CHAPTER SEVEN

ON RELATIVE ORDER-ORIENTED RESULTS OF ENTIRE FUNCTIONS OF TWO COMPLEX VARIABLES
Chapter 7

ON RELATIVE ORDER-ORIENTED RESULTS OF ENTIRE FUNCTIONS OF TWO COMPLEX VARIABLES

7.1 Introductory remarks.

Extending the idea of entire functions of one complex variable, we intend to establish some growth properties of entire functions of two complex variables in this chapter.

Let \( f \) be an entire function of two complex variables holomorphic in the closed polydisc

\[ U = \{(z_1, z_2) : \ |z_i| \leq r_i, \ i = 1, 2 \text{ for all } r_1 \geq 0, r_2 \geq 0\} \]

and \( M_f(r_1, r_2) = \max \{|f(z_1, z_2)| : |z_i| \leq r_i, \ i = 1, 2\} \). Then in view of maximum modulus principal and Hartogs’s theorem \([41], \text{p.21, p.51}\),

\( M_f(r_1, r_2) \) is an increasing function of \( r_1, r_2 \) and for given two entire functions \( f \) and \( g \) of two complex variables, the ratio \( \frac{M_f(r_1, r_2)}{M_g(r_1, r_2)} \) as \( r_1, r_2 \to \infty \) is called the growth of \( f \) with respect to \( g \).

In connection to the above discussion, Banerjee and Dutta \([10]\) proved the following results:

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Some results of this chapter have been accepted for publication and to appear in the International Journal of Pure and Applied Mathematics, see \([36]\) and the remaining portions are communicated, see \([38]\) and \([39]\).
**Theorem 7.1.1** [10] Let $f_1$ and $f_2$ be any two entire functions of two complex variables having order $v_2 \rho_g (f_1)$ and $v_2 \rho_g (f_2)$ respectively where $g$ is also an entire function of two complex variables. Then

$$v_2 \rho_g (f_1 \pm f_2) \leq \max \{v_2 \rho_g (f_1), v_2 \rho_g (f_2)\}.$$  

The equality holds when $v_2 \rho_g (f_1) \neq v_2 \rho_g (f_2)$.

**Theorem 7.1.2** [10] Let $f_1$ and $f_2$ be any two entire functions of two complex variables having order $v_2 \rho_g (f_1)$ and $v_2 \rho_g (f_2)$ respectively where $g$ is also an entire function of two complex variables satisfying Property (A). Then

$$v_2 \rho_g (f_1 f_2) \leq \max \{v_2 \rho_g (f_1), v_2 \rho_g (f_2)\}.$$  

But they remained silent about the reverse inequality of Theorem 7.1.2. In this chapter we wish to establish the condition of equality of Theorem 7.1.2. Further we prove that the similar results of Theorem 7.1.2 holds for the quotient $\frac{f_1}{f_2}$. Similarly, in case of relative lower order, it therefore seems reasonable to study a parallel investigations on its basic properties, which is also the prime concern of this chapter. The existing literature, relevant definitions and related examples have already been discussed in Chapter One and Chapter Two respectively.

### 7.2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

**Lemma 7.2.1** [10] Suppose $f$ be an entire function of two complex variables, $\alpha > 1, 0 < \beta < \alpha, s > 1, 0 < \mu < \lambda$ and $n$ is a positive integer. Then

(a) $M_f (\alpha r_1, \alpha r_2) > \beta M_f (r_1, r_2)$,
(b) There exists $K = K(s, f) > 0$ such that $(M_f (r_1, r_2))^s \leq KM_f (r_1^s, r_2^s)$ for $r > 0$

and (c) $\lim_{r_1, r_2 \to \infty} \frac{M_f (r_1^s, r_2^s)}{M_f (r_1, r_2)} = \infty = \lim_{r_1, r_2 \to \infty} \frac{M_f (r_1, r_2)}{M_f (r_1^\mu, r_2^\mu)}$.

**Lemma 7.2.2** [10] Let $f$ be an entire function satisfying the Property (A). Then for any positive integer $n$ and for all sufficiently large $r_1, r_2$,

$$[M_f (r_1, r_2)]^n \leq M_f (r_1^\sigma, r_2^\sigma)$$
Lemma 7.2.3 Let $f, g$ and $h$ be any three entire functions of two complex variables. Then for $M_g(r_1, r_2) \leq M_h(r_1, r_2)$ and all sufficiently large values of $r_1, r_2$,
\[ v_2 \lambda_h(f) \leq v_2 \lambda_g(f), \]
where $l \geq 1$.

**Proof.** As $M_g(r_1, r_2) \leq M_h(r_1, r_2)$ and $M_f(r_1, r_2)$ is an increasing function of $r_1, r_2$ we get for all sufficiently large values of $r_1, r_2$ that
\[
M_h^{-1}(r_1, r_2) \leq M_g^{-1}(r_1, r_2)
\]
i.e., $M_h^{-1}M_f(r_1, r_2) \leq M_g^{-1}M_f(r_1, r_2)$
i.e., \[
\liminf_{r \to \infty} \frac{\log M_h^{-1}M_f(r_1, r_2)}{\log (r_1r_2)} \leq \liminf_{r \to \infty} \frac{\log M_g^{-1}M_f(r_1, r_2)}{\log (r_1r_2)}
\]
i.e., $v_2 \lambda_h(f) \leq v_2 \lambda_g(f)$.

This proves the lemma. 

### 7.3 Theorems.

In this section we present the main results of the chapter.

**Theorem 7.3.1** Let $f, g$ and $h$ be any three entire functions of two complex variables such that $v_2 \rho_h(f) < \infty$ and $v_2 \lambda_h(f \circ g) = \infty$. Then
\[
\lim_{r_1, r_2 \to \infty} \frac{\log M_h^{-1}M_f \circ g(r_1, r_2)}{\log M_h^{-1}M_f(r_1, r_2)} = \infty.
\]

**Proof.** Let us suppose that the conclusion of the theorem do not hold. Then we can find a constant $\beta > 0$ such that for a sequence of values of $r_1, r_2$ tending to infinity,
\[
\log M_h^{-1}M_f \circ g(r_1, r_2) \leq \beta \log M_h^{-1}M_f(r_1, r_2).
\]
Again from the definition of $v_2 \rho_h(f)$, it follows for all sufficiently large values of $r_1, r_2$ that
\[
\log M_h^{-1}M_f(r_1, r_2) \leq (v_2 \rho_h(f) + \epsilon) \log (r_1r_2).
\]
Thus from (7.1) and (7.2), we have for a sequence of values of \( r_1, r_2 \) tending to infinity that
\[
\log M^{-1}_h M_{fog}(r_1, r_2) \leq \beta (v_2 \rho_h (f) + \epsilon) \log (r_1 r_2)
\]
i.e.,
\[
\frac{\log M^{-1}_h M_{fog}(r_1, r_2)}{\log (r_1 r_2)} \leq \frac{\beta (v_2 \rho_h (f) + \epsilon) \log (r_1 r_2)}{\log (r_1 r_2)}
\]
i.e.,
\[
\liminf_{r_1, r_2 \to \infty} \frac{\log M^{-1}_h M_{fog}(r_1, r_2)}{\log (r_1 r_2)} = v_2 \lambda_{fog} < \infty.
\]
This is a contradiction.
Thus the theorem follows. ■

**Remark 7.3.1** Theorem 7.3.1 is also valid with “limit superior” instead of “limit” if \( v_2 \lambda_{fog} = \infty \) is replaced by \( v_2 \rho_{fog} = \infty \) and the other conditions remain the same.

**Corollary 7.3.1** Under the assumptions of Theorem 7.3.1 and Remark 7.3.1,
\[
\lim_{r_1, r_2 \to \infty} \frac{M^{-1}_h M_{fog}(r_1, r_2)}{M^{-1}_h M_f(r_1, r_2)} = \infty \quad \text{and} \quad \limsup_{r_1, r_2 \to \infty} \frac{M^{-1}_h M_{fog}(r_1, r_2)}{M^{-1}_h M_f(r_1, r_2)} = \infty
\]
respectively hold.

The proof is omitted.

Analogously, one may also state the following theorem and corollaries without their proofs as those may be carried out in the line of Remark 7.3.1, Theorem 7.3.1 and Corollary 7.3.1 respectively.

**Theorem 7.3.2** Let \( f, h \) be any two analytic functions of \( n \) complex variables and \( g \) be entire of \( n \) complex variables in \( U \) such that \( v_2 \rho_h (g) < \infty \) and \( v_2 \rho_h (f \circ g) = \infty \). Then
\[
\limsup_{r_1, r_2 \to \infty} \frac{\log M^{-1}_h M_{fog}(r_1, r_2)}{\log M^{-1}_h M_g(r_1, r_2)} = \infty.
\]

**Remark 7.3.2** Theorem 7.3.2 is also valid with “limit” instead of “limit superior” if \( v_2 \rho_h (f \circ g) = \infty \) is replaced by \( v_2 \lambda_h (f \circ g) = \infty \) and the other conditions remain the same.
Corollary 7.3.2 Under the assumptions of Theorem 7.3.2 and Remark 7.3.2,
\[
\limsup_{r_1,r_2 \to \infty} \frac{M^{-1}_h M_{f g}(r_1, r_2)}{M^{-1}_h M_g(r_1, r_2)} = \infty \quad \text{and} \quad \lim_{r_1,r_2 \to \infty} \frac{M^{-1}_h M_{f g}(r_1, r_2)}{M^{-1}_h M_g(r_1, r_2)} = \infty
\]
respectively hold.

Theorem 7.3.3 Let \(f\) and \(g\) be any two entire functions of two complex variables. Then
\[
\frac{v_2 \lambda_f}{v_2 \rho_g} \leq v_2 \lambda_g(f) \leq \min \left\{ \frac{v_2 \lambda_f}{v_2 \rho_g}, \frac{v_2 \rho_f}{v_2 \lambda_g} \right\} \leq \max \left\{ \frac{v_2 \lambda_f}{v_2 \rho_g}, \frac{v_2 \rho_f}{v_2 \lambda_g} \right\} \leq v_2 \rho_g(f) \leq \frac{v_2 \rho_f}{v_2 \lambda_g}.
\]

Proof. From definitions of \(v_2 \rho_f\) and \(v_2 \lambda_f\) we have, for all sufficiently large values of \(r_1, r_2\) that
\[
M_f(r_1, r_2) \leq \exp \left\{ (v_2 \rho_f + \varepsilon) \log (r_1 r_2) \right\}, \quad (7.3)
\]
\[
M_f(r_1, r_2) \geq \exp \left\{ (v_2 \lambda_f - \varepsilon) \log (r_1 r_2) \right\} \quad (7.4)
\]
and also for a sequence of values of \(r_1, r_2\) tending to infinity we get that
\[
M_f(r_1, r_2) \geq \exp \left\{ (v_2 \rho_f - \varepsilon) \log (r_1 r_2) \right\}, \quad (7.5)
\]
\[
M_f(r_1, r_2) \leq \exp \left\{ (v_2 \lambda_f + \varepsilon) \log (r_1 r_2) \right\}. \quad (7.6)
\]
Similarly from the definitions of \(v_2 \rho_g\) and \(v_2 \lambda_g\), it follows for all sufficiently large values of \(r_1, r_2\) that
\[
M_g(r_1, r_2) \leq \exp \left\{ (v_2 \rho_g + \varepsilon) \log (r_1 r_2) \right\}
\]
\(i.e.,\) \(M^{-1}_g \exp \left\{ (v_2 \rho_g + \varepsilon) \log (r_1 r_2) \right\} \]
i.e., \(M^{-1}_g(r_1, r_2) \geq \exp \left[ \frac{\log (r_1 r_2)}{(v_2 \rho_g + \varepsilon)} \right] \quad (7.7)
\[
M_g(r_1, r_2) \geq \exp \left\{ (v_2 \lambda_g - \varepsilon) \log (r_1 r_2) \right\}
\]
i.e., \(M^{-1}_g \exp \left\{ (v_2 \lambda_g - \varepsilon) \log (r_1 r_2) \right\} \]
i.e., \(M^{-1}_g(r_1, r_2) \leq \exp \left[ \frac{\log (r_1 r_2)}{(v_2 \lambda_g - \varepsilon)} \right] \quad (7.8)\]
and for a sequence of values of \( r_1, r_2 \) tending to infinity we obtain that
\[
M_g(r_1, r_2) \geq \exp \left\{ (v_2 \rho_g - \varepsilon) \log (r_1 r_2) \right\}
\]
i.e., \( (r_1 r_2) \geq M_g^{-1} \left[ \exp \left\{ (v_2 \rho_g - \varepsilon) \log (r_1 r_2) \right\} \right] \)
i.e., \( M_g^{-1}(r_1, r_2) \leq \exp \left[ \frac{\log (r_1 r_2)}{(v_2 \rho_g - \varepsilon)} \right] \),
(7.9)

\[
M_g(r_1, r_2) \leq \exp \left\{ (v_2 \lambda_g + \varepsilon) \log (r_1 r_2) \right\}
\]
i.e., \( (r_1 r_2) \leq M_g^{-1} \left[ \exp \left\{ (v_2 \lambda_g + \varepsilon) \log (r_1 r_2) \right\} \right] \)
i.e., \( M_g^{-1}(r_1, r_2) \geq \exp \left[ \frac{\log (r_1 r_2)}{(v_2 \lambda_g + \varepsilon)} \right] \).
(7.10)

Now from (7.5) and in view of (7.7), we get for a sequence of values of \( r_1, r_2 \) tending to infinity that
\[
\log M_g^{-1} M_f(r_1, r_2) \geq \log M_g^{-1} \left[ \exp \left\{ (v_2 \rho_f - \varepsilon) \log (r_1 r_2) \right\} \right]
\]
i.e., \( \log M_g^{-1} M_f(r_1, r_2) \geq \log \exp \left[ \frac{\log \left\{ (v_2 \rho_f - \varepsilon) \log (r_1 r_2) \right\}}{(v_2 \rho_g + \varepsilon)} \right] \)
i.e., \( \log M_g^{-1} M_f(r_1, r_2) \geq \frac{(v_2 \rho_f - \varepsilon)}{(v_2 \rho_g + \varepsilon)} \log (r_1 r_2) \)
i.e., \( \frac{\log M_g^{-1} M_f(r_1, r_2)}{\log (r_1 r_2)} \geq \frac{(v_2 \rho_f - \varepsilon)}{(v_2 \rho_g + \varepsilon)}. \)

As \( \varepsilon (> 0) \) is arbitrary, it follows that
\[
\limsup_{r_1, r_2 \to \infty} \frac{\log M_g^{-1} M_f(r_1, r_2)}{\log (r_1 r_2)} \geq \frac{v_2 \rho_f}{v_2 \rho_g}
\]
i.e., \( \rho_{g(p,q)}(f) \geq \frac{v_2 \rho_f}{v_2 \rho_g}. \)
(7.11)

Analogously from (7.4) and in view of (7.10), it follows for a sequence of values of \( r_1, r_2 \) tending to infinity that
\[
\log M_g^{-1} M_f(r_1, r_2) \geq \log M_g^{-1} \left[ \exp \left\{ (v_2 \lambda_f - \varepsilon) \log (r_1 r_2) \right\} \right]
\]
i.e., \( \log M_g^{-1} M_f(r_1, r_2) \geq \log \exp \left[ \frac{\log \left\{ (v_2 \lambda_f - \varepsilon) \log (r_1 r_2) \right\}}{(v_2 \lambda_g + \varepsilon)} \right] \)
\[
\begin{align*}
\text{i.e., } \log M_g^{-1} M_f (r_1, r_2) & \geq \frac{(v_2 \lambda_f - \varepsilon)}{(v_2 \lambda_g + \varepsilon)} \log (r_1 r_2) \\
\text{i.e., } \frac{\log M_g^{-1} M_f (r_1, r_2)}{\log (r_1 r_2)} & \geq \frac{(v_2 \lambda_f - \varepsilon)}{(v_2 \lambda_g + \varepsilon)}.
\end{align*}
\]

Since \(\varepsilon (>0)\) is arbitrary, we get from above that
\[
\limsup_{r_1, r_2 \to \infty} \frac{\log M_g^{-1} M_f (r_1, r_2)}{\log (r_1 r_2)} \geq \frac{v_2 \lambda_f}{v_2 \lambda_g}.
\]

\text{i.e., } \rho_g (f) \geq \frac{v_2 \lambda_f}{v_2 \lambda_g}.

(7.12)

Again in view of (7.8), we have from (7.3) for all sufficiently large values of \(r_1, r_2\) that
\[
\begin{align*}
\log M_g^{-1} M_f (r_1, r_2) & \leq \log M_g^{-1} \left[ \exp \left\{ (v_2 \rho_f + \varepsilon) \log (r_1 r_2) \right\} \right] \\
\text{i.e., } \log M_g^{-1} M_f (r_1, r_2) & \leq \log \exp \left[ \frac{\log \exp \left\{ (v_2 \rho_f + \varepsilon) \log (r_1 r_2) \right\}}{(v_2 \lambda_f - \varepsilon)} \right]
\end{align*}
\]

\begin{align*}
\text{i.e., } \log M_g^{-1} M_f (r_1, r_2) & \leq \frac{(v_2 \rho_f + \varepsilon)}{(v_2 \lambda_f - \varepsilon)} \log (r_1 r_2) \\
\text{i.e., } \frac{\log M_g^{-1} M_f (r_1, r_2)}{\log (r_1 r_2)} & \leq \frac{(v_2 \rho_f + \varepsilon)}{(v_2 \lambda_f - \varepsilon)}.
\end{align*}

Since \(\varepsilon (>0)\) is arbitrary, we obtain that
\[
\begin{align*}
\limsup_{r_1, r_2 \to \infty} \frac{\log M_g^{-1} M_f (r_1, r_2)}{\log (r_1 r_2)} & \leq \frac{v_2 \rho_f}{v_2 \lambda_g} \\
\text{i.e., } \rho_g (f) & \leq \frac{v_2 \rho_f}{v_2 \lambda_g}.
\end{align*}
\]

(7.13)

Again from (7.4) and in view of (7.7), we get for all sufficiently large values of \(r_1, r_2\) that
\[
\begin{align*}
\log M_g^{-1} M_f (r_1, r_2) & \geq \log M_g^{-1} \left[ \exp \left\{ (v_2 \lambda_f - \varepsilon) \log (r_1 r_2) \right\} \right] \\
\text{i.e., } \log M_g^{-1} M_f (r_1, r_2) & \geq \log \exp \left[ \frac{\log \exp \left\{ (v_2 \lambda_f - \varepsilon) \log (r_1 r_2) \right\}}{(v_2 \rho_g + \varepsilon)} \right]
\end{align*}
\]
\[ i.e., \quad \log M_g^{-1}M_f(r_1, r_2) \geq \frac{(v_2 \lambda_f - \varepsilon)}{(v_2 \rho_g + \varepsilon)} \log (r_1 r_2) \]

\[ i.e., \quad \frac{\log M_g^{-1}M_f(r_1, r_2)}{\log (r_1 r_2)} \geq \frac{(v_2 \lambda_f - \varepsilon)}{(v_2 \rho_g + \varepsilon)}. \]

As \( \varepsilon (> 0) \) is arbitrary, it follows from above that

\[
\liminf_{r_1, r_2 \to \infty} \frac{\log M_g^{-1}M_f(r_1, r_2)}{\log (r_1 r_2)} \geq \frac{v_2 \lambda_f}{v_2 \rho_g},
\]

\[ i.e., \quad \lambda_g (f) \geq \frac{v_2 \lambda_f}{v_2 \rho_g}. \] (7.14)

Also in view of (7.9), we get from (7.3) for a sequence of values of \( r_1, r_2 \) tending to infinity that

\[
\log M_g^{-1}M_f(r_1, r_2) \leq \log M_g^{-1} \left[ \exp \{(v_2 \rho_f + \varepsilon) \log (r_1 r_2)\} \right]
\]

\[ i.e., \quad \log M_g^{-1}M_f(r_1, r_2) \leq \log \exp \left[ \frac{\log \exp \{(v_2 \rho_f + \varepsilon) \log (r_1 r_2)\}}{(v_2 \rho_g - \varepsilon)} \right] \]

\[ i.e., \quad \log M_g^{-1}M_f(r_1, r_2) \leq \frac{(v_2 \rho_f + \varepsilon)}{(v_2 \rho_g - \varepsilon)} \log (r_1 r_2) \]

\[ i.e., \quad \frac{\log M_g^{-1}M_f(r_1, r_2)}{\log (r_1 r_2)} \leq \frac{(v_2 \rho_f + \varepsilon)}{(v_2 \rho_g - \varepsilon)}. \]

Since \( \varepsilon (> 0) \) is arbitrary, we get from above that

\[
\liminf_{r_1, r_2 \to \infty} \frac{\log M_g^{-1}M_f(r_1, r_2)}{\log (r_1 r_2)} \leq \frac{v_2 \rho_f}{v_2 \rho_g},
\]

\[ i.e., \quad \lambda_g (f) \leq \frac{v_2 \rho_f}{v_2 \rho_g}. \] (7.15)

Similarly from (7.6) and in view of (7.8), it follows for a sequence of values of \( r_1, r_2 \) tending to infinity that

\[
\log M_g^{-1}M_f(r_1, r_2) \leq \log M_g^{-1} \left[ \exp \{(v_2 \lambda_f + \varepsilon) \log (r_1 r_2)\} \right]
\]

\[ i.e., \quad \log M_g^{-1}M_f(r_1, r_2) \leq \log \exp \left[ \frac{\log \exp \{(v_2 \lambda_f + \varepsilon) \log (r_1 r_2)\}}{(v_2 \lambda_g - \varepsilon)} \right]. \]
i.e., \( \log M_g^{-1} M_f (r_1, r_2) \leq \frac{(v_2 \lambda_f + \varepsilon)}{(v_2 \lambda_g - \varepsilon)} \log (r_1 r_2) \)

i.e., \( \frac{\log M_g^{-1} M_f (r_1, r_2)}{\log (r_1 r_2)} \leq \frac{(v_2 \lambda_f + \varepsilon)}{(v_2 \lambda_g - \varepsilon)} \).

As \( \varepsilon (> 0) \) is arbitrary, we obtain from above that

\[
\liminf_{r_1, r_2 \to \infty} \frac{\log M_g^{-1} M_f (r_1, r_2)}{\log (r_1 r_2)} \leq \frac{v_2 \lambda_f}{v_2 \lambda_g}.
\]

i.e., \( \lambda_g (f) \leq \frac{v_2 \lambda_f}{v_2 \lambda_g}. \) \( (7.16) \)

Thus the theorem follows from \( (7.11) \), \( (7.12) \), \( (7.13) \), \( (7.14) \), \( (7.15) \) and \( (7.16) \). \( \blacksquare \)

**Corollary 7.3.3** Let \( f \) and \( g \) be any two entire functions of two complex variables such that \( g \) is of regular growth. Then

\[
v_2 \lambda_g (f) = \frac{v_2 \lambda_f}{v_2 \rho_g} \quad \text{and} \quad v_2 \rho_g (f) = \frac{v_2 \rho_f}{v_2 \rho_g}.
\]

In addition, if \( \rho_f = \rho_g \) then

\[
v_2 \rho_g (f) = v_2 \lambda_f (g) = 1.
\]

**Corollary 7.3.4** Let \( f \) and \( g \) be any two entire functions of two complex variables with regular growth respectively. Then

\[
v_2 \lambda_g (f) = v_2 \rho_g (f) = \frac{v_2 \rho_f}{v_2 \rho_g}.
\]

**Corollary 7.3.5** Let \( f \) and \( g \) be any two entire functions of two complex variables with regular growth respectively. Also suppose that \( v_2 \rho_f = v_2 \rho_g \). Then

\[
v_2 \lambda_g (f) = v_2 \rho_g (f) = v_2 \lambda_f (g) = v_2 \rho_f (g) = 1.
\]

**Corollary 7.3.6** Let \( f \) and \( g \) be any two entire functions of two complex variables with regular growth respectively. Then

\[
v_2 \rho_g (f) \cdot v_2 \rho_f (g) = v_2 \lambda_g (f) \cdot v_2 \lambda_f (g) = 1.
\]
Corollary 7.3.7 Let \( f \) and \( g \) be any two entire functions of two complex variables such that either \( f \) is not of regular growth or \( g \) is not of regular growth. Then

\[
v_2 \lambda_g(f) \cdot v_2 \lambda_f(g) < 1 < v_2 \rho_g(f) \cdot v_2 \rho_f(g).
\]

Corollary 7.3.8 Let \( f \) and \( g \) be any two entire functions of two complex variables. Then

(i) \( v_2 \lambda_g(f) = \infty \) when \( v_2 \rho_g = 0 \),

(ii) \( v_2 \rho_g(f) = \infty \) when \( v_2 \lambda_g = 0 \),

(iii) \( v_2 \lambda_g(f) = 0 \) when \( v_2 \rho_g = \infty \)

and

(iv) \( v_2 \rho_g(f) = 0 \) when \( v_2 \lambda_g = \infty \).

Corollary 7.3.9 Let \( f \) and \( g \) be any two entire functions of two complex variables. Then

(i) \( v_2 \rho_g(f) = 0 \) when \( v_2 \rho_f = 0 \),

(ii) \( v_2 \lambda_g(f) = 0 \) when \( v_2 \lambda_f = 0 \),

(iii) \( v_2 \rho_g(f) = \infty \) when \( v_2 \rho_f = \infty \)

and

(iv) \( v_2 \lambda_g(f) = \infty \) when \( v_2 \lambda_f = \infty \).

Theorem 7.3.4 Let \( f_1 \) and \( f_2 \) be any two entire functions of two complex variables having orders \( v_2 \rho_g(f_1) \) and \( v_2 \rho_g(f_2) \) respectively where \( g \) is also an entire function of two complex variables having Property (A). Then

\[
v_2 \rho_g(f_1 \cdot f_2) = \max \{v_2 \rho_g(f_1), v_2 \rho_g(f_2)\}
\]

if \( v_2 \rho_g(f_1) \neq v_2 \rho_g(f_2) \).

Proof. Let us suppose that \( k, h, h_1 \) and \( h_2 \) be any four entire functions of two complex variables such that \( k \) has Property (A), \( h = \frac{h_1}{h_2} \) and \( v_2 \rho_k(h) \leq v_2 \rho_k(h_2) \). Now \( h_1 = h \cdot h_2 \) and in view of Theorem 7.1.2, we get that

\[
v_2 \rho_k(h_1) \leq \max \{v_2 \rho_k(h), v_2 \rho_k(h_2)\}
\]

i.e., \( v_2 \rho_k(h_1) \leq v_2 \rho_k(h_2) \).
Therefore from above and in view of the condition \( v_2 \rho_k (h) \leq v_2 \rho_k (h) \), it follows that

\[
v_2 \rho_k (h) = v_2 \rho_k \left( \frac{h_1}{h_2} \right) \leq \max \{ v_2 \rho_k (h_1), v_2 \rho_k (h_2) \} . \tag{7.17}
\]

Now let \( f = f_1 \cdot f_2 \) which implies that \( f_2 = \frac{f}{f_1} \). Also without loss of any generality, we may consider that \( v_2 \rho_g (f_1) < v_2 \rho_g (f_2) \). Therefore in view of Theorem 7.1.2, we have

\[
v_2 \rho_g (f) = v_2 \rho_g (f_1 \cdot f_2) \leq \max \{ v_2 \rho_g (f_1), v_2 \rho_g (f_2) \} = v_2 \rho_g (f_2) . \tag{7.18}
\]

As \( f_2 = \frac{f}{f_1} \), we obtain in view of (7.17) that

\[
v_2 \rho_g (f_2) = v_2 \rho_g \left( \frac{f}{f_1} \right) \leq \max \{ v_2 \rho_g (f), v_2 \rho_g (f_1) \} .
\]

Since \( v_2 \rho_g (f_1) < v_2 \rho_g (f_2) \), it follows from above that

\[
v_2 \rho_g (f_2) \leq v_2 \rho_g (f) . \tag{7.19}
\]

Thus the theorem follows from (7.18) and (7.19) \( \blacksquare \)

**Theorem 7.3.5** Let \( f_1 \) and \( f_2 \) be any two entire functions of two complex variables having orders \( v_2 \rho_g (f_1) \) and \( v_2 \rho_g (f_2) \) respectively where \( g \) is also an entire function of two complex variables having Property (A). Then

\[
v_2 \rho_g \left( \frac{f_1}{f_2} \right) \leq \max \{ v_2 \rho_g (f_1), v_2 \rho_g (f_2) \} .
\]

Equality holds if \( v_2 \rho_g (f_1) \neq v_2 \rho_g (f_2) \).

**Proof.** Suppose \( h = \frac{f_1}{f_2} \). Then \( f_1 = h \cdot f_2 \). Further let \( v_2 \rho_g (f_1) \leq v_2 \rho_g (f_2) \). Now if possible let \( v_2 \rho_g (h) > v_2 \rho_g (f_2) \). Therefore in view of Theorem 7.1.2 and Theorem 7.3.4, we obtain that \( v_2 \rho_g (f_1) = v_2 \rho_g (h) > v_2 \rho_g (f_2) \) which contradicts the hypothesis \( v_2 \rho_g (f_1) \leq v_2 \rho_g (f_2) \). So \( v_2 \rho_g (h) = v_2 \rho_g \left( \frac{f_1}{f_2} \right) \leq v_2 \rho_g (f_2) = \max \{ v_2 \rho_g (f_1), v_2 \rho_g (f_2) \} \).

Further, let \( v_2 \rho_g (f_1) < v_2 \rho_g (f_2) \). Also if possible, let \( v_2 \rho_g (h) < v_2 \rho_g (f_2) \).
Then \( v_2 \rho_g (f_1) = \max \{ v_2 \rho_g (h), v_2 \rho_g (f_2) \} = v_2 \rho_g (f_2) \) which is also a contradiction. Therefore \( v_2 \rho_g \left( \frac{f_1}{f_2} \right) = v_2 \rho_g (f_2) = \max \{ v_2 \rho_g (f_1), v_2 \rho_g (f_2) \} \). This proves the theorem.

Now we prove the following theorems which are based on the basic properties of relative lower order of entire functions of two complex variables:

**Theorem 7.3.6** If \( f_1, f_2, \ldots, f_n \ (n \geq 2) \) and \( g \) are entire functions of two complex variables, then

\[
v_2 \lambda_f (g) \geq v_2 \lambda_{f_i} (g),
\]

where \( f = f_1 \pm \sum_{k=2}^{n} f_k \) and \( v_2 \lambda_{f_i} (g) = \min \{ v_2 \lambda_{f_k} (g) \mid k = 1, 2, \ldots, n \} \). The sign of equality holds when \( v_2 \lambda_{f_i} (g) \neq \{ v_2 \lambda_{f_k} (g) \mid k = 1, 2, \ldots, n \text{ and } k \neq i \} \).

**Proof.** If \( v_2 \lambda_f (g) = \infty \) then the result is obvious. So we suppose that \( v_2 \lambda_f (g) < \infty \).

We can clearly assume that \( v_2 \lambda_{f_i} (g) \) is finite. Also suppose that \( v_2 \lambda_{f_i} (g) \leq v_2 \lambda_{f_k} (g) \) where \( k = 1, 2, \ldots, i, \ldots, n \).

Now for any arbitrary \( \varepsilon > 0 \), we get for all sufficiently large values of \( r_1, r_2 \) that

\[
M_{f_k} \left( r_1^{\frac{1}{v_2 \lambda_{f_k} (g) - \varepsilon}}, r_2^{\frac{1}{v_2 \lambda_{f_k} (g) - \varepsilon}} \right) < M_g (r_1, r_2) \quad \text{where } k = 1, 2, \ldots, n
\]

i.e., \( M_{f_k} (r_1, r_2) < M_g \left( r_1^{\frac{1}{v_2 \lambda_{f_k} (g) - \varepsilon}}, r_2^{\frac{1}{v_2 \lambda_{f_k} (g) - \varepsilon}} \right) \) where \( k = 1, 2, \ldots, n \)

i.e., \( M_{f_k} (r_1, r_2) \leq M_g \left( r_1^{\frac{1}{v_2 \lambda_{f_i} (g) - \varepsilon}}, r_2^{\frac{1}{v_2 \lambda_{f_i} (g) - \varepsilon}} \right) \) \tag{7.20}

where \( k = 1, 2, \ldots, n \).
Now for all sufficiently large values of $r_1, r_2$,

$$M_f (r_1, r_2) < \sum_{k=1}^{n} M_{f_k} (r_1, r_2)$$

i.e.,

$$M_f (r_1, r_2) < \sum_{k=1}^{n} M_g \left( \frac{1}{r_1^{(v_2 \lambda_{f_k} (g)-\varepsilon)}}, \frac{1}{r_2^{(v_2 \lambda_{f_k} (g)-\varepsilon)}} \right)$$

i.e.,

$$M_f (r_1, r_2) < nM_g \left( \frac{1}{r_1^{(v_2 \lambda_f (g)-\varepsilon)}}, \frac{1}{r_2^{(v_2 \lambda_f (g)-\varepsilon)}} \right). \quad (7.21)$$

Now in view of the first part of Lemma 7.2.1 we obtain from (7.21) for all sufficiently large values of $r_1, r_2$ that

$$M_f (r_1, r_2) < M_g \left( (n+1)r_1^{(v_2 \lambda_f (g)-\varepsilon)}, (n+1)r_2^{(v_2 \lambda_f (g)-\varepsilon)} \right)$$

i.e.,

$$M_f \left[ \left( \frac{r_1}{n+1} \right)^{(v_2 \lambda_f (g)-\varepsilon)} \left( \frac{r_2}{n+1} \right)^{(v_2 \lambda_f (g)-\varepsilon)} \right] < M_g (r_1, r_2)$$

i.e.,

$$\left( \frac{r_1}{n+1} \right)^{(v_2 \lambda_f (g)-\varepsilon)} \left( \frac{r_2}{n+1} \right)^{(v_2 \lambda_f (g)-\varepsilon)} < M_f^{-1} M_g (r_1, r_2)$$

i.e.,

$$(v_2 \lambda_f (g) - \varepsilon) \log \left( \frac{r_1}{n+1} \cdot \frac{r_2}{n+1} \right) < \log M_f^{-1} M_g (r_1, r_2)$$

i.e.,

$$(v_2 \lambda_f (g) - \varepsilon) < \frac{\log M_f^{-1} M_g (r_1, r_2)}{\log (r_1 r_2) + O(1)}$$

i.e.,

$$\frac{\log M_f^{-1} M_g (r_1, r_2)}{\log (r_1 r_2) + O(1)} > (v_2 \lambda_f (g) - \varepsilon).$$

So

$$v_2 \lambda_f (g) = \liminf_{r_1, r_2 \to \infty} \frac{\log M_f^{-1} M_g (r_1, r_2)}{\log (r_1 r_2) + O(1)} \geq v_2 \lambda_f (g) - \varepsilon.$$ 

Since $\varepsilon > 0$ is arbitrary,

$$v_2 \lambda_f (g) \geq v_2 \lambda_f (g). \quad (7.22)$$
Next let \( v_2 \lambda_{f_i}(g) < v_2 \lambda_{f_k}(g) \) where \( k = 1, 2, \ldots, n \) and \( k \neq i \).

As \( \varepsilon (>0) \) is arbitrary, from the definition of generalized lower order it follows for a sequence of values of \( r_1, r_2 \) tending to infinity that

\[
M_g(r_1, r_2) < M_{f_i}\left( r_1^{\frac{1}{\lambda_{f_i}(g)+\varepsilon}}, r_2^{\frac{1}{\lambda_{f_i}(g)+\varepsilon}} \right)
\]

i.e.,

\[
M_g\left( r_1^{\frac{1}{\lambda_{f_i}(g)+\varepsilon}}, r_2^{\frac{1}{\lambda_{f_i}(g)+\varepsilon}} \right) < M_{f_i}(r_1, r_2).
\] (7.23)

Since \( v_2 \lambda_{f_i}(g) < v_2 \lambda_{f_k}(g) \) where \( k = 1, 2, \ldots, n \) and \( k \neq i \), then in view of the third part of Lemma 3.2.1 we obtain that

\[
\lim_{r_1, r_2 \to \infty} \frac{M_g\left( r_1^{\frac{1}{\lambda_{f_i}(g)+\varepsilon}}, r_2^{\frac{1}{\lambda_{f_i}(g)+\varepsilon}} \right)}{M_g\left( r_1^{\frac{1}{\lambda_{f_k}(g)-\varepsilon}}, r_2^{\frac{1}{\lambda_{f_k}(g)-\varepsilon}} \right)} = \infty
\] (7.24)

where \( k = 1, 2, \ldots, n \) and \( k \neq i \).

Therefore from (7.24), we obtain for all sufficiently large values of \( r_1, r_2 \) that

\[
M_g\left( r_1^{\frac{1}{\lambda_{f_i}(g)+\varepsilon}}, r_2^{\frac{1}{\lambda_{f_i}(g)+\varepsilon}} \right) > nM_g\left( r_1^{\frac{1}{\lambda_{f_k}(g)-\varepsilon}}, r_2^{\frac{1}{\lambda_{f_k}(g)-\varepsilon}} \right),
\] (7.25)

where \( k = 1, 2, \ldots, n \) and \( k \neq i \).

Thus from (7.20), (7.23) and (7.25) we get for a sequence of values of \( r_1, r_2 \) tending to infinity that

\[
M_{f_i}(r_1, r_2) > M_g\left( r_1^{\frac{1}{\lambda_{f_i}(g)+\varepsilon}}, r_2^{\frac{1}{\lambda_{f_i}(g)+\varepsilon}} \right)
\]

i.e.,

\[
M_{f_i}(r_1, r_2) > nM_g\left( r_1^{\frac{1}{\lambda_{f_k}(g)+\varepsilon}}, r_2^{\frac{1}{\lambda_{f_k}(g)+\varepsilon}} \right)
\]

i.e.,

\[
M_{f_i}(r_1, r_2) > nM_{f_k}(r_1, r_2) \quad \text{where} \ k = 1, 2, \ldots, n \ \text{and} \ k \neq i.
\] (7.26)
So from \((7.23)\) and \((7.26)\) and in view of the first part of Lemma \(7.2.1\), it follows for a sequence of values of \(r_1, r_2\) tending to infinity that

\[
M_f (r_1, r_2) \geq M_{f_1} (r_1, r_2) - \sum_{k=1}^{n} M_{f_k} (r_1, r_2)
\]

i.e.,
\[
M_f (r_1, r_2) \geq M_{f_1} (r_1, r_2) - \frac{1}{n} \sum_{k=1}^{n} M_{f_i} (r_1, r_2)
\]

i.e.,
\[
M_f (r_1, r_2) > M_{f_1} (r_1, r_2) - \left(\frac{n - 1}{n}\right) M_{f_1} (r_1, r_2)
\]

i.e.,
\[
M_f (r_1, r_2) > \left(\frac{1}{n}\right) M_{f_1} (r_1, r_2)
\]

i.e.,
\[
M_f (r_1, r_2) > \left(\frac{1}{n}\right) M_g \left[ \frac{1}{r_1^{\frac{1}{v_2 \lambda_{f_i}(g) + \varepsilon}}} , \frac{1}{r_2^{\frac{1}{v_2 \lambda_{f_i}(g) + \varepsilon}}} \right]
\]

i.e.,
\[
M_f (r_1, r_2) > M_g \left[ \frac{1}{n + 1} , \frac{1}{n + 1} \right]
\]

This gives for a sequence of values of \(r_1, r_2\) tending to infinity that

\[
M_f \left\{ (n + 1) r_1 \right\}^{\frac{1}{v_2 \lambda_{f_i}(g) + \varepsilon}} \cdot \left\{ (n + 1) r_2 \right\}^{\frac{1}{v_2 \lambda_{f_i}(g) + \varepsilon}} > M_g (r_1, r_2)
\]

i.e.,
\[
\left\{ (n + 1) r_1 \right\}^{\frac{1}{v_2 \lambda_{f_i}(g) + \varepsilon}} \cdot \left\{ (n + 1) r_2 \right\}^{\frac{1}{v_2 \lambda_{f_i}(g) + \varepsilon}} > M_f^{-1} M_g (r_1, r_2)
\]

i.e.,
\[
(v_2 \lambda_{f_i}(g) + \varepsilon) > \log M_f^{-1} M_g (r_1, r_2) \quad \log \left\{ (n + 1) r_1 \cdot (n + 1) r_2 \right\}
\]

i.e.,
\[
(v_2 \lambda_{f_i}(g) + \varepsilon) > \log M_f^{-1} M_g (r_1, r_2) \quad \log (r_1 r_2) + O(1)
\]

i.e.,
\[
v_2 \lambda_{f_i}(g) \geq \liminf_{r \to \infty} \frac{\log M_f^{-1} M_g (r_1, r_2)}{\log (r_1 r_2) + O(1)}
\]

i.e.,
\[
v_2 \lambda_{f_i}(g) = \liminf_{r \to \infty} \frac{\log M_f^{-1} M_g (r_1, r_2)}{\log (r_1 r_2)} \leq v_2 \lambda_{f_i}(g).
\] (7.27)
Proof. Suppose that $v_2 \lambda_f (g) = v_2 \lambda_{f_i} (g)$, when $v_2 \lambda_{f_i} (g) \neq \{v_2 \lambda_{f_k} (g) \mid k = 1, 2, \ldots n \text{ and } k \neq i \}$. □

**Theorem 7.3.7** If $f_1, f_2, \ldots f_n \ (n \geq 2)$, $g$ are entire functions of two complex variables and $g$ has the Property (A), then

$$v_2 \lambda_f (g) \geq v_2 \lambda_{f_i} (g)$$

where $f = \prod_{k=1}^{n} f_k$ and $v_2 \lambda_{f_i} (g) = \min \{v_2 \lambda_{f_k} (g) \mid k = 1, 2, \ldots, n \}$. The sign of equality holds when $v_2 \lambda_{f_i} (g) \neq v_2 \lambda_{f_k} (g) \ (k = 1, 2, \ldots, n \text{ and } k \neq i)$. Finally, assume that $F_1$ and $F_2$ are entire functions such that $f = \frac{F_1}{F_2}$ is also an entire function. Then $v_2 \lambda_f (g) = \min \{v_2 \lambda_{F_1} (g), v_2 \lambda_{F_2} (g)\}$.

**Proof.** Suppose that $v_2 \lambda_f (g) < \infty$. Otherwise if $v_2 \lambda_f (g) = \infty$ then the result is obvious.

We can clearly assume that $v_2 \lambda_{f_i} (g)$ is finite. Also suppose that $v_2 \lambda_{f_i} (g) \leq v_2 \lambda_{f_k} (g)$ where $k = 1, 2, \ldots, n$.

Now for any arbitrary $\varepsilon > 0$, we have for all sufficiently large values of $r_1, r_2$ that

$$M_{f_k} \left( r_1^{\frac{v_2 \lambda_{f_k} (g) - \frac{\varepsilon}{2}}{1}}, r_2^{\frac{v_2 \lambda_{f_k} (g) - \frac{\varepsilon}{2}}{1}} \right) < M_g (r_1, r_2) \quad \text{where } k = 1, 2, \ldots, n$$

i.e.,

$$M_{f_k} (r_1, r_2) < M_g \left[ r_1^{\frac{1}{v_2 \lambda_{f_k} (g) - \frac{\varepsilon}{2}}}, r_2^{\frac{1}{v_2 \lambda_{f_k} (g) - \frac{\varepsilon}{2}}} \right] \quad \text{where } k = 1, 2, \ldots, n$$

i.e.,

$$M_{f_k} (r_1, r_2) \leq M_g \left[ r_1^{\frac{1}{v_2 \lambda_{f_k} (g) - \frac{\varepsilon}{2}}}, r_2^{\frac{1}{v_2 \lambda_{f_k} (g) - \frac{\varepsilon}{2}}} \right] \quad (7.28)$$

where $k = 1, 2, \ldots, n$.

Now we consider the expression

$$\frac{v_2 \lambda_{f_i} (g) - \frac{\varepsilon}{2}}{\log(r_1 r_2)}$$

for all sufficiently large values of $r_1, r_2$.

Thus for any $\delta > 1$, it follows from the above expression for all sufficiently large values of $r_1, r_2$, say $r_1 \geq r_{11} \geq r_{10}, r_2 \geq r_{21} \geq r_{20}$ that

$$\frac{v_2 \lambda_{f_i} (g) - \frac{\varepsilon}{2}}{\log(r_{11} r_{20})} = \delta \quad (7.29)$$
Now from (7.28), we have for all sufficiently large values of \( r_1, r_2 \) that

\[
M_f (r_1, r_2) < \prod_{k=1}^{n} M_{f_k} (r_1, r_2)
\]

\[\text{i.e., } M_f (r_1, r_2) < \prod_{k=1}^{n} M_g \left( r_1^{\frac{1}{\omega_k (g) - \frac{\pi}{2}}} r_2^{\frac{1}{\omega_k (g) - \frac{\pi}{2}}} \right)\]

\[\text{i.e., } M_f (r_1, r_2) < \left[ M_g \left( r_1^{\frac{1}{\omega_k (g) - \frac{\pi}{2}}} r_2^{\frac{1}{\omega_k (g) - \frac{\pi}{2}}} \right) \right]^n. \quad (7.30)\]

Since \( g(z_1, z_2) \) has the Property (A), in view of Lemma 7.2.2 and (7.29), we obtain from (7.30) for all sufficiently large values of \( r_1, r_2 \) that

\[
M_f (r_1, r_2) < M_g \left( r_1^{\frac{1}{\omega_k (g) - \frac{\pi}{2}}} r_2^{\frac{1}{\omega_k (g) - \frac{\pi}{2}}} \right)^{\delta}
\]

\[\text{i.e., } M_f (r_1, r_2) < M_g \left[ r_1^{\frac{1}{\omega_k (g) - \frac{\pi}{2}}} r_2^{\frac{1}{\omega_k (g) - \frac{\pi}{2}}} \right] \]

\[\text{i.e., } M_f \left[ r_1^{\omega_k (g) - \frac{\epsilon}{\omega_k (g) - \frac{\pi}{2}}} r_2^{\omega_k (g) - \frac{\epsilon}{\omega_k (g) - \frac{\pi}{2}}} \right] < M_g (r_1, r_2) \]

\[\text{i.e., } r_1^{\omega_k (g) - \frac{\epsilon}{\omega_k (g) - \frac{\pi}{2}}} r_2^{\omega_k (g) - \frac{\epsilon}{\omega_k (g) - \frac{\pi}{2}}} < M_f^{-1} M_g (r_1, r_2) \]

\[\text{i.e., } (\omega_k (g) - \frac{\epsilon}{\omega_k (g) - \frac{\pi}{2}}) \log (r_1 r_2) < \log M_f^{-1} M_g (r_1, r_2) \]

\[\text{i.e., } (\omega_k (g) - \frac{\epsilon}{\omega_k (g) - \frac{\pi}{2}}) < \frac{\log M_f^{-1} M_g (r_1, r_2)}{\log (r_1 r_2)}. \]

So

\[
v_2 \omega_f (g) = \liminf_{r_1, r_2 \to \infty} \frac{\log M_f^{-1} M_g (r_1, r_2)}{\log (r_1 r_2)} \geq v_2 \omega_f (g) - \frac{\epsilon}{\omega_f (g) - \frac{\pi}{2}}. \]

Since \( \epsilon > 0 \) is arbitrary,

\[
v_2 \omega_f (g) \geq v_2 \omega_f (g). \quad (7.31)\]

Thus the theorem follows from (7.31).

Now let us suppose that \( k, h, h_1, h_2, \ldots, h_n \) \((n \geq 2)\) are all entire functions of two complex variables such that \( k \) has the Property (A), \( h = \frac{h_1}{h_2 h_3 \cdots h_n} \).
and \( v_2 \lambda_h (k) \geq \min \{ v_2 \lambda_{h_k} (k) \mid k = 2, ..., n \} \). Now \( h_1 = h \cdot h_2 \cdot h_3 \cdots h_n \) and in view of (7.31), we get that

\[
v_2 \lambda_{h_1} (k) \geq \min \{ v_2 \rho_h (k), v_2 \lambda_{h_k} (k) \mid k = 2, ..., n \}
\]
i.e., \( v_2 \lambda_{h_1} (k) \geq \min \{ v_2 \lambda_{h_k} (k) \mid k = 2, ..., n \} \).

As \( v_2 \lambda_h (k) \geq \min \{ v_2 \lambda_{h_k} (k) \mid k = 2, ..., n \} \), therefore from the above it follows that

\[
v_2 \lambda_h (k) = v_2 \lambda \frac{k_1}{h_2 h_3 \cdots h_n} (k) \geq \min \{ v_2 \lambda_{h_k} (k) \mid k = 1, 2, ..., n \} \quad (7.32)
\]

Since \( f = \prod_{k=1}^{n} f_k \), we get that \( f_i = \frac{f}{\prod_{k=1, k \neq i}^{f} f_k} \). Also without loss of any generality, we may consider that \( v_2 \lambda_{f_i} (g) < v_2 \lambda_{f_k} (g) \) where \( k = 1, 2, ... n \) and \( k \neq i \). Therefore in view of (7.32), we get that

\[
v_2 \lambda_{f_i} (g) \geq v_2 \lambda \frac{f}{\prod_{k=1, k \neq i}^{f} f_k} (g) \geq \min \{ v_2 \lambda_f (g), v_2 \lambda_{f_k} (g) \mid k = 1, 2, ..., n \text{ and } k \neq i \}.
\]

Since \( v_2 \lambda_{f_i} (g) < v_2 \lambda_{f_k} (g) \), where \( k = 1, 2, ... n \) and \( k \neq i \), it follows from above that

\[
v_2 \lambda_{f_i} (g) \geq v_2 \lambda_f (g) \quad (7.33)
\]

So from (7.31) and (7.33), we finally obtain that

\[
v_2 \lambda_{f_1} (g) = v_2 \lambda_f (g),
\]

if one assumes that \( v_2 \lambda_{f_i} (g) \neq v_2 \lambda_{f_k} (g) \) for all \( k \in \{ 1, 2, ..., n \} \setminus \{ i \} \).

Let now \( f = \frac{F_1}{F_2} \) with \( F_1, F_2, f \) entire, and suppose \( v_2 \lambda_{F_1} (g) \geq v_2 \lambda_{F_2} (g) \). We have \( F_1 = \vec{f} \cdot F_2 \). Thus \( v_2 \lambda_{F_1} (g) = v_2 \lambda_f (g) \) if \( v_2 \lambda_f (g) < v_2 \lambda_{F_2} (g) \).

So it follows that \( v_2 \lambda_{F_1} (g) < v_2 \lambda_{F_2} (g) \), which contradicts the hypothesis "\( v_2 \lambda_{F_1} (g) \geq v_2 \lambda_{F_2} (g) \)." Hence \( v_2 \lambda_f (g) = v_2 \lambda_{F_{f_1}} (g) \geq v_2 \lambda_{F_2} (g) \) = \( \min \{ v_2 \lambda_{F_1} (g), v_2 \lambda_{F_2} (g) \} \). Also suppose that \( v_2 \lambda_{F_1} (g) > v_2 \lambda_{F_2} (g) \).

Then \( v_2 \lambda_{F_1} (g) = \min \{ v_2 \lambda_f (g), v_2 \lambda_{F_2} (g) \} = v_2 \lambda_{F_2} (g) \), if \( v_2 \lambda_f (g) > v_2 \lambda_{F_2} (g) \), which is also a contradiction. Thus

\[
v_2 \lambda_f (g) = v_2 \lambda_{F_{f_1}} (g) = \min \{ v_2 \lambda_{F_1} (g), v_2 \lambda_{F_2} (g) \}.
\]

This proves the theorem. \( \blacksquare \)
Theorem 7.3.8 If $n > 1$ be a positive integer, then
\[
\frac{1}{n} v_2 \lambda_f (g) \leq v_2 \lambda_{f^n} (g) \leq v_2 \lambda_f (g).
\]

Proof. From the first and second part of Lemma 7.2.1, we obtain that
\[
\{M_f (r_1, r_2)\}^n \leq K M_f (r_1^n, r_2^n) < M_f ((K + 1) r_1^n, (K + 1) r_2^n) \tag{7.34}
\]
for $n > 1, r_1 > 0$ and $r_2 > 0$ where $K = K (n, f) > 0$.
Therefore from (7.34), we obtain that
\[
M_f^{-1} (r_1^n, r_2^n) < (K + 1) \left\{ M_f^{-1} (r_1, r_2) \right\}^n \tag{i.e.,}
\]
\[
\frac{1}{(K + 1)} M_f^{-1} (r_1^n, r_2^n) < \left\{ M_f^{-1} (r_1, r_2) \right\}^n.
\]
So
\[
v_2 \lambda_{f^n} (g) \geq \frac{\log (K f^{-1} M_f (r_1^n, r_2^n))}{\log (r_1 r_2)^n}
\]
i.e., $v_2 \lambda_{f^n} (g) \geq \frac{1}{n} v_2 \lambda_f (g)$. \tag{7.35}

On the other hand, since $\{M_f (r_1, r_2)\}^n > M_f (r_1, r_2)$ for all sufficiently large values of $r$, we have by Lemma 7.2.3
\[
v_2 \lambda_{f^n} (g) \leq v_2 \lambda_f (g). \tag{7.36}
\]
Thus the theorem follows from (7.35) and (7.36). \qed

Corollary 7.3.10 If $n > 1$ be a positive integer, then
\[
\frac{1}{n} v_2 \rho_f (g) \leq v_2 \rho_{f^n} (g) \leq v_2 \rho_f (g).
\]

Theorem 7.3.9 Let $f$, $g$ and $h$ be any three entire functions of two complex variables such that the relative order ($\lambda_f$) and the relative lower order ($\rho_f$) of $f$ with respect to $h$ and the relative order ($\lambda_g$) and the relative lower order ($\rho_g$) of $f$ with respect to $h$ are $v_2 \lambda_f (f)$ ($v_2 \lambda_h (f)$) and $v_2 \rho_f (g)$ ($v_2 \lambda_h (g)$) respectively. Then
\[
\frac{v_2 \lambda_h (f)}{v_2 \rho_h (g)} \leq v_2 \lambda_f (g) \leq \min \left\{ \frac{v_2 \lambda_h (f)}{v_2 \rho_h (g)}, \frac{v_2 \rho_h (f)}{v_2 \rho_h (g)} \right\}
\]
\[
\leq \max \left\{ \frac{v_2 \lambda_h (f)}{v_2 \rho_h (g)}, \frac{v_2 \rho_h (f)}{v_2 \rho_h (g)} \right\} \leq \frac{v_2 \rho_h (f)}{v_2 \lambda_h (g)}.
\]
Proof. From the definitions of \( v_2 \rho_h (f) \) and \( v_2 \lambda_h (f) \), we have for all sufficiently large values of \( r_1, r_2 \) that

\[
M_h^{-1} M_f (r_1, r_2) \leq \exp \{ (v_2 \rho_h (f) + \varepsilon) \log (r_1 r_2) \}
\]

i.e., \( M_f (r_1, r_2) \leq M_h \left[ \exp \{ (v_2 \rho_h (f) + \varepsilon) \log (r_1 r_2) \} \right], \tag{7.37}
\]

and also for a sequence of values of \( r_1, r_2 \) tending to infinity we get that

\[
M_h^{-1} M_f (r_1, r_2) \geq \exp \{ (v_2 \rho_h (f) - \varepsilon) \log (r_1 r_2) \}
\]

i.e., \( M_f (r_1, r_2) \geq M_h \left[ \exp \{ (v_2 \rho_h (f) - \varepsilon) \log (r_1 r_2) \} \right]. \tag{7.38}
\]

Similarly from the definitions of \( v_2 \rho_h (g) \) and \( v_2 \lambda_h (g) \), it follows for all sufficiently large values of \( r_1, r_2 \) that

\[
M_h^{-1} M_g (r_1, r_2) \leq \exp \{ (v_2 \rho_h (g) + \varepsilon) \log (r_1 r_2) \}
\]

i.e., \( M_g (r_1, r_2) \leq M_h \left[ \exp \{ (v_2 \rho_h (g) + \varepsilon) \log (r_1 r_2) \} \right]\)

i.e., \( M_h (r_1, r_2) \geq M_g \left[ \exp \left\{ \frac{\log (r_1 r_2)}{(v_2 \rho_h (g) + \varepsilon)} \right\} \right]. \tag{7.42}
\]

and for a sequence of values of \( r_1, r_2 \) tending to infinity we obtain that

\[
M_h^{-1} M_g (r_1, r_2) \geq \exp \{ (v_2 \rho_h (g) - \varepsilon) \log (r_1 r_2) \}
\]

i.e., \( M_g (r_1, r_2) \geq M_h \left[ \exp \{ (v_2 \rho_h (g) - \varepsilon) \log (r_1 r_2) \} \right]\)

i.e., \( M_h (r_1, r_2) \leq M_g \left[ \exp \left\{ \frac{\log (r_1 r_2)}{(v_2 \rho_h (g) - \varepsilon)} \right\} \right]. \tag{7.43}
\]

and for a sequence of values of \( r_1, r_2 \) tending to infinity we obtain that

\[
M_h^{-1} M_g (r_1, r_2) \geq \exp \{ (v_2 \rho_h (g) - \varepsilon) \log (r_1 r_2) \}
\]

i.e., \( M_g (r_1, r_2) \geq M_h \left[ \exp \{ (v_2 \rho_h (g) - \varepsilon) \log (r_1 r_2) \} \right]\)

i.e., \( M_h (r_1, r_2) \leq M_g \left[ \exp \left\{ \frac{\log (r_1 r_2)}{(v_2 \rho_h (g) - \varepsilon)} \right\} \right]. \tag{7.44}
\]
\[ M_h^{-1} M_g (r_1, r_2) \leq \exp \{ (v_2 \lambda_h (g) + \varepsilon) \log (r_1 r_2) \} \]

i.e., \[ M_g (r_1, r_2) \leq M_h \left[ \exp \left\{ (v_2 \lambda_h (g) + \varepsilon) \log (r_1 r_2) \right\} \right] \]

i.e., \[ M_h (r_1, r_2) \geq M_g \left[ \exp \left\{ \frac{\log (r_1 r_2)}{(v_2 \lambda_h (g) + \varepsilon)} \right\} \right] . \tag{7.45} \]

Now from (7.39) and in view of (7.42), we get for a sequence of values of \( r_1, r_2 \) tending to infinity that

\[ \log M_g^{-1} M_f (r_1, r_2) \geq \log M_g^{-1} M_h \left[ \exp \left\{ (v_2 \rho_h (f) - \varepsilon) \log (r_1 r_2) \right\} \right] \]

i.e., \[ \log M_g^{-1} M_f (r_1, r_2) \geq \log M_g^{-1} M_h \left[ \exp \left\{ \frac{\log \exp \left\{ (v_2 \rho_h (f) - \varepsilon) \log (r_1 r_2) \right\}}{(v_2 \rho_h (g) + \varepsilon)} \right\} \right] \]

i.e., \[ \log M_g^{-1} M_f (r_1, r_2) \geq \frac{(v_2 \rho_h (f) - \varepsilon)}{(v_2 \rho_h (g) + \varepsilon)} \log (r_1 r_2) \]

i.e., \[ \frac{\log M_g^{-1} M_f (r_1, r_2)}{\log (r_1 r_2)} \geq \frac{(v_2 \rho_h (f) - \varepsilon)}{(v_2 \rho_h (g) + \varepsilon)} . \tag{7.46} \]

As \( \varepsilon (> 0) \) is arbitrary, it follows that

\[ \limsup_{r_1, r_2 \to \infty} \frac{\log M_g^{-1} M_f (r_1, r_2)}{\log (r_1 r_2)} \geq \frac{v_2 \rho_h (f)}{v_2 \rho_h (g)} \]

i.e., \[ v_2 \rho_g (f) \geq \frac{v_2 \rho_h (f)}{v_2 \rho_h (g)} . \tag{7.46} \]

Analogously, from (7.38) and in view of (7.45) it follows for a sequence of values of \( r_1, r_2 \) tending to infinity that

\[ \log M_g^{-1} M_f (r_1, r_2) \geq \log M_g^{-1} M_h \left[ \exp \left\{ (v_2 \lambda_h (f) - \varepsilon) \log (r_1 r_2) \right\} \right] \]

i.e., \[ \log M_g^{-1} M_f (r_1, r_2) \]

\[ \geq \log M_g^{-1} M_h \left[ \exp \left\{ \frac{\log \exp \left\{ (v_2 \lambda_h (f) - \varepsilon) \log (r_1 r_2) \right\}}{(v_2 \lambda_h (g) + \varepsilon)} \right\} \right] \]
\[ i.e., \quad \log M_g^{-1} M_f (r_1, r_2) \geq \frac{(v_2 \lambda_h (f) - \varepsilon)}{(v_2 \lambda_h (g) + \varepsilon)} \log (r_1 r_2) \]

\[ i.e., \quad \frac{\log M_g^{-1} M_f (r_1, r_2)}{\log (r_1 r_2)} \geq \frac{(v_2 \lambda_h (f) - \varepsilon)}{(v_2 \lambda_h (g) + \varepsilon)}. \]

Since \( \varepsilon (>0) \) is arbitrary, we get from above that

\[ \limsup_{r_1, r_2 \to \infty} \frac{\log M_g^{-1} M_f (r_1, r_2)}{\log (r_1 r_2)} \geq \frac{v_2 \lambda_h (f)}{v_2 \lambda_h (g)}. \]

\[ i.e., \quad v_2 \rho_g (f) \geq \frac{v_2 \lambda_h (f)}{v_2 \lambda_h (g)}. \] (7.47)

Again in view of (7.43), we have from (7.37) for all sufficiently large values of \( r_1, r_2 \) that

\[ \log M_g^{-1} M_f (r_1, r_2) \leq \log M_g^{-1} M_h \left[ \exp \left\{ \frac{(v_2 \rho_h (f) + \varepsilon) \log (r_1 r_2)}{(v_2 \lambda_h (g) - \varepsilon)} \right\} \right] \]

\[ i.e., \quad \log M_g^{-1} M_f (r_1, r_2) \leq \log M_g^{-1} M_g \left[ \exp \left\{ \frac{(v_2 \rho_h (f) + \varepsilon) \log (r_1 r_2)}{(v_2 \lambda_h (g) - \varepsilon)} \right\} \right] \]

\[ i.e., \quad \log M_g^{-1} M_f (r_1, r_2) \leq \frac{(v_2 \rho_h (f) + \varepsilon)}{(v_2 \lambda_h (g) - \varepsilon)} \log (r_1 r_2) \]

\[ i.e., \quad \frac{\log M_g^{-1} M_f (r_1, r_2)}{\log (r_1 r_2)} \leq \frac{(v_2 \rho_h (f) + \varepsilon)}{(v_2 \lambda_h (g) - \varepsilon)}. \]

Since \( \varepsilon (>0) \) is arbitrary, we obtain that

\[ \limsup_{r_1, r_2 \to \infty} \frac{\log M_g^{-1} M_f (r_1, r_2)}{\log (r_1 r_2)} \leq \frac{v_2 \rho_h (f)}{v_2 \lambda_h (g)}. \]

\[ i.e., \quad v_2 \rho_g (f) \leq \frac{v_2 \rho_h (f)}{v_2 \lambda_h (g)}. \] (7.48)

Again from (7.38) and in view of (7.42), we get for all sufficiently large values of \( r_1, r_2 \) that

\[ \log M_g^{-1} M_f (r_1, r_2) \geq \log M_g^{-1} M_h \left[ \exp \left\{ (v_2 \lambda_h (f) - \varepsilon) \log (r_1 r_2) \right\} \right] \]
\[ i.e., \log M_y^{-1} M_f (r_1, r_2) \geq \log M_y^{-1} M_g \left[ \exp \left\{ \frac{\log \exp \{(v_2 \lambda_h (f) - \varepsilon) \log (r_1 r_2)\}}{(v_2 \rho_h (g) + \varepsilon)} \right\} \right] \]

\[ i.e., \log M_y^{-1} M_f (r_1, r_2) \geq \frac{(v_2 \lambda_h (f) - \varepsilon)}{(v_2 \rho_h (g) + \varepsilon)} \log (r_1 r_2) \]

\[ i.e., \frac{\log M_y^{-1} M_f (r_1, r_2)}{\log (r_1 r_2)} \geq \frac{(v_2 \lambda_h (f) - \varepsilon)}{(v_2 \rho_h (g) + \varepsilon)}. \]

As \( \varepsilon (> 0) \) is arbitrary, it follows from above that

\[ \liminf_{r_1, r_2 \to \infty} \frac{\log M_y^{-1} M_f (r_1, r_2)}{\log (r_1 r_2)} \geq \frac{v_2 \lambda_h (f)}{v_2 \rho_h (g)} \]

\[ i.e., v_2 \lambda_g (f) \geq \frac{v_2 \lambda_h (f)}{v_2 \rho_h (g)}. \quad (7.49) \]

Also in view of (7.44), we get from (7.37) for a sequence of values of \( r_1, r_2 \) tending to infinity that

\[ \log M_y^{-1} M_f (r_1, r_2) \leq \log M_y^{-1} M_h \left[ \exp \{(v_2 \rho_h (f) + \varepsilon) \log (r_1 r_2)\} \right] \]

\[ i.e., \log M_y^{-1} M_f (r_1, r_2) \leq \log M_y^{-1} M_g \left[ \exp \left\{ \frac{\log \exp \{(v_2 \rho_h (f) + \varepsilon) \log (r_1 r_2)\}}{(v_2 \rho_h (g) - \varepsilon)} \right\} \right] \]

\[ i.e., \log M_y^{-1} M_f (r_1, r_2) \leq \frac{(v_2 \rho_h (f) + \varepsilon)}{(v_2 \rho_h (g) - \varepsilon)} \log (r_1 r_2) \]

\[ i.e., \frac{\log M_y^{-1} M_f (r_1, r_2)}{\log (r_1 r_2)} \leq \frac{(v_2 \rho_h (f) + \varepsilon)}{(v_2 \rho_h (g) - \varepsilon)}. \]

Since \( \varepsilon (> 0) \) is arbitrary, we get from above that

\[ \liminf_{r_1, r_2 \to \infty} \frac{\log M_y^{-1} M_f (r_1, r_2)}{\log (r_1 r_2)} \leq \frac{v_2 \rho_h (f)}{v_2 \rho_h (g)} \]

\[ i.e., v_2 \lambda_g (f) \leq \frac{v_2 \rho_h (f)}{v_2 \rho_h (g)}. \quad (7.50) \]
Similarly from (7.41) and in view of (7.43), it follows for a sequence of values of \( r_1, r_2 \) tending to infinity that

\[
\log M_g^{-1} M_f (r_1, r_2) \leq \log M_g^{-1} M_h \left[ \exp \left\{ (v_2 \lambda_h (f) + \varepsilon) \log (r_1 r_2) \right\} \right]
\]

i.e.,

\[
\log M_g^{-1} M_f (r_1, r_2) \leq \log M_g^{-1} M_g \left[ \exp \left\{ \frac{(v_2 \lambda_h (f) + \varepsilon) \log (r_1 r_2)}{(v_2 \lambda_h (g) - \varepsilon)} \right\} \right]
\]

i.e.,

\[
\log M_g^{-1} M_f (r_1, r_2) \leq \frac{(v_2 \lambda_h (f) + \varepsilon) \log (r_1 r_2)}{(v_2 \lambda_h (g) - \varepsilon)}
\]

i.e.,

\[
\frac{\log M_g^{-1} M_f (r_1, r_2)}{\log (r_1 r_2)} \leq \frac{(v_2 \lambda_h (f) + \varepsilon)}{(v_2 \lambda_h (g) - \varepsilon)}.
\]

As \( \varepsilon (> 0) \) is arbitrary, we obtain from above that

\[
\liminf_{r_1, r_2 \to \infty} \frac{\log M_g^{-1} M_f (r_1, r_2)}{\log (r_1 r_2)} \leq \frac{v_2 \lambda_h (f)}{v_2 \lambda_h (g)}
\]

i.e.,

\[
v_2 \lambda_g (f) \leq \frac{v_2 \lambda_h (f)}{v_2 \lambda_h (g)}.
\]  \hspace{1cm} (7.51)

Thus the theorem follows from (7.46), (7.47), (7.48), (7.49), (7.50) and (7.51).

In view of Theorem 7.3.1, one can easily verify the following corollaries:

**Corollary 7.3.11** Let \( f \) be an entire function of two complex variables with regular relative growth with respect to an entire function \( h \) of two complex variables and \( g \) be another entire function of two complex variables. Then

\[
v_2 \lambda_g (f) = \frac{v_2 \rho_h (f)}{v_2 \rho_h (g)} \quad \text{and} \quad v_2 \rho_g (f) = \frac{v_2 \rho_h (f)}{v_2 \lambda_h (g)}.
\]

In addition, if \( v_2 \rho_h (f) = v_2 \rho_h (g) \) then

\[
v_2 \lambda_g (f) = v_2 \rho_f (g) = 1.
\]
Corollary 7.3.12 Let $f$, $g$, $h$ be any three entire functions of two complex variables such that $g$ is of regular relative growth with respect to an entire function $h$. Then

$$v_2\lambda_g(f) = \frac{v_2\lambda_h(f)}{v_2\rho_h(g)} \quad \text{and} \quad v_2\rho_g(f) = \frac{v_2\rho_h(f)}{v_2\rho_h(g)}.$$ 

In addition, if $v_2\rho_h(f) = v_2\rho_h(g)$ then

$$v_2\rho_g^{(p,q)}(f) = v_2\lambda_f^{(q,p)}(g) = 1.$$

Corollary 7.3.13 Let $f$ and $g$ be any two entire functions of two complex variables with the regular relative growth with respect to another entire function $h$ of two complex variables respectively. Then

$$v_2\lambda_g(f) = v_2\rho_g(f) = \frac{v_2\rho_h(f)}{v_2\rho_h(g)}.$$ 

Corollary 7.3.14 Let $f$ and $g$ be any two entire functions of two complex variables with regular relative growth with respect to another entire function $h$ of two complex variables respectively. Also suppose that

$$v_2\rho_h(f) = v_2\rho_h(g).$$

Then

$$v_2\lambda_g(f) = v_2\rho_g(f) = v_2\lambda_f(g) = v_2\rho_f(g) = 1.$$ 

Corollary 7.3.15 Let $f$, $g$ and $h$ be any three entire functions of two complex variables such that either $f$ is not of regular relative growth or $g$ is not of regular relative growth with respect to $h$. Then

$$v_2\rho_g(f) \cdot v_2\rho_f(g) \geq 1.$$ 

When $f$ and $g$ are both of regular relative growth with respect to $h$ respectively, then

$$v_2\rho_g(f) \cdot v_2\rho_f(g) = 1.$$
Corollary 7.3.16 Let $f$, $g$ and $h$ be any three entire functions of two complex variables such that either $f$ is not of regular relative growth or $g$ is not of regular relative growth with respect to $h$. Then

$$v_2 \lambda_g (f) \cdot v_2 \lambda_f (g) \leq 1.$$ 

When $f$ and $g$ are both of regular relative growth with respect to $h$ respectively, then

$$v_2 \lambda_g (f) \cdot v_2 \lambda_f (g) = 1.$$

Corollary 7.3.17 Let $f$ and $g$ be any two entire functions of two complex variables. Then

(i) $v_2 \lambda_g (f) = \infty$ when $v_2 \rho_h (g) = 0$,
(ii) $v_2 \rho_g (f) = \infty$ when $v_2 \lambda_h (g) = 0$,
(iii) $v_2 \lambda_g (f) = 0$ when $v_2 \rho_h (g) = \infty$

and

(iv) $v_2 \rho_g (f) = 0$ when $v_2 \lambda_h (g) = \infty$.

Corollary 7.3.18 Let $f$ and $g$ be any two entire functions of two complex variables. Then

(i) $v_2 \rho_g (f) = 0$ when $v_2 \rho_h (f) = 0$,
(ii) $v_2 \lambda_g (f) = 0$ when $v_2 \lambda_h (f) = 0$,
(iii) $v_2 \rho_g (f) = \infty$ when $v_2 \rho_h (f) = \infty$

and

(iv) $v_2 \lambda_g (f) = \infty$ when $v_2 \lambda_h (f) = \infty$. 

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