CHAPTER EIGHT

A NOTE ON ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES AND THEIR EXPONENT OF CONVERGENCE OF ZEROS
Chapter 8

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8.1 Introductory remarks.

Let $f$ be an entire function of one complex variable in the finite complex plane with zeros $z_1, z_2, z_3, \ldots$ such that $0 < |z_1| \leq |z_2| \leq |z_3| \cdots \text{ and } |z_k| = r_k \to \infty$ as $k \to \infty$. Then the exponent of convergence of the zeros of $f$ denoted by $\rho_1$ is defined as

$$\rho_1 = \inf \left\{ \alpha > 0 : \sum_{n} \frac{1}{(r_n)^\alpha} \text{ is convergent} \right\}.$$ 

Lahiri and Banerjee (cf. [57]) proved the existence of a continuum number of entire functions of single complex variable each having an exponent of convergence of its zeros. Datta and Jha (cf. [17]) extended the theorem to $(p,q)$-th exponent of convergence of zeros of entire functions where $p$ and $q$ are any two positive integers with $p > q$. Extending the idea of

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entire functions of one complex variable to several complex variables, in this chapter we would like to revisit the result of Datta and Jha (2008) for entire functions of several complex variables. The existing literature, relevent definitions and related examples have already been discussed in Chapter One and Chapter Two respectively.

8.2 Lemma.

In this section we present a lemma which will be needed in the sequel.

**Lemma 8.2.1** [66] Let \( \{g_n\} \) be a non decreasing sequence of real numbers with \( g_n \to \infty \) as \( n \to \infty \). Let \( b = \limsup_{n \to \infty} \frac{\log n}{g_n} > 0 \) (\( b \) may be \( +\infty \)) and let \( 0 < t < b \). Then there exists a subsequence \( \{g_{j_n}\} \) of \( \{g_n\} \) such that

\[
\limsup_{n \to \infty} \frac{\log n}{g_{j_n}} = t.
\]

8.3 Theorem.

In this section we present the main result of the chapter.

**Theorem 8.3.1** Let \( f \) be an entire function of \( n(\geq 2) \) complex variables with zeros \( z^{(1)}, z^{(2)}, z^{(3)}, \cdots \) such that \( 0 < |z^{(1)}| \leq |z^{(2)}| \leq |z^{(3)}| \cdots \) and \( |z^{(k)}| = r^{(k)}_n \to \infty \) as \( k \to \infty \). Let \( \rho^s_1(p,q) > 0 \) be the \( (p,q) \)-th exponent of convergence of the zeros of \( f \) where \( p \) and \( q \) are any two positive integers with \( p > q \). If \( 0 \leq \beta < \rho^s_1(p,q) \), then there exists a continuum number of entire functions of \( n \) complex variables each having \( \beta \) as the \( (p,q) \)-th exponent of convergence of its zeros.

**Proof.** We first show that there exists an entire function which has \( \beta \) as the \( (p,q) \)-th exponent of convergence of its zeros.

**Case I.** \( \beta = 0 \).

Since \( r^{(k)}_n \to \infty \) as \( k \to \infty \), there exists a sequence \( \{k_m\} \) of positive integers with \( k_1 < k_2 < k_3 \cdots \), such that

\[
r^{(k_m)}_n > \exp^{[q]} \left( k \log^{[p]} k \right) \text{ for } k = 1, 2, 3, \cdots
\]
i.e.,
\[
\log[q] r^{(k_m)} > k \log[p] k \text{ for } k = 1, 2, 3, \ldots
\]
which gives that
\[
\limsup_{k \to \infty} \frac{\log[p] k}{\log[q] r^{(k_m)}} = 0.
\]
Let \( z^{(k_m)} \) be a point in the complex plane such that \( |z^{(k_m)}| = r^{(k_m)} \) for \( m = 1, 2, 3, \ldots \).
So in view of the extended notion of Weierstrass theorem, it follows that there exists an entire function \( \psi(z) \) of \( n \) complex variables which has zeros only at the points \( z^{(k_m)} \) for \( k = 1, 2, 3, \ldots \) and therefore \( \psi(z) \) is the desired function.

**Case II.** \( 0 < \beta < \rho^p_q(p, q) \).
Let us choose \( h_k = \log[q] r^{(k)} \) and \( s = \log[p-1] k \). Then the sequence \( \{h_k\} \) satisfies the conditions of extended notion of Lemma 8.1.1 involving several complex variables.
Hence there exists a subsequence \( \{h_{k_n}\} \) of \( \{h_k\} \) such that
\[
\beta = \limsup_{k \to \infty} \frac{\log s}{h_{k_m}}.
\]
Thus there exists an entire function \( \varphi(z) \) which serves the purpose.

Now, it is easy to verify that the set
\[
\left\{ z^{(k_m)} : z^{(k_m)} \in \mathbb{C}^n \& |z^{(k_m)}| = r^{(k_m)} \to \infty \text{ as } k \to \infty \right\}
\]
has the cardinality \( c \). Also we know that if an entire function \( \varphi(z) \) has zeros at \( z^{(k_m)} \) for \( m = 1, 2, 3, \ldots \) then for any entire function \( \psi(z) \), the function of the form \( e^{\psi(z)} \varphi(z) \) is an entire function. Thus we obtain a family of entire functions having the cardinality \( c \), where each member of the family has \( \beta \) as the \((p, q)\)-th exponent of convergence of its zeros where \( p, q \) are positive integers and \( p > q \). This proves the theorem. \( \blacksquare \)

For the particular values of \( p \) and \( q \), we have the following two corollaries as deduced from Theorem 8.3.1.

For \( p = 2 \) and \( q = 1 \),
Corollary 8.3.1  Let $f$ be an entire function of $n \ (\geq 2)$ complex variables with zeros $z^{(1)}, z^{(2)}, z^{(3)}, \cdots$ such that $0 < |z^{(1)}| \leq |z^{(2)}| \leq |z^{(3)}| \cdots$ and $|z^{(k)}| = r_n^{(k)} \to \infty$ as $k \to \infty$. Let $\rho_1^s (2, 1) > 0$ be the hyper exponent of convergence of the zeros of $f$. If $0 \leq \gamma < \rho_1^s (2, 1)$, then there exists a continuum number of entire functions of $n$ complex variables each having $\gamma$ as the hyper exponent of convergence of its zeros.

Remark 8.3.1  In fact Corollary 8.3.1 is an extension of Theorem 2 of [17].

For $p=1$ and $q=1$,

Corollary 8.3.2  Let $f$ be an entire function with zeros $z_1, z_2, z_3, \cdots$ such that $0 < |z_1| \leq |z_2| \leq |z_3| \cdots$ and $|z_n| = r_n \to \infty$ as $n \to \infty$. Let $\rho_1 (2, 1) > 0$ be the hyper exponent of convergence of the zeros of $f$. If $0 \leq \gamma \leq \rho_1 (2, 1)$, then there exists a continuum number of entire functions each having $\gamma$ as the hyper exponent of convergence of its zeros.

Remark 8.3.2  Actually Corollary 8.3.2 extends the notion as employed in Theorem 1 of [57].

Remark 8.3.3  The following example ensures Theorem 8.3.1 for $n=1$.

Example 8.3.1  Let $f(z) = \sin z$.

Then the exponent of convergence of zeros of $f$ is equal to 1.

Let $\alpha$ be a real number such that $0 \leq \alpha \leq 1$.

Now we show that there exists an entire function having exponent of convergence of zeros as $\alpha$.

Let $g(z) = \sin (z^\alpha)$. The zeros of $g$ are $0, \pi^{1/\alpha}, (2\pi)^{1/\alpha}, \ldots$.

Therefore the exponent of convergence of zeros of $g$

$$\rho_1(g) = \limsup_{n \to \infty} \frac{\log n}{\log r_n}$$

$$= \limsup_{n \to \infty} \frac{\log n}{\log (n\pi)^{1/\alpha}}$$

$$= \alpha.$$