G_{11/3})^2 = 2 + \sqrt{3} = \left( \frac{1 + \sqrt{3}}{\sqrt{2}} \right)^2. \]  

Employing (3.4.27) in (3.4.24), we find that

\[ \left( \frac{G_{33}}{G_{11/3}} \right)^6 + \left( \frac{G_{11/3}}{G_{33}} \right)^6 = 20. \]
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Solving (3.4.28) for \( \left( \frac{G_{33}}{G_{11/3}} \right)^6 \), we obtain

\[
\left( \frac{G_{33}}{G_{11/3}} \right)^6 = 10 + 3\sqrt{11} = \left( \frac{3 + \sqrt{11}}{\sqrt{2}} \right)^2.
\]

From (3.4.27) and (3.4.29), we obtain

\[
G_{12}^{12} = \left( \frac{1 + \sqrt{3}}{\sqrt{2}} \right)^6 \left( \frac{3 + \sqrt{11}}{\sqrt{2}} \right)^2,
\]

and

\[
G_{11/3}^{12} = \left( \frac{1 + \sqrt{3}}{\sqrt{2}} \right)^6 \left( \frac{\sqrt{2}}{3 + \sqrt{11}} \right)^2.
\]

From (3.4.30) and (3.4.31), we can easily find the values of \( G_{11} \) and \( G_{11/3} \).

**Theorem 3.4.4**

\[
G_{39} = 2^{1/4} \left( \frac{\sqrt{13} + 3}{2} \right)^{1/6} \left( \sqrt{\frac{5 + \sqrt{13}}{8}} + \sqrt{\frac{\sqrt{13} - 3}{8}} \right)
\]

and

\[
G_{13/3} = 2^{1/4} \left( \frac{\sqrt{13} + 3}{2} \right)^{1/6} \left( \sqrt{\frac{5 + \sqrt{13}}{8}} - \sqrt{\frac{\sqrt{13} - 3}{8}} \right).
\]

**Proof:** As above, if

\[
G_n = (4\alpha(1 - \alpha))^{-1/24}
\]

and \( \beta, \gamma, \) and \( \delta \) have degrees 3, 13, and 39, respectively, over \( \alpha \), then

\[
G_{9n} = (4\beta(1 - \beta))^{-1/24}, \quad G_{169n} = (4\gamma(1 - \gamma))^{-1/24} \quad \text{and} \quad G_{1521n} = (4\delta(1 - \delta))^{-1/24}.
\]

Putting \( n = 1/39 \) in (3.4.14), we find that

\[
\left( \frac{G_{39}}{G_{13/3}} \right)^6 + \left( \frac{G_{13/3}}{G_{39}} \right)^6 = 2\sqrt{2} \left[ \left( G_{39}G_{13/3} \right)^3 - \frac{1}{\left( G_{39}G_{13/3} \right)^3} \right].
\]
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Now, by Lemma 3.2.6, we obtain

\[ Q^2 + \frac{1}{Q^2} - 3 \left( Q + \frac{1}{Q} \right) - \left( R + \frac{1}{R} \right) = 0, \]  

(3.4.34)

where

\[ Q = \frac{G_{9n}G_{169n}}{G_nG_{1521n}} \quad \text{and} \quad T = \frac{G_nG_{9n}}{G_{169n}G_{1521n}}. \]

Putting \( n = 1/39 \) in (3.4.34), we find that

\[ \left( \frac{G_{39}}{G_{13/3}} \right)^4 + \left( \frac{G_{13/3}}{G_{39}} \right)^4 - 3 \left( \frac{G_{39}}{G_{13/3}} \right)^2 + \left( \frac{G_{13/3}}{G_{39}} \right)^2 + 1 = 0. \]  

(3.4.35)

Therefore, we obtain

\[ \left( \frac{G_{39}}{G_{13/3}} \right)^2 + \left( \frac{G_{13/3}}{G_{39}} \right)^2 = \frac{3 + \sqrt{13}}{2}. \]  

(3.4.36)

Solving (3.4.36) for \( \left( \frac{G_{39}}{G_{13/3}} \right)^2 \), we find that

\[ \left( \frac{G_{39}}{G_{13/3}} \right)^2 = \frac{1}{2} \left[ \frac{3 + \sqrt{13}}{2} + \sqrt{\frac{3 + 3\sqrt{13}}{2}} \right]. \]  

(3.4.37)

Employing (3.4.37) in (3.4.33), we find that

\[ \left( \frac{G_{39}G_{13/3}}{G_{39}} \right)^3 - \frac{1}{\left( \frac{G_{39}G_{13/3}}{G_{39}} \right)^3} = \frac{27 + 7\sqrt{13}}{4\sqrt{2}}. \]  

(3.4.38)

Solving (3.4.38) for \( \left( \frac{G_{39}G_{13/3}}{G_{39}} \right)^3 \), we find that

\[ \left( \frac{G_{39}G_{13/3}}{G_{39}} \right)^3 = \frac{1}{2} \left[ \frac{27 + 7\sqrt{13}}{4\sqrt{2}} + \sqrt{\frac{747 + 189\sqrt{13}}{16}} \right]. \]  

(3.4.39)

Since, \( 747^2 - 13.189^2 = 306^2 \), by Lemma 3.2.8, we note that

\[ \sqrt{747 + 189\sqrt{13}} = \sqrt{\frac{747 + 306}{2}} + \sqrt{\frac{747 - 306}{2}} = \frac{21 + 9\sqrt{13}}{\sqrt{2}}. \]
Thus, from (3.4.39), we find that

\[ (G_{39}G_{13/3})^3 = \sqrt{2}(3 + \sqrt{13}). \]  

(3.4.40)

From (3.4.37) and (3.4.40), we obtain

\[ G_{39}^{12} = (3 + \sqrt{13})^2 \left[ \frac{3 + \sqrt{13}}{2} + \sqrt{3 + 3\sqrt{13}} \right]^3. \]  

(3.4.41)

Therefore,

\[ G_{39} = 2^{1/4}(3 + \sqrt{13})^{1/6} \left[ \frac{3 + \sqrt{13}}{4} + \frac{1}{2} \sqrt{3 + 3\sqrt{13}} \right]^{1/4}. \]  

(3.4.42)

Now, putting \( a = (1 + \sqrt{13})/8 \) in Lemma 3.2.9, we obtain

\[ \left( \frac{3 + \sqrt{13}}{4} + \sqrt{3 + 3\sqrt{13}} \right)^{1/4} = \sqrt{\frac{5 + \sqrt{13}}{8}} + \sqrt{\frac{\sqrt{13} - 3}{8}}. \]

Thus, we arrive at the required value of \( G_{39} \). Similarly, we can get the value of \( G_{13/3} \).

**Theorem 3.4.5**

\[ G_{55} = 2^{1/4}(2 + \sqrt{5})^{1/4} \left( \frac{\sqrt{7 + \sqrt{5}}}{8} + \sqrt{\frac{5 - 1}{8}} \right) \]

and

\[ G_{11/5} = 2^{1/4}(2 + \sqrt{5})^{1/4} \left( \frac{\sqrt{7 + \sqrt{5}}}{8} - \sqrt{\frac{5 - 1}{8}} \right). \]

**Proof:** As in the previous proofs, if

\[ G_n = (4\alpha(1 - \alpha))^{-1/24} \]

and \( \beta, \gamma, \) and \( \delta \) have degrees 5, 11, and 55, respectively, over \( \alpha \), then by (3.1.4)

\[ G_{25n} = (4\beta(1 - \beta))^{-1/24}, \quad G_{121n} = (4\gamma(1 - \gamma))^{-1/24} \quad \text{and} \quad G_{3025n} = (4\delta(1 - \delta))^{-1/24}. \]  

(3.4.43)
By putting \( n = 1/55 \) in (3.4.2), we deduce that
\[
\left( \frac{G_{55}}{G_{11/5}} \right)^3 + \left( \frac{G_{11/5}}{G_{55}} \right)^3 = 2 \left[ \left( G_{55} G_{11/5} \right)^2 - \frac{1}{\left( G_{55} G_{11/5} \right)^2} \right].
\] (3.4.44)

By Theorem 3.1.8, we deduce that
\[
T^3 + \frac{1}{T^3} - 5 \left( T^2 + \frac{1}{T^2} \right) + 10 \left( T + \frac{1}{T} \right) \left( T^2 + \frac{1}{T^2} - 1 \right)\]
\[
- 4 \left( P^4 + \frac{1}{P^4} \right) + 10 \left( P^2 + \frac{1}{P^2} \right) - 25 = 0,
\] (3.4.45)

where
\[
T = \frac{G_n G_{121n}}{G_{25n} G_{3025n}} \quad \text{and} \quad P^2 = 1/ \left( G_{55} G_{11/5} \right)^2.
\]

Putting \( n = 1/55 \) in (3.4.45), we find that
\[
4 \left( P^4 + \frac{1}{P^4} \right) - 30 \left( P^2 + \frac{1}{P^2} \right) + 53 = 0,
\] (3.4.46)

where, now, \( P^2 = 1/ \left( G_{55} G_{11/5} \right)^2 \).

From (3.4.46), we deduce that
\[
P^2 + \frac{1}{P^2} = \frac{15 + 3\sqrt{5}}{4}.
\] (3.4.47)

Solving for \( 1/P^2 \), we find that
\[
\frac{1}{P^2} = \frac{15 + 3\sqrt{5}}{4} + \frac{1}{4} \sqrt{103 + 45\sqrt{5}}.
\] (3.4.48)

Since, \( 103^2 - 5 \cdot 45^2 = 22^2 \), we see from Lemma 3.2.8 that
\[
\sqrt{103 + 45\sqrt{5}} = \sqrt{\frac{125}{2}} + \sqrt{\frac{81}{2}} = \frac{9 + 5\sqrt{5}}{\sqrt{2}}.
\]

Thus
\[
\frac{1}{P^2} = 3 + \sqrt{5}.
\] (3.4.49)
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Employing (3.4.49) in (3.4.44), we find that

\[
\left( \frac{G_{55}}{G_{11/5}} \right)^3 + \left( \frac{G_{11/5}}{G_{55}} \right)^3 = \frac{9 + 5\sqrt{5}}{2}. \tag{3.4.50}
\]

Solving (3.4.50) for \( \left( \frac{G_{55}}{G_{11/5}} \right)^3 \), we obtain

\[
\left( \frac{G_{55}}{G_{11/5}} \right)^3 = \frac{1}{2} \left( \frac{9 + 5\sqrt{5}}{2} + \sqrt{\frac{95 + 45\sqrt{5}}{2}} \right). \tag{3.4.51}
\]

From (3.4.49) and (3.4.51), we deduce that

\[
G_{55}^{12} = (3 + \sqrt{5})^3 \left( \frac{9 + 5\sqrt{5}}{4} + \sqrt{\frac{95 + 45\sqrt{5}}{8}} \right)^2. \tag{3.4.52}
\]

Thus,

\[
G_{55} = 2^{1/4}(2 + \sqrt{5})^{1/6} \left( \frac{9 + 5\sqrt{5}}{4} + \sqrt{\frac{95 + 45\sqrt{5}}{8}} \right)^{1/6}. \tag{3.4.53}
\]

Now, substituting \( b = (3 + \sqrt{5})/8 \) in Lemma 3.2.9, we see that

\[
\left( \frac{9 + 5\sqrt{5}}{4} + \sqrt{\frac{95 + 45\sqrt{5}}{8}} \right)^{1/6} = \sqrt{\frac{7 + \sqrt{5}}{8}} + \sqrt{\frac{\sqrt{5} - 1}{8}}. \tag{3.4.54}
\]

This completes the evaluation of \( G_{55} \). The value of \( G_{11/5} \) can be deduced similarly by using (3.4.49) and (3.4.51).

**Theorem 3.4.6**

\[
G_{65} = \left( \frac{3 + \sqrt{13}}{2} \right)^{1/4} \left( \frac{\sqrt{5} + 1}{2} \right)^{1/4} \left( \frac{9 + \sqrt{65}}{8} + \frac{1 + \sqrt{65}}{8} \right)^{1/2}
\]

and

\[
G_{13/5} = \left( \frac{\sqrt{13} - 3}{2} \right)^{1/4} \left( \frac{\sqrt{5} - 1}{2} \right)^{1/4} \left( \frac{9 + \sqrt{65}}{8} + \frac{1 + \sqrt{65}}{8} \right)^{1/2}.
\]
Proof: Here also, if
\[ G_n = (4\alpha(1 - \alpha))^{-1/24} \]
and \( \beta, \gamma, \) and \( \delta \) have degrees 5, 13, and 65, respectively, over \( \alpha \), then as in the previous proofs, by (3.1.4)
\[
G_{25n} = (4\beta(1 - \beta))^{-1/24}, \quad G_{169n} = (4\gamma(1 - \gamma))^{-1/24} \quad \text{and} \quad G_{4225n} = (4\delta(1 - \delta))^{-1/24}. \quad (3.4.55)
\]
By putting \( n = 1/65 \) in (3.4.2), we deduce that
\[
\left( \frac{G_{65}}{G_{13/5}} \right)^3 + \left( \frac{G_{13/5}}{G_{65}} \right)^3 = 2 \left( G_{65}G_{13/5} \right)^2 - \frac{1}{\left( G_{65}G_{13/5} \right)^2} \quad (3.4.56)
\]
By Lemma 3.2.7, we deduce that
\[
Q^3 + \frac{1}{Q^3} - 5 \left( Q + \frac{1}{Q} + 2 \right) \left( R + \frac{1}{R} + 2 \right) - \left( R^2 + \frac{1}{R^2} \right) = 0, \quad (3.4.57)
\]
where, now,
\[
Q = \frac{G_{25n}G_{169n}}{G_nG_{4225n}} \quad \text{and} \quad R = \frac{G_nG_{25n}}{G_{169n}G_{4225n}}.
\]
Putting \( n = 1/65 \) in (3.4.57), we find that
\[
\left( Q^3 + \frac{1}{Q^3} \right) - 20 \left( Q + \frac{1}{Q} \right) - 42 = 0, \quad (3.4.58)
\]
where, now, \( Q = \left( G_{65}/G_{13/5} \right)^2 \).
From (3.4.58), we obtain
\[
Q + \frac{1}{Q} = \frac{3 + \sqrt{65}}{2}. \quad (3.4.59)
\]
Solving for \( G_{65}/G_{13/5} \), we find that
\[
\frac{G_{65}}{G_{13/5}} = \sqrt{\frac{7 + \sqrt{65}}{8} + \sqrt{\frac{\sqrt{65} - 1}{8}}}. \quad (3.4.60)
\]
Invoking (3.4.60) in (3.4.56), we find that
\[
2 \left[ \left( G_{65}^{13/5} \right)^2 - \frac{1}{G_{65}^{13/5}} \right] = \sqrt{74 + 10\sqrt{65}}.
\] (3.4.61)

Solving (3.4.61) for \( \left( G_{65}^{13/5} \right)^2 \), we find that
\[
\left( G_{65}^{13/5} \right)^2 = \frac{1}{4} \left( \sqrt{74 + 10\sqrt{65}} + \sqrt{90 + 10\sqrt{65}} \right).
\] (3.4.62)

Thus, we deduce that
\[
G_{65}^{13/5} = \frac{1}{2} \left( \sqrt{74 + 10\sqrt{65}} + \sqrt{90 + 10\sqrt{65}} \right)^{1/2}.
\] (3.4.63)

Since, \( 90^2 - 65.10^2 = 40^2 \), from Lemma 3.2.8, we see that
\[
\sqrt{90 + 10\sqrt{65}} = \sqrt{\frac{130}{2}} + \sqrt{\frac{50}{2}} = 5 + \sqrt{65}.
\]

Hence,
\[
G_{65}^{13/5} = \frac{\sqrt{9 + \sqrt{65}}}{8} + \frac{\sqrt{1 + \sqrt{65}}}{8}.
\] (3.4.64)

From (3.4.60) and (3.4.64), we deduce that
\[
G_{65}^2 = \left( \sqrt{\frac{7 + \sqrt{65}}{8}} + \sqrt{\frac{\sqrt{65} - 1}{8}} \right) \left( \sqrt{\frac{9 + \sqrt{65}}{8}} + \sqrt{\frac{1 + \sqrt{65}}{8}} \right).
\] (3.4.65)

and
\[
G_{13/5}^2 = \left( \sqrt{\frac{7 + \sqrt{65}}{8}} - \sqrt{\frac{\sqrt{65} - 1}{8}} \right) \left( \sqrt{\frac{9 + \sqrt{65}}{8}} + \sqrt{\frac{1 + \sqrt{65}}{8}} \right).
\] (3.4.66)

Now, simple calculation shows that
\[
\left( \sqrt{\frac{7 + \sqrt{65}}{8}} \pm \sqrt{\frac{\sqrt{65} - 1}{8}} \right)^2 = \left( \frac{\sqrt{13} \pm 3}{2} \right) \left( \frac{\sqrt{5} \pm 1}{2} \right).
\] (3.4.67)

Using (3.4.67) in (3.4.65) and (3.4.66), we easily arrive at the required class invariants.

As mentioned in the Introduction, we have seen that our evaluation of \( G_{65} \) is much easier than that of Berndt, Chan and Zhang \[18], \[25].