MEASURE OF RELATIVE (p,q)-th ORDER-BASED GROWTH OF COMPOSITE ENTIRE FUNCTIONS
Chapter 6

MEASURE OF RELATIVE (p,q)-th ORDER-BASED GROWTH OF COMPOSITE ENTIRE FUNCTIONS

6.1 Introductory remarks.

Extending some results of Tu, Chen and Zheng [70], in this chapter we deduce some results on the growth properties of composite entire functions in the light of their relative (p, q)-th orders which are defined newly in terms of index pairs as already employed in Chapter Five. The existing literature, relevent definitions and related examples in connection with above have already been discussed in Chapter One and Chapter Two respectively and therefore are omitted here.

6.2 Lemma.

In this section we present a lemma which will be needed in the sequel.

Lemma 6.2.1 [11] If f and g are any two entire functions then for all sufficiently large values of r,

\[ M_f \left( \frac{1}{8} M_g \left( \frac{r}{2} \right) - |g(0)| \right) \leq M_{f \circ g}(r) \leq M_f \left( M_g(r) \right). \]

Communicated, see [40]
6.3 Theorems.

In this section we present the main results of the chapter.

**Theorem 6.3.1** Let \( f \) and \( g \) be any two entire functions with index-pairs \((p, q)\) and \((m, n)\) respectively where \( p, q, m, n \) are all positive integers such that \( p \geq q \) and \( m \geq n \). Then

(i) the index-pair of \( f \circ g \) is \((p, n)\) when \( q = m \) and either \( \lambda_f (p, q) > 0 \) or \( \lambda_g (m, n) > 0 \). Also

(a) \( \lambda_f (p, q) \cdot \rho_g (m, n) \leq \rho_{f\circ g} (p, q) \leq \rho_f (p, q) \cdot \rho_g (m, n) \) if \( \lambda_f (p, q) > 0 \) and

(b) \( \lambda_f (p, q) \cdot \rho_g (m, n) \leq \rho_{f\circ g} (p, q) \leq \rho_f (p, q) \cdot \rho_g (m, n) \) if \( \lambda_g (m, n) > 0 \);

(ii) the index-pair of \( f \circ g \) is \((p, q + n - m)\) when \( q > m \) and either \( \lambda_f (p, q) > 0 \) or \( \lambda_g (m, n) > 0 \). Also

(a) \( \lambda_f (p, q) \cdot \rho_g (p, q + n - m) \leq \rho_{f\circ g} (p, q) \leq \rho_f (p, q) \cdot \rho_g (m, n) \) if \( \lambda_f (p, q) > 0 \) and

(b) \( \rho_{f\circ g} (p, q + n - m) = \rho_f (p, q) \) if \( \lambda_g (m, n) > 0 \);

(iii) the index-pair of \( f \circ g \) is \((p + m - q, n)\) when \( q < m \) and either \( \lambda_f (p, q) > 0 \) or \( \lambda_g (m, n) > 0 \). Also

(a) \( \rho_{f\circ g} (p + m - q, n) = \rho_g (m, n) \) if \( \lambda_f (p, q) > 0 \) and

(b) \( \lambda_g (m, n) \leq \rho_{f\circ g} (p + m - q, n) \leq \rho_g (m, n) \) if \( \lambda_g (m, n) > 0 \).

**Proof.** In view of the first part of Lemma 6.2.1, it follows for all sufficiently large values of \( r \) that

\[
\log^{[p]} M_{f\circ g} (r) \geq (\lambda_f (p, q) - \varepsilon) \log^{[q]} M_g \left( \frac{r}{2} \right) + O(1) \tag{6.1}
\]

and also for a sequence of values of \( r \) tending to infinity that

\[
\log^{[p]} M_{f\circ g} (r) \geq (\rho_f (p, q) - \varepsilon) \log^{[q]} M_g \left( \frac{r}{2} \right) + O(1) \tag{6.2}
\]
Similarly, in view of the second part of Lemma 6.2.1, we have for all sufficiently large values of \( r \) that
\[
\log^p M_{fog}(r) \leq (\rho_f(p,q) + \varepsilon) \log^q M_g(r) . \tag{6.3}
\]

Now the following two cases may arise:

**Case I.** Let \( q = m \).

Now we have from (6.3) for all sufficiently large values of \( r \) that
\[
\log^p M_{fog}(r) \leq (\rho_f(p,q) + \varepsilon) (\rho_g(m,n) + \varepsilon)
\]
\[\text{i.e., } \lim_{r \to \infty} \frac{\log^p M_{fog}(r)}{\log^n r} \leq \rho_f(p,q) \cdot \rho_g(m,n) . \tag{6.4}\]

Also from (6.1), we obtain for a sequence of values of \( r \) tending to infinity that
\[
\log^p M_{fog}(r) \geq (\lambda_f(p,q) - \varepsilon) (\rho_g(m,n) - \varepsilon) \log^n r + O(1)
\]
\[\text{i.e., } \limsup_{r \to \infty} \frac{\log^p M_{fog}(r)}{\log^n r} \geq \lambda_f(p,q) \cdot \rho_g(m,n) . \tag{6.5}\]

Moreover, we have from (6.2) for a sequence of values of \( r \) tending to infinity that
\[
\log^p M_{fog}(r) \geq (\rho_f(p,q) - \varepsilon) (\lambda_g(m,n) - \varepsilon) \log^n r + O(1)
\]
\[\text{i.e., } \limsup_{r \to \infty} \frac{\log^p M_{fog}(r)}{\log^n r} \geq \rho_f(p,q) \cdot \lambda_g(m,n) . \tag{6.6}\]

Therefore from (6.4) and (6.5), we get for \( \lambda_f(p,q) > 0 \) that
\[
\lambda_f(p,q) \cdot \rho_g(m,n) \leq \limsup_{r \to \infty} \frac{\log^p M_{fog}(r)}{\log^n r} \leq \rho_f(p,q) \cdot \rho_g(m,n)
\]
\[\text{i.e., } \lambda_f(p,q) \cdot \rho_g(m,n) \leq \rho_{fog}(p,n) \leq \rho_f(p,q) \cdot \rho_g(m,n) . \tag{6.7}\]

Likewise, from (6.4) and (6.6) we obtain for \( \lambda_g(m,n) > 0 \) that
\[
\rho_f(p,q) \cdot \lambda_g(m,n) \leq \limsup_{r \to \infty} \frac{\log^p M_{fog}(r)}{\log^n r} \leq \rho_f(p,q) \cdot \rho_g(m,n)
\]
\[\text{i.e., } \rho_f(p,q) \cdot \lambda_g(m,n) \leq \rho_{fog}(p,n) \leq \rho_f(p,q) \cdot \rho_g(m,n) . \tag{6.8}\]
Also from \((6.7)\) and \((6.8)\), one can easily verify that \(\rho_{f \circ g}(p-1, n) = \infty\), \(\rho_{f \circ g}(p, n-1) = 0\) and \(\rho_{f \circ g}(p+1, n+1) = 1\) and therefore we obtain that the index-pair of \(f \circ g\) is \((p, n)\) when \(q = m\) and either \(\lambda_f (p, q) > 0\) or \(\lambda_g (m, n) > 0\) and thus the first part of the theorem is established.

**Case II.** Let \(q > m\).

Now we obtain from \((7.37)\) for all sufficiently large values of \(r\) that

\[
\log^{[p]} M_{f \circ g}(r) \leq (\rho_f (p, q) + \varepsilon) \log^{[q-m]} \log^{[m]} M_g (r)
\]

\[i.e., \quad \log^{[p]} M_{f \circ g}(r) \leq (\rho_f (p, q) + \varepsilon) \log^{[q-m]} \left[(\rho_g (m, n) + \varepsilon) \log^{[n]} r\right]\]

\[i.e., \quad \log^{[p]} M_{f \circ g}(r) \leq (\rho_f (p, q) + \varepsilon) \log^{[q+n-m]} r + O(1)\]

\[i.e., \quad \lim_{r \to \infty} \frac{\log^{[p]} M_{f \circ g}(r)}{\log^{[q+n-m]} r} \leq \rho_f (p, q) \cdot \quad (6.9)\]

Also from \((6.1)\), we have for a sequence of values of \(r\) tending to infinity that

\[
\log^{[p]} M_{f \circ g}(r) \geq (\lambda_f (p, q) - \varepsilon) \log^{[q-m]} \left[(\rho_g (m, n) - \varepsilon) \log^{[n]} \left(\frac{r}{2}\right)\right]
\]

\[+ O(1)\]

\[i.e., \quad \log^{[p]} M_{f \circ g}(r) \geq (\lambda_f (p, q) - \varepsilon) \log^{[q+m-n]} r + O(1)\]

\[i.e., \quad \limsup_{r \to \infty} \frac{\log^{[p]} M_{f \circ g}(r)}{\log^{[q+n-m]} r} \geq \lambda_f (p, q) \cdot \quad (6.10)\]

Further, we get from \((6.2)\) for a sequence of values of \(r\) tending to infinity that

\[
\log^{[p]} M_{f \circ g}(r) \geq (\rho_f (p, q) - \varepsilon) \log^{[q-m]} \left[(\lambda_g (m, n) - \varepsilon) \log^{[n]} \left(\frac{r}{2}\right)\right]
\]

\[+ O(1)\]

\[i.e., \quad \log^{[p]} M_{f \circ g}(r) \geq (\rho_f (p, q) - \varepsilon) \log^{[q+n-m]} r + O(1)\]

\[i.e., \quad \limsup_{r \to \infty} \frac{\log^{[p]} M_{f \circ g}(r)}{\log^{[q+n-m]} r} \geq \rho_f (p, q) \cdot \quad (6.11)\]
Therefore from (6.9) and (6.10), we get for \( \lambda_f (p, q) > 0 \) that

\[
\lambda_f (p, q) \leq \limsup_{r \to \infty} \frac{\log^p M_{f \circ g} (r)}{\log^{q+n-m} r} \leq \rho_f (p, q)
\]

i.e., \( \lambda_f (p, q) \leq \rho_{f \circ g} (p, q + n - m) \leq \rho_f (p, q) \). \( (6.12) \)

Likewise, from (7.46) and (7.48) we get for \( \lambda_g (m, n) > 0 \) that

\[
\rho_f (p, q) \leq \limsup_{r \to \infty} \frac{\log^p M_{f \circ g} (r)}{\log^{q+n-m} r} \leq \rho_f (p, q)
\]

i.e., \( \rho_{f \circ g} (p, q + n - m) = \rho_f (p, q) \). \( (6.13) \)

Hence from (6.12) and (6.13), one can easily verify that \( \rho_{f \circ g} (p - 1, q + n - m) = \infty \), \( \rho_{f \circ g} (p, q + n - m - 1) = 0 \) and \( \rho_{f \circ g} (p + 1, q + n - m + 1) = 1 \) and therefore we get that the index-pair of \( f \circ g \) is \( (p, q + n - m) \) when \( q > m \) and either \( \lambda_f (p, q) > 0 \) or \( \lambda_g (m, n) > 0 \) and thus the second part of the theorem follows.

**Case III.** Let \( q < m \).

Then we obtain from (6.3) for all sufficiently large values of \( r \) that

\[
\log^{[p+m-q]} M_{f \circ g} (r) \leq \log^{[m]} M_g (r) + O(1)
\]

i.e., \( \log^{[p+m-q]} M_{f \circ g} (r) \leq (\rho_g (m, n) + \varepsilon) \log^{[n]} r + O(1) \)

i.e., \( \lim_{r \to \infty} \frac{\log^{[p+m-q]} M_{f \circ g} (r)}{\log^{[n]} r} \leq \rho_g (m, n) \). \( (6.14) \)

Also from (6.1), we have for a sequence of values of \( r \) tending to infinity that

\[
\log^{[p+m-q]} M_{f \circ g} (r) \geq \log^{[m]} M_g \left( \frac{r}{2} \right) + O(1)
\]

i.e., \( \log^{[p+m-q]} M_{f \circ g} (r) \geq (\rho_g (m, n) - \varepsilon) \log^{[n]} r + O(1) \)

\[ \limsup_{r \to \infty} \frac{\log^{[p+m-q]} M_{f \circ g} (r)}{\log^{[n]} r} \geq \rho_g (m, n) \]. \( (6.15) \)

Further, we get from (6.2) for a sequence of values of \( r \) tending to infinity
that

$$\log^{[p+m-q]} M_{fog} (r) \geq \log^{[m]} M_g \left( \frac{r}{2} \right) + O(1)$$

i.e.,

$$\log^{[p+m-q]} M_{fog} (r) \geq (\lambda_g (m, n) - \varepsilon) \log^{[n]} r + O(1)$$

$$\limsup_{r \to \infty} \frac{\log^{[p+m-q]} M_{fog} (r)}{\log^{[n]} r} \geq \lambda_g (m, n) .$$

(6.16)

Therefore from (6.14) and (6.15), we obtain for $$\lambda_f (p, q) > 0$$ that

$$\rho_g (m, n) \leq \frac{\log^{[p+m-q]} M_{fog} (r)}{\log^{[n]} r} \leq \rho_g (m, n)$$

i.e.,

$$\rho_{fog} (p + m - q, n) = \rho_g (m, n) .$$

(6.17)

Similarly, from (6.14) and (6.16) we get for $$\lambda_g (m, n) > 0$$ that

$$\lambda_g (m, n) \leq \limsup_{r \to \infty} \frac{\log^{[p+m-q]} M_{fog} (r)}{\log^{[n]} r} \leq \rho_g (m, n)$$

i.e.,

$$\lambda_g (m, n) \leq \rho_{fog} (p + m - q, n) \leq \rho_g (m, n) .$$

(6.18)

So from (6.17) and (6.18), one can easily verify that $$\rho_{fog} (p + m - q - 1, n) = \infty, \rho_{fog} (p + m - q, n - 1) = 0$$ and $$\rho_{fog} (p + m - q + 1, n + 1) = 1$$ and therefore we obtain that the index-pair of $$fog$$ is $$(p + m - q, n)$$ when $$q < m$$ and either $$\lambda_f (p, q) > 0$$ or $$\lambda_g (m, n) > 0$$ and thus the third part of the theorem is established. ■

Remark 6.3.1 Theorem 6.3.1 can be treated as an extension of Theorem 3.1 and Theorem 3.2 of Tu, Chen and Zheng [70].

Theorem 6.3.2 Let $$f$$ and $$g$$ be any two entire functions with index-pairs $$(p, q)$$ and $$(m, n)$$ respectively where $$p, q, m, n$$ are all positive integers such that $$p \geq q$$ and $$m \geq n$$. Then

(i) $$\lambda_f (p, q) \cdot \lambda_g (m, n) \leq \lambda_{fog} (p, n)$$

$$\leq \min \{ \rho_f (p, q) \cdot \lambda_g (m, n), \lambda_f (p, q) \cdot \rho_g (m, n) \}$$

if $$q = m$$, $$\lambda_f (p, q) > 0$$ and $$\lambda_g (m, n) > 0;$$
(ii) \( \lambda_{f \circ g} (p, q + n - m) = \lambda_f (p, q) \)
\[
\text{if } q > m, \text{ } \lambda_f (p, q) > 0 \text{ and } \lambda_g (m, n) > 0
\]

and

(iii) \( \lambda_{f \circ g} (p + m - q, n) = \lambda_g (m, n) \)
\[
\text{if } q < m, \text{ } \lambda_f (p, q) > 0 \text{ and } \lambda_g (m, n) > 0 .
\]

In the line of Theorem 6.3.1, one can easily deduce the conclusion of Theorem 6.3.2 and so its proof is omitted.

**Theorem 6.3.3** Let \( f, g, h \) and \( k \) be any four entire functions with index-pairs \( (p, q), (m, n), (a, b) \) and \( (c, d) \) respectively where \( a, b, c, d, p, q, m, n \) are all positive integers such that \( a \geq b, c \geq d, p \geq q \) and \( m \geq n \).

(i) If either \((q = m, a = c = p, q \geq n)\) or \((q < m, c = p, a = p + m - q, q \geq n)\) holds and \( \lambda_f (p, q) > 0, 0 < \lambda_{h, (b, n)}^\left( f \circ g \right) \leq \rho_{h, (b, n)}^\left( f \circ g \right) < \infty, 0 < \lambda_{k, (d, q)}^\left( f \circ g \right) \leq \rho_{k, (d, q)}^\left( f \circ g \right) < \infty \)

\[
\frac{\lambda_{h, (b, n)}^\left( f \circ g \right)}{\rho_{k, (d, q)}^\left( f \circ g \right)} \leq \liminf_{r \to \infty} \frac{\log \left[ b \right] M_{h, (b, n)}^\left( f \circ g \right)}{\log \left[ d \right] M_{k, (d, q)}^\left( f \circ g \right)} \leq \frac{\lambda_{h, (b, n)}^\left( f \circ g \right)}{\lambda_{k, (d, q)}^\left( f \circ g \right)}
\]

and

(ii) If \( q > m, a = c = p, \lambda_f (p, q) > 0, 0 < \lambda_{h, (b, q + n - m)}^\left( f \circ g \right) \leq \rho_{h, (b, q + n - m)}^\left( f \circ g \right) \leq \infty \) and \( 0 < \lambda_{k, (d, q)}^\left( f \circ g \right) \leq \rho_{k, (d, q)}^\left( f \circ g \right) \leq \infty \)

\[
\frac{\lambda_{h, (b, q + n - m)}^\left( f \circ g \right)}{\rho_{k, (d, q)}^\left( f \circ g \right)} \leq \liminf_{r \to \infty} \frac{\log \left[ b \right] M_{h, (b, q + n - m)}^\left( f \circ g \right)}{\log \left[ d \right] M_{k, (d, q)}^\left( f \circ g \right)} \leq \frac{\lambda_{h, (b, q + n - m)}^\left( f \circ g \right)}{\lambda_{k, (d, q)}^\left( f \circ g \right)}
\]
Proof. Let either \((q = m, a = c = p, q \geq n)\) or \((q < m, c = p, a = p + m - q, q \geq n)\) hold and \(\lambda_f (p,q) > 0\). Then in view of Theorem \[6.3.1\], the index-pair of \(f \circ g\) is \((p,n)\) or \((p + m - q, n)\) respectively and therefore by definition of \(\rho^{(b,n)}_h (f \circ g)\) \(\left(\text{respectively } \lambda^{(b,n)}_h (f \circ g)\right)\) and \(\rho^{(d,q)}_k (f)\) \(\left(\text{respectively } \lambda^{(d,q)}_k (f)\right)\) exist.

Now from the definitions of \(\rho^{(d,q)}_k (f)\) \(\text{and } \lambda^{(b,n)}_h (f \circ g)\), we have for arbitrary positive \(\varepsilon\) and for all sufficiently large values of \(r\) that

\[
\log^{[b]} M^{-1}_h M_{f \circ g} (r) \geq \left(\lambda^{(b,n)}_h (f \circ g) - \varepsilon\right) \log^n r \tag{6.19}
\]

and

\[
\log^{[d]} M^{-1}_k M_f \left(\exp^{[q-n]} r\right) \leq \left(\rho^{(d,q)}_k (f) + \varepsilon\right) \log^n r . \tag{6.20}
\]

Now from \eqref{6.19} and \eqref{6.20}, it follows for all sufficiently large values of \(r\) that

\[
\frac{\log^{[b]} M^{-1}_h M_{f \circ g} (r)}{\log^{[d]} M^{-1}_k M_f \left(\exp^{[q-n]} r\right)} \geq \frac{\left(\lambda^{(b,n)}_h (f \circ g) - \varepsilon\right) \log^n r}{\left(\rho^{(d,q)}_k (f) + \varepsilon\right) \log^n r} .
\]

As \(\varepsilon (>0)\) is arbitrary, we obtain that

\[
\liminf_{r \to \infty} \frac{\log^{[b]} M^{-1}_h M_{f \circ g} (r)}{\log^{[d]} M^{-1}_k M_f \left(\exp^{[q-n]} r\right)} \geq \frac{\lambda^{(b,n)}_h (f \circ g)}{\rho^{(d,q)}_k (f)} . \tag{6.21}
\]

Again, we get for a sequence of values of \(r\) tending to infinity that

\[
\log^{[b]} M^{-1}_h M_{f \circ g} (r) \leq \left(\lambda^{(b,n)}_h (f \circ g) + \varepsilon\right) \log^n r \tag{6.22}
\]

and for all sufficiently large values of \(r\) that

\[
\log^{[d]} M^{-1}_k M_f \left(\exp^{[q-n]} r\right) \geq \left(\lambda^{(d,q)}_k (f) - \varepsilon\right) \log^n r . \tag{6.23}
\]

Combining \eqref{6.22} and \eqref{6.23}, we obtain for a sequence of values of \(r\) tending to infinity that

\[
\frac{\log^{[b]} M^{-1}_h M_{f \circ g} (r)}{\log^{[d]} M^{-1}_k M_f \left(\exp^{[q-n]} r\right)} \leq \frac{\left(\lambda^{(b,n)}_h (f \circ g) + \varepsilon\right) \log^n r}{\left(\lambda^{(d,q)}_k (f) - \varepsilon\right) \log^n r} .
\]
Since \( \varepsilon (>0) \) is arbitrary, it follows that
\[
\liminf_{r \to \infty} \frac{\log^[b] M_h^{-1} M_{fog}(r)}{\log^[d] M_k^{-1} M_f \left( \exp^{[q-n]} r \right)} \leq \frac{\lambda_h^{(b,n)}(f \circ g)}{\lambda_k^{(d,q)}(f)}.
\] (6.24)

Also, for a sequence of values of \( r \) tending to infinity,
\[
\log^[d] M_k^{-1} M_f \left( \exp^{[q-n]} r \right) \leq \left( \lambda_k^{(d,q)}(f) + \varepsilon \right) \log^[n] r.
\] (6.25)

Now, from (6.19) and (6.25), we obtain for a sequence of values of \( r \) tending to infinity that
\[
\frac{\log^[b] M_h^{-1} M_{fog}(r)}{\log^[d] M_k^{-1} M_f \left( \exp^{[q-n]} r \right)} \geq \frac{\left( \lambda_h^{(b,n)}(f \circ g) - \varepsilon \right)}{\left( \lambda_k^{(d,q)}(f) + \varepsilon \right)} \log^[n] r.
\]

As \( \varepsilon (>0) \) is arbitrary, we get from above that
\[
\limsup_{r \to \infty} \frac{\log^[b] M_h^{-1} M_{fog}(r)}{\log^[d] M_k^{-1} M_f \left( \exp^{[q-n]} r \right)} \geq \frac{\lambda_h^{(b,n)}(f \circ g)}{\lambda_k^{(d,q)}(f)}.
\] (6.26)

Also, we obtain for all sufficiently large values of \( r \) that
\[
\log T_h^{-1} T_{fog}(r) \leq \left( \rho_h^{(b,n)}(f \circ g) + \varepsilon \right) \log^[n] r.
\] (6.27)

Now, it follows from (6.23) and (6.27) for all sufficiently large values of \( r \) that
\[
\frac{\log^[b] M_h^{-1} M_{fog}(r)}{\log^[d] M_k^{-1} M_f \left( \exp^{[q-n]} r \right)} \leq \frac{\left( \rho_h^{(b,n)}(f \circ g) + \varepsilon \right)}{\left( \lambda_k^{(d,q)}(f) - \varepsilon \right)} \log^[n] r.
\]

Since \( \varepsilon (>0) \) is arbitrary, we obtain that
\[
\limsup_{r \to \infty} \frac{\log^[b] M_h^{-1} M_{fog}(r)}{\log^[d] M_k^{-1} M_f \left( \exp^{[q-n]} r \right)} \leq \frac{\rho_h \left( f \rho_h^{(b,n)}(f \circ g) \circ g \right)}{\lambda_k^{(d,q)}(f)}.
\] (6.28)

Thus the first part of the theorem follows from (6.21), (6.24), (6.26) and (6.28). Similarly, one can easily derive the second part of the theorem. ■

The following theorem can be proved in the line of Theorem 6.3.3 and so its proof is omitted.
Theorem 6.3.4 Let \( f, g, h \) and \( l \) be any four entire functions with index-pairs \((p, q), (m, n), (a, b)\) and \((x, y)\) respectively where \( a, b, p, q, m, n, x, y \) are all positive integers such that \( a \geq b, p \geq q, m \geq n \) and \( x \geq y \).

(i) If either \((q = m = x, a = p)\) or \((q < m = x, a = p + m - q)\) holds and 
\( \lambda_{g}(m, n) > 0, 0 < \lambda_{h}^{(b,n)}(f \circ g) \leq \rho_{h}^{(b,n)}(f \circ g) < \infty, 0 < \lambda_{l}^{(y,n)}(g) \leq \rho_{l}^{(y,n)}(g) < \infty \) then

\[
\frac{\lambda_{h}^{(b,n)}(f \circ g)}{\rho_{l}^{(y,n)}(g)} \leq \liminf_{r \to \infty} \frac{\log^{[b]} M_{h}^{-1} M_{f \circ g}(r)}{\log^{[y]} M_{l}^{-1} M_{g}(r)} \leq \frac{\lambda_{h}^{(b,n)}(f \circ g)}{\lambda_{l}^{(y,n)}(g)}
\]

\[
\leq \limsup_{r \to \infty} \frac{\log^{[b]} M_{h}^{-1} M_{f \circ g}(r)}{\log^{[y]} M_{l}^{-1} M_{g}(r)} \leq \frac{\rho_{h}^{(b,n)}(f \circ g)}{\lambda_{l}^{(y,n)}(g)}
\]

and

(ii) If \( q > m = x, a = p, \lambda_{g}(m, n) > 0, 0 < \lambda_{h}^{(b,q+n-m)}(f \circ g) \leq \rho_{h}^{(b,q+n-m)}(f \circ g) < \infty, 0 < \lambda_{l}^{(y,n)}(g) \leq \rho_{l}^{(y,n)}(g) < \infty \) then

\[
\frac{\lambda_{h}^{(b,q+n-m)}(f \circ g)}{\rho_{l}^{(y,n)}(g)} \leq \liminf_{r \to \infty} \frac{\log^{[b]} M_{h}^{-1} M_{f \circ g}(\exp^{[q-m]} r)}{\log^{[y]} M_{l}^{-1} M_{g}(r)} \leq \frac{\lambda_{h}^{(b,q+n-m)}(f \circ g)}{\lambda_{l}^{(y,n)}(g)}
\]

\[
\leq \limsup_{r \to \infty} \frac{\log^{[b]} M_{h}^{-1} M_{f \circ g}(\exp^{[q-m]} r)}{\log^{[y]} M_{l}^{-1} M_{g}(r)} \leq \frac{\rho_{h}^{(b,q+n-m)}(f \circ g)}{\lambda_{l}^{(y,n)}(g)}
\]

Theorem 6.3.5 Let \( f, g, h \) and \( k \) be any four entire functions with index-pairs \((p, q), (m, n), (a, b)\) and \((c, d)\) respectively where \( a, b, c, d, p, q, m, n \) are all positive integers with \( a \geq b, c \geq d, p \geq q \) and \( m \geq n \).

(i) If either \((q = m, a = c = p, q \geq n)\) or \((q < m, c = p, a = p + m - q, q \geq n)\) holds and \( \lambda_{f}(p, q) > 0, 0 < \rho_{h}^{(b,n)}(f \circ g) < \infty, 0 < \rho_{k}^{(d,q)}(f) < \infty \) then

\[
\liminf_{r \to \infty} \frac{\log^{[b]} M_{h}^{-1} M_{f \circ g}(r)}{\log^{[d]} M_{k}^{-1} M_{f}(\exp^{[q-n]} r)} \leq \frac{\rho_{h}^{(b,n)}(f \circ g)}{\rho_{k}^{(d,q)}(f)}
\]

\[
\leq \limsup_{r \to \infty} \frac{\log^{[b]} M_{h}^{-1} M_{f \circ g}(r)}{\log^{[d]} M_{k}^{-1} M_{f}(\exp^{[q-n]} r)}
\]
and

(ii) If \( q > m \), \( a = c = p, \lambda_f(p, q) > 0, 0 < \rho_{h^{(b,q+n-m)}}(f \circ g) < \infty \) and \( 0 < \rho_{k^{(d,q)}}(f) < \infty \) then

\[
\liminf_{r \to \infty} \frac{\log[b] M_{h^{-1} M_f \circ g}(r)}{\log[d] M_{k^{-1} M_f} \left( \exp^{[m-n]} r \right)} \leq \frac{\rho_{h^{(b,q+n-m)}}(f \circ g)}{\rho_{k^{(d,q)}}(f)} \leq \limsup_{r \to \infty} \frac{\log[b] M_{h^{-1} M_f \circ g}(r)}{\log[d] M_{k^{-1} M_f} \left( \exp^{[m-n]} r \right)}.
\]

**Proof.** Let either \((q = m, a = c = p, q \geq n)\) or \((q < m, c = p, a = p + m - q, q \geq n)\) hold and also \( \lambda_f(p, q) > 0 \). Therefore in view of Theorem 6.3.1, the index-pair of \( f \circ g \) is \((p, n)\) or \((p + m - q, n)\) respectively. Hence by Definition 2.2.9, \( \rho_{h^{(b,n)}}(f \circ g) \) and \( \rho_{k^{(d,q)}}(f) \) exist.

Now from the definition of \( \rho_{k^{(d,q)}}(f) \), we get for a sequence of values of \( r \) tending to infinity that

\[
\log[d] M_{k^{-1} M_f} \left( \exp^{[q-n]} r \right) \geq \left( \rho_{k^{(d,q)}}(f) - \varepsilon \right) \log[n] r
\]

i.e.,

\[
\log T_{P[b]}^{-1} T_{P[f]}(r) \geq \left( \rho_{k^{(d,q)}}(f) - \varepsilon \right) \log[n] r. \tag{6.29}
\]

Now from (6.27) and (6.29), it follows for a sequence of values of \( r \) tending to infinity that

\[
\frac{\log[b] M_{h^{-1} M_f \circ g}(r)}{\log[d] M_{k^{-1} M_f} \left( \exp^{[q-n]} r \right)} \leq \frac{\left( \rho_{h^{(b,n)}}(f \circ g) + \varepsilon \right) \log[n] r}{\left( \rho_{k^{(d,q)}}(f) - \varepsilon \right) \log[n] r}.
\]

As \( \varepsilon (> 0) \) is arbitrary, we obtain that

\[
\liminf_{r \to \infty} \frac{\log[b] M_{h^{-1} M_f \circ g}(r)}{\log[d] M_{k^{-1} M_f} \left( \exp^{[q-n]} r \right)} \leq \frac{\rho_{h^{(b,n)}}(f \circ g)}{\rho_{k^{(d,q)}}(f)} \leq \limsup_{r \to \infty} \frac{\log[b] M_{h^{-1} M_f \circ g}(r)}{\log[d] M_{k^{-1} M_f} \left( \exp^{[q-n]} r \right)}.
\]

(6.30)

Again, we obtain for a sequence of values of \( r \) tending to infinity that

\[
\log[b] M_{h^{-1} M_f \circ g}(r) \geq \left( \rho_{h^{(b,n)}}(f \circ g) - \varepsilon \right) \log[n] r. \tag{6.31}
\]
So combining (6.20) and (6.31), we get for a sequence of values of $r$ tending to infinity that

$$\frac{\log^b M_h^{-1} f \circ g (r)}{\log^d M_k^{-1} f \left( \exp^{[q-n]} r \right)} \geq \frac{\left( \rho_h^{(b,n)} (f \circ g) - \varepsilon \right) \log^n r}{\left( \rho_k^{(d,q)} (f) + \varepsilon \right) \log^n r}.$$  

Since $\varepsilon (> 0)$ is arbitrary, it follows that

$$\limsup_{r \to \infty} \frac{\log^b M_h^{-1} f \circ g (r)}{\log^d M_k^{-1} f \left( \exp^{[q-n]} r \right)} \geq \frac{\rho_h^{(b,n)} (f \circ g)}{\rho_k^{(d,q)} (f)}.$$  

(6.32)

Thus the first part of the theorem follows from (6.30) and (6.32). Analogously, the second part of the theorem can be derived in a like manner. ■

The following theorem can be carried out in the line of Theorem 6.3.5 and therefore we omit its proof.

**Theorem 6.3.6** Let $f, g, h$ and $l$ be any four entire functions with index-pairs $(p, q), (m, n), (a, b)$ and $(x, y)$ respectively where $a, b, p, q, m, n, x, y$ are all positive integers such that $a \geq b$, $p \geq q$, $m \geq n$ and $x \geq y$.

(i) If either $(q = m = x, a = p)$ or $(q < m = x, a = p + m - q)$ holds and $\lambda_g (m, n) > 0, 0 < \rho_h^{(b,n)} (f \circ g) < \infty, 0 < \rho_l^{(y,n)} (g) < \infty$ then

$$\liminf_{r \to \infty} \frac{\log^b M_h^{-1} f \circ g (r)}{\log^y M_l^{-1} g (r)} \leq \frac{\rho_h^{(b,n)} (f \circ g)}{\rho_l^{(y,n)} (g)} \leq \limsup_{r \to \infty} \frac{\log^b M_h^{-1} f \circ g (r)}{\log^y M_l^{-1} g (r)}$$

and

(ii) if $q > m = x, a = p, \lambda_g (m, n) > 0, 0 < \rho_h^{(b,q+n-m)} (f \circ g) < \infty, 0 < \rho_l^{(y,n)} (g) < \infty$ then

$$\liminf_{r \to \infty} \frac{\log^b M_h^{-1} f \circ g \left( \exp^{[q-m]} r \right)}{\log^y M_l^{-1} g (r)} \leq \frac{\rho_h^{(b,q+n-m)} (f \circ g)}{\rho_l^{(y,n)} (g)} \leq \limsup_{r \to \infty} \frac{\log^b M_h^{-1} f \circ g \left( \exp^{[q-m]} r \right)}{\log^y M_l^{-1} g (r)}.$$
The following theorem is a natural consequence of Theorem 6.3.3 and Theorem 6.3.5:

**Theorem 6.3.7** Let \( f, g, h \) and \( k \) be any four entire functions with index-pairs \((p, q), (m, n), (a, b)\) and \((c, d)\) respectively where \( a, b, c, d, p, q, m, n \) are all positive integers such that \( a \geq b, c \geq d, p \geq q \) and \( m \geq n \).

(i) If either \((q = m, a = c = p, q \geq n)\) or \((q < m, c = p, a = p + m - q, q \geq n)\) holds and \( \lambda_f (p, q) > 0, 0 < \lambda_h^{(b, n)} (f \circ g) \leq \rho_h^{(b, n)} (f \circ g) < \infty, 0 < \lambda_h^{(d, q)} (f) \leq \rho_k^{(d, q)} (f) < \infty \) then

\[
\liminf_{r \to \infty} \frac{\log[b] M_{h^{-1}M_{f \circ g}} (r)}{\log[d] M_{k^{-1}M_{f}} (\exp[q-n] r)} \leq \min \left\{ \frac{\lambda_h^{(b, n)} (f \circ g)}{\lambda_k^{(d, q)} (f)}, \frac{\rho_h^{(b, n)} (f \circ g)}{\rho_k^{(d, q)} (f)} \right\}
\]

\[
\leq \max \left\{ \frac{\lambda_h^{(b, n)} (f \circ g)}{\lambda_k^{(d, q)} (f)}, \frac{\rho_h^{(b, n)} (f \circ g)}{\rho_k^{(d, q)} (f)} \right\}
\]

\[
\leq \limsup_{r \to \infty} \frac{\log[b] M_{h^{-1}M_{f \circ g}} (r)}{\log[d] M_{k^{-1}M_{f}} (\exp[q-n] r)}
\]

and

(ii) if \( q > m, a = c = p, \lambda_f (p, q) > 0, 0 < \lambda_h^{(b, q+n-m)} (f \circ g) \leq \rho_h^{(b, q+n-m)} (f \circ g) < \infty \) and \( 0 < \lambda_h^{(d, q)} (f) \leq \rho_k^{(d, q)} (f) < \infty \) then

\[
\liminf_{r \to \infty} \frac{\log[b] M_{h^{-1}M_{f \circ g}} (r)}{\log[d] M_{k^{-1}M_{f}} (\exp[m-n] r)} \leq \min \left\{ \frac{\lambda_h^{(b, q+n-m)} (f \circ g)}{\lambda_k^{(d, q)} (f)}, \frac{\rho_h^{(b, q+n-m)} (f \circ g)}{\rho_k^{(d, q)} (f)} \right\}
\]
\[ \leq \max \left\{ \frac{\lambda_h^{(b, q+n-m)} (f \circ g)}{\lambda_k^{(d,q)} (f)}, \frac{\rho_h^{(b, q+n-m)} (f \circ g)}{\rho_k^{(d,q)} (f)} \right\} \]

\[ \leq \limsup_{r \to \infty} \frac{\log[b] M_h^{-1} M_{f \circ g} (r)}{\log[d] M_k^{-1} M_f (\exp[m-n] r)}. \]

The proof is omitted.

Analogously, one may state the following theorem without its proof.

**Theorem 6.3.8** Let \( f, g, h \) and \( l \) be any four entire functions with index-pairs \((p, q), (m, n), (a, b)\) and \((x, y)\) respectively where \( a, b, p, q, m, n, x, y \) are all positive integers such that \( a \geq b, p \geq q, m \geq n \) and \( x \geq y \).

(i) If either \((q = m = x, a = p)\) or \((q < m = x, a = p + m - q)\) holds and \( \lambda_g (m, n) > 0, 0 < \lambda_h^{(b, n)} (f \circ g) \leq \rho_h^{(b, n)} (f \circ g) < \infty, 0 < \lambda_l^{(y, n)} (g) \leq \rho_l^{(y, n)} (g) < \infty \) then

\[ \liminf_{r \to \infty} \frac{\log[b] M_h^{-1} M_{f \circ g} (r)}{\log[y] M_l^{-1} M_g (r)} \leq \min \left\{ \frac{\lambda_h^{(b, n)} (f \circ g)}{\lambda_l^{(y, n)} (g)}, \frac{\rho_h^{(b, n)} (f \circ g)}{\rho_l^{(y, n)} (g)} \right\} \]

\[ \leq \max \left\{ \frac{\lambda_h^{(b, n)} (f \circ g)}{\lambda_l^{(y, n)} (g)}, \frac{\rho_h^{(b, n)} (f \circ g)}{\rho_l^{(y, n)} (g)} \right\} \]

\[ \leq \limsup_{r \to \infty} \frac{\log[b] M_h^{-1} M_{f \circ g} (r)}{\log[y] M_l^{-1} M_g (r)} \]

and

(ii) if \( q > m = x, a = p, \lambda_g (m, n) > 0, 0 < \lambda_h^{(b, q+n-m)} (f \circ g) \leq \rho_h^{(b, q+n-m)} (f \circ g) < \infty, 0 < \lambda_l^{(y, n)} (g) \leq \rho_l^{(y, n)} (g) < \infty \) then

\[ \liminf_{r \to \infty} \frac{\log[b] M_h^{-1} M_{f \circ g} (\exp[q-m] r)}{\log[y] M_l^{-1} M_g (r)} \leq \min \left\{ \frac{\lambda_h^{(b, q+n-m)} (f \circ g)}{\lambda_l^{(y, n)} (g)}, \frac{\rho_h^{(b, q+n-m)} (f \circ g)}{\rho_l^{(y, n)} (g)} \right\} \]
\[
\leq \max \left\{ \frac{\lambda_h^{(b,q+n-m)} (f \circ g)}{\lambda_l^{(y,n)} (g)} \cdot \frac{\rho_h^{(b,q+n-m)} (f \circ g)}{\rho_l^{(y,n)} (g)} \right\} \\
\leq \limsup_{r \to \infty} \frac{\log^{[b]} M_h^{-1} M_f (g) \left( \exp^{[q-m]} r \right)}{\log^{[b]} M_l^{-1} M_g (r)}.
\]

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