CHAPTER 5

ROBUSTNESS ANALYSIS OF THE CONTROLLER

5.1 INTRODUCTION

Robust control is a branch of control theory that explicitly deals with uncertainty in its approach to controller design. It also refers to the control of unknown plants with unknown dynamics subject to unknown disturbances. It is designed to function properly so long as uncertain parameters or disturbances are within some (typically compact) set. Controllers designed using robust control methods tend to be able to cope with small differences between the true system and the nominal model used for design.

The early methods of Bode and others were fairly robust. The state-space methods invented in the 1960s and 1970s were sometimes found to lack robustness, prompting research to improve them. This was the start of the theory of robust control, which took shape in the 1980s and 1990s and is still active today. In contrast with an adaptive control policy, a robust control policy is static, rather than adapting to measurement of variations, the controller is designed to work assuming that certain variables will be unknown but, bounded.

Figure 5.1 shows an expanded view of the simple control loop. Uncertainty is shown entering the system in three places. There is uncertainty in the model of the plant. There are disturbances that occur in the plant.
system. Also there is noise which is read on the sensor inputs. Each of these uncertainties can have an additive or multiplicative component. Robust methods aim to achieve robust performance and/or stability in the presence of small modeling errors.

![Plant control loop with uncertainty diagram]

**Figure 5.1 Plant control loop with uncertainty**

Figure 5.1 also shows the separation of the computer control system with that of the plant. It is important to understand that the control system designer has little control of the uncertainty in the plant. The designer creates a control system that is based on a model of the plant. However, the implemented control system must interact with the actual plant, not the model of the plant.

### 5.2 EFFECTS OF UNCERTAINTY

Control system engineers are concerned with three main topics: observability, controllability and stability. Observability is the ability to observe all of the parameters or state variables in the system. Controllability is the ability to move a system from any given state to any desired state. Stability is often phrased as the bounded response of the system to any
bounded input. Any successful control system will have and maintain all three of these properties. Uncertainty presents a challenge to the control system engineer who tries to maintain these properties using limited information.

One method to deal with uncertainty in the past is stochastic control. In stochastic control, uncertainties in the system are modeled as probability distributions. These distributions are combined to yield the control law. This method deals with the expected value of control. Abnormal situations may arise that deliver results that are not necessarily close to the expected value. This may not be acceptable for embedded control systems that have safety implications.

Robust control methods seek to bound the uncertainty rather than express it in the form of a distribution. Given a bound on the uncertainty, the control can deliver results that meet the control system requirements in all cases.

A plant model is rarely a fully accurate description of the real plant. Neglected dynamics, approximately known parameters, and changes in operating conditions all contribute to plant-modeling errors jeopardize controller performance. A controller is ‘robust’ when it maintains its performance regardless of plant modeling errors. The robust control toolbox provides a systematic approach to design robust, multivariable feedback control systems. This approach involves the following steps:

- Modeling and quantifying plant uncertainty
- Performing robustness analysis
- Synthesizing robust, multivariable controllers
- Reducing controller and plant model order
5.2.1 Modeling and Quantifying Plant Uncertainty

With the robust control toolbox, one can capture the typical of ‘nominal’ behaviors of the plant but also the amount of uncertainty and variability. Plant model uncertainty result from,

- Model parameters with approximately known values
- Neglected or poorly known dynamics, such as high frequency dynamics
- Changes in operating conditions
- Linear approximation of non linear behaviors
- Estimation errors in a model identified from measured data

5.2.2 Performing Robustness Analysis

Using the robust control toolbox, one can analyze the effect of model uncertainty on the closed loop stability and performance of the control system. In particular, one can determine whether your control system will perform adequately over your entire operating range, and what source of uncertainty is most likely to jeopardize performance. The robust control toolbox provides tools to access:

- Worst-case gain/phase margins one loop at a time
- Worst-case stability margins taking loop interactions into account
- Worst-case gain between any two points in the closed –loop system
- Worst-case sensitivity to external disturbances
These tools also provide sensitivity information to help to identify which uncertain elements contribute most to performance degradation. It is also possible to determine whether a more accurate model, tighter manufacturing tolerances, or a better sensor would most improve control system robustness.

### 5.2.3 Synthesizing Robust Multivariable Controllers

The robust control toolbox provides a variety of controller synthesis algorithms based on loop shaping $H_\infty$ or $\mu$-synthesis, and LMI techniques. These algorithms are applicable to SISO and MIMO control systems. MIMO controller synthesis does not involve sequential loop closure, and is therefore well suited to multi loop control systems with strong loop interaction and challenging I/O decoupling requirements.

Performance and robustness requirements can often be expressed in terms of the open-loop response gain. For example, high gain at low frequency roll-off improves stability where the plant model is poor. Loop shaping aims at designing a controller that achieves a particular loop shape. The robust control toolbox provides tools to automate computing controllers that best match user defined loop shape. In MIMO systems, these tools operate on the singular values of the open-loop response to increase or decrease the gain in all input/output directions.

### 5.2.4 Reducing Controller and Plant Model Order

The robust control toolbox tools used for reducing the number of states of a plant or controller model while preserving its essential dynamics. Detailed first-principles or finite-element plant models may have a high order. Similarly, $H_\infty$ or $\mu$-synthesis algorithms tend to produce high-order controllers with superfluous states. In both cases, model reduction allows to develop
approximate plant and controller models that are reliable and cost-effective to implement.

Conventional control theory has allowed man to control and automate his environment for centuries. Modern control techniques have allowed engineers to optimize the control systems they build for cost and performance. However, optimal control algorithms are not always tolerant to changes in the control system or the environment. Robust control theory is a method to measure the performance changes of a control system with changing system parameters. Application of this technique is important to build dependable embedded systems. The goal is to allow exploration of the design space for alternatives that are insensitive to changes in the system and can maintain their stability and performance.

To gain a perspective for robust control, it is useful to examine some basic concepts from control theory. Control theory can be broken down historically into two main areas: conventional control and modern control. Conventional control became interesting with the development of feedback theory. Feedback is used in order to stabilize the control system.

Conventional control relies upon developing a model of the control system using differential equations. Laplace transforms are then used to express the system equations in the frequency domain where they can be manipulated algebraically. Due to the complexity of the mathematics, conventional control methods were used mostly for Single-Input-Single-Output (SISO) systems.

The development that was key to future developments in robust control was the root-locus method. In the frequency domain, G(s) and H(s) were expressed as ratios of polynomials in the complex frequency variables. Nyquist, Bode and others realized that the roots of the denominator
polynomial determine the stability of the control system. These roots were referred to as "poles" of the transfer functions. The location of these poles had to be in the left half-plane of the complex frequency plot to guarantee stability. Root locus was developed as a method to graphically show the movements of poles in the frequency domain as the coefficients of the s-polynomial were changed. Movement into the right half plane meant an unstable system. Thus systems could be judged by their sensitivity to small changes in the denominator coefficients. Modern control methods were developed with a realization that control system equations could be structured in such a way that computers could efficiently solve them. Modern control methods were extremely successful because they could be efficiently implemented on computers, they could handle Multiple-Input-Multiple-Output (MIMO) systems, and they could be optimized.

5.3 MODELING

One of the most difficult parts of designing a good control system is modeling the behavior of the plant. There are a variety of reasons for why modeling is difficult.

- **Imperfect plant data** - Often, little hard data is available about the plant. Many control systems are designed concurrently with the plant. Even if there are similar plants in existence, each plant is slightly different because of the tolerances associated with individual components.

- **Time varying plants** - The dynamics of some plants vary over time. A fixed control model may not accurately depict the plant at all times.

- **Higher order dynamics** - Some plants have a high frequency dynamic that is often neglected in the nominal plant model.
For instance, vibration may cause unwanted affects at high frequencies. Sometimes this dynamic is unknown and sometimes it is deliberately ignored in order to simplify the model.

- **Non-linearity** - Most control systems are designed assuming linear time invariant systems. This is done because it greatly simplifies the analysis of the system. However, all of the systems encountered in the real world have some non-linear component. Thus the model will always be an approximation of the real world behavior.

- **Complexity** - Mechanical and electrical systems are inherently complex to model. Even a simple system with a simple requires complex differential equations to describe its behavior.

One technique for handling the model uncertainty that often occurs at high frequencies is to balance performance and robustness in the system through gain scheduling. A high gain means that the system will respond quickly to differences between the desired state and the actual state of the plant.

At low frequencies where the plant is accurately modeled, this high gain (near 1) results in high performance of the system. This region of operation is called the performance band. At high frequencies where the plant is not modeled accurately, the gain is lower. A low gain at high frequencies results in a larger error term between the measured output and the reference signal. This region is called the robustness band.

In this region the feedback from the output is essentially ignored. The method for changing the gain over different frequencies is through the
transfer function. This involves setting the poles and zeros of the transfer function to achieve a filter. Between these two regions, performance and robustness, there is a transition region. In this region the controller does not perform well for either performance or robustness. The transition region cannot be made arbitrarily small because it depends on the number of poles and zeros of the transfer function. Adding terms to the transfer function increases the complexity of the control system. Thus, there is a trade-off between the simplicity of the model and the minimal size of the transition band.

There are a variety of techniques that have been developed for robust control. These techniques are difficult to understand and tedious to implement. An adaptive control system sets up observers for each significant state variable in the system. The system can adjust each observer to account for time varying parameters of the system. In an adaptive system, there is always a dual role of the control system. The output is to be brought closer to the desired input while, at the same time, the system continues to learn about changes in the system parameters. This method sometimes suffers from problems in convergence for the system parameters.

$H_2$ or $H_{\infty}$ - Hankel norms are used to measure control system properties. A norm is an abstraction of the concept of length. Both of these techniques are frequency domain techniques. $H_2$ control seeks to bound the power gain of the system while $H_{\infty}$ control seeks to bound the energy gain of the system. Gains in power or energy in the system indicate operation of the system near a pole in the transfer function. These situations are unstable.

Parameter estimation techniques establish boundaries in the frequency domain that cannot be crossed to maintain stability. These boundaries are evaluated by given uncertainty vectors. This technique is
graphical. It has some similarities to the root locus technique. The advancement of this technique is based upon computational simplifications in evaluating whether multiple uncertainties cause the system to cross a stability boundary. These techniques claim to give the user clues on how to change the system to make it more insensitive to uncertainties.

Lyaponov functions are constructed, which are described as energy like functions that model the behavior of real systems. These functions are evaluated along the system trajectory to see if the first derivative is always dissipative in energy. Any gain in energy represents the system is operating near a pole and will therefore be unstable.

Fuzzy control is based upon the construction of fuzzy sets to describe the uncertainty inherent in all variables and a method of combining these variables called fuzzy logic. Fuzzy control is applicable to robust control because it is a method of handling the uncertainty of the system. Fuzzy control is a controversial issue. Its proponents claim the ability to control without the requirement for complex mathematical modeling. It appears to have applications where there are a large number of variables to be controlled and it is intuitively obvious (but not mathematically obvious) how to control the system.

5.3.1 Model Identification and Robustness

A control system must always have some robustness property. A robust controller is such that its properties do not change much if applied to a system slightly different from the mathematical one used for its synthesis. This specification is important: no real physical system truly behaves like the series of differential equations used to represent it mathematically.
The process of determining the equations that govern the model dynamics is called system identification. This can be done off-line: for example, executing a series of measures from which to calculate an approximated mathematical model, typically its transfer function or matrix. Such identification from the output, however, cannot take account of unobservable dynamics. Some advanced control techniques include an "on-line" identification process. The parameters of the model are calculated while the controller itself is running.

Analysis of the robustness of a SISO control system can be performed in the frequency domain, considering the system's transfer function and using Nyquist and Bode diagrams. For MIMO and, in general, more complicated control systems one must consider the theoretical results devised for each control technique. i.e., if particular robustness qualities are needed, the engineer must shift his attention to a control technique including them in its properties.

A particular robustness issue is the requirement for a control system to perform properly in the presence of input and state constraints. In the physical world every signal is limited. It could happen that a controller will send control signals that cannot be followed by the physical system: for example, trying to rotate a valve at excessive speed. This can produce undesired behavior of the closed-loop system, or even break actuators or other subsystems. Specific control techniques are available to solve the problem.

Kharitonov's theorem is a result used in control theory to assess the stability of a dynamical system when the physical parameters of the system are not known precisely. When the coefficients of the characteristic polynomial are known, the Routh-Hurwitz stability criterion can be used to check if the system is stable (i.e. if all roots have negative real parts). Kharitonov's theorem can be used in the case where the coefficients are only
known to be within specified ranges. It provides a test of stability for a so-called interval polynomial, while Routh-Hurwitz is concerned with an ordinary polynomial.

A number of papers on robustness analysis of uncertain systems have been published in the past few years. Kharitonov’s theorem states that the Hurwitz stability of a real (or complex) interval polynomial family can be guaranteed by the Hurwitz stability of four (or eight) prescribed critical vertex polynomials in this family. This result is significant since it reduces checking stability of infinitely many polynomials to checking stability of finitely many polynomials, and the number of critical vertex polynomials need to be checked is independent of the order of the polynomial family. An important extension of Kharitonov’s theorem is the edge theorem discovered by Bartlett, Hollot and Huang. The edge theorem states that the stability of a polytope of polynomials can be guaranteed by the stability of its one-dimensional exposed edge polynomials. The significance of the edge theorem is that it allows some dependency among polynomial coefficients, and applies to more general stability regions, e.g., unit circle, left sector, shifted half plane, hyperbola region, etc. When the dependency among polynomial coefficients is nonlinear, however, Ackerman(1997) shows that checking a subset of a polynomial family generally can not guarantee the stability of the entire family. In this paper, we study the frequency response of uncertain systems using Kharitonov’s stability theory on first order complex polynomial set.

5.3.2 Modern Theory of Robust Control

In mathematics, stability theory addresses the stability of solutions of differential equations and of trajectories of dynamical systems under small perturbations of initial conditions. Informally, a controller designed for a particular set of parameters is said to be robust if it would also work well under a different set of assumptions. High-gain feedback is a simple example
of a robust control method; with sufficiently high gain, the effect of any parameter variations will be negligible.

Robustness again disturbances and model uncertainty is at the heart of control practice. Indeed, in the case where both all external disturbances and a model of the system to be controlled are exactly known, there is no need for feedback. Optimal performance can be achieved with an open loop controller. For many real processes a controller design has to cope with the effect of uncertainties, which very often cause a poor performance or even instability of closed-loop systems. The reason for that is a perpetual time change of parameters (due to aging, influence of environment, working point changes etc.), as well as un modeled dynamics. The former uncertainty type is denoted as the parametric uncertainty and the latter one the dynamic uncertainty. A controller ensuring closed-loop stability under both of these uncertainty types is called a robust controller.

A problem of great interest in control theory is the design of controller which can guarantee some level of performance in the presence of plant parameter uncertainty. Kharitonov’s theorem provides a necessary and sufficient analysis test for determining the robust stability of polynomials with perturbed coefficients, however, there are few results that exploit kharitonov’s theorem for synthesizing robust controllers.

5.4 PERFORMANCE ANALYSIS OF THE SYSTEM

Analysis of control system with parametric uncertainties has attracted much attention in recent years. Hence for any closed loop system, it is necessary to analyze the stability and robustness for uncertainties in the process. The two classical measures of stability margin are gain margin and phase margin were described be Doyle et al (1992). A control system is said to be robust if the close loop system is stable even when the model parameters
of the actual process are different than that used for controller design. The term robust is commonly used to indicate a closed loop system which is stable and meets the performance specifications even though the model parameter varies.

The feedback system is internally stable with plant P and the controller C. The plant is perturbed to $k_p$, with $k$ is a positive real number. The upper gain ($k_{\text{max}}$) is the first number such that the internal stability holds for $1 \leq k < k_{\text{max}}$. If there is no such number, then $k_{\text{max}} = \infty$. Similarly, the lower gain $k_{\text{min}}$, is the least non negative number such that internal stability hold for $k_{\text{min}} < k \leq 1$. The phase margin should be $0 \leq \phi < \phi_{\max}$. For a given process and controller the stability and robustness of the closed loop system is obtained by phase margin and gain margins. If either of them is small, the system is close to instability.

The extend to which the system withstands the uncertainty in model parameters is obtained by using Kharitonov’s theorem. According to Kharitonov’s theorem, every polynomial in the family is stable if and only if the following four great polynomials are stable. To compare the robustness of the different controller design methods, the range of uncertainty in each of the model parameters for which the controller is stable is to be calculated. The robustness of the closed loop system for the perturbation separately in time delay, time constant and process gain is analysed theoretically by kharitonov’s method (Sinha 1994). In this method, the stability of the four equations are formed from Kharitonov’s polynomials are to be checked. The characteristic equation of the system using second order pade’s approximation for delay is,

$$P(s) = a_0 + a_1s + a_2s^2 + a_3s^3 + a_4s^4$$  (5.1)
where

\[ a_0 = k_\varepsilon k_p \]

\[ a_i = k_\varepsilon k_p (\tau_i - 0.5\tau_d) - \tau_i \]

\[ a_2 = k_\varepsilon k_p (0.0833\tau^2_d - 0.5\tau_d\tau_i + \tau_D\tau_i) - \tau_i - 0.5\tau_d\tau_i \]

\[ a_3 = k_\varepsilon k_p (0.0833\tau_i\tau^2_d - 0.5\tau_d\tau_i\tau_D) + 0.5\tau_d\tau_i - 0.0833\tau^2_d\tau_i \]

\[ a_4 = 0.0833k_\varepsilon k_p \tau_i\tau_D\tau^2_d + 0.08335\tau_i\tau^2_d \]

Kharitonov’s equations for \( a_i^- \leq a_i^+ \) (\( i = 0,1,2,3,4 \)) are given below, where \( a_i^- \) and \( a_i^+ \) are the lower and upper bound for \( a_i \), respectively:

\[ a_0^+ + a_1^+ s + a_2^- s^2 + a_3^- s^3 + a_4^+ s^4 = 0 \] (5.2)

\[ a_0^- + a_1^- s + a_2^+ s^2 + a_3^+ s^3 + a_4^- s^4 = 0 \] (5.3)

\[ a_0^- + a_1^- s + a_2^- s^2 + a_3^+ s^3 + a_4^- s^4 = 0 \] (5.4)

\[ a_0^+ + a_1^+ s + a_2^- s^2 + a_3^- s^3 + a_4^+ s^4 = 0 \] (5.5)

For fixed values of \( k_p \) and \( \tau \), a perturbation in time delay \( (\tau_d - \Delta\tau_d) \leq \tau_d \leq (\tau_d + \Delta\tau_d) \) i.e., when \( (\tau_d - \Delta\tau_d) \leq \tau_d \leq (\tau_d + \Delta\tau_d) \) is substituted in the coefficient of equations (5.2) to (5.4) and Kharitonov’s equations are checked for stability using Routh–Hurwitz method (Kharitonov 2003; Sinha 1994).
Table 5.1  ISE, IAE values for FOPTD1 system under parameter uncertainty using PID controller

<table>
<thead>
<tr>
<th>Method</th>
<th>ISE values for uncertainty in</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>+20% kp</td>
</tr>
<tr>
<td>FLC</td>
<td>5.89</td>
</tr>
<tr>
<td>DS</td>
<td>6.00</td>
</tr>
<tr>
<td>Dual Loop PID</td>
<td>6.01</td>
</tr>
<tr>
<td>MRC</td>
<td>6.25</td>
</tr>
<tr>
<td>IMC</td>
<td>21.8</td>
</tr>
<tr>
<td>ZN</td>
<td>9.29</td>
</tr>
</tbody>
</table>

Table 5.2  Stability region for k_p, τ and τ_d for the process with transfer function having (k_p = 3.1818; τ=1.0659; τ_d =0.673 )

<table>
<thead>
<tr>
<th>Process Parameter</th>
<th>FLC</th>
<th>DS Method</th>
<th>Dual PID</th>
<th>MRC</th>
<th>IMC</th>
<th>ZN</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kp</td>
<td>±24</td>
<td>±0.23</td>
<td>±0.25</td>
<td>±0.21</td>
<td>±0.13</td>
<td>±0.12</td>
</tr>
<tr>
<td>τ</td>
<td>±26</td>
<td>±0.24</td>
<td>±0.22</td>
<td>±0.19</td>
<td>±0.09</td>
<td>±0.10</td>
</tr>
<tr>
<td>τ_d</td>
<td>±30</td>
<td>±0.31</td>
<td>±0.30</td>
<td>±0.29</td>
<td>±0.10</td>
<td>±0.06</td>
</tr>
</tbody>
</table>

Similarly, perturbation in k_p (for fixed τ and τ_d) and τ (k_p and for fixed τ_d) is evaluated and the stability ranges are listed in Tables 5.1 and 5.2. The stability region for the controller designed by the FLC, DS and Dual loop PID is found to be more than the controllers designed by IMC and other methods. Thus, the controller designed by the Dual loop and DS method is more robust than the other methods.
5.5 SUMMARY

The robustness of the controller due to parametric uncertainties has been evaluated by using Kharitonov’s method. It has been demonstrated that Kharitonov’s polynomial can be effectively used for quantifying the robustness level of the controller. The FLC, DS, dual, MRC, ZN and IMC tuning methods have been implemented on transfer function of the position control of the web guide system and a comparison of the control performance using these methods has been simulated. An analysis of the performance characteristics for all the control loops shows that the FLC method outperforms all the tuning techniques.