4.1 INTRODUCTION

The term stuttering Poisson distribution was introduced by Galliher et al. (1959) in connection with a class of distributions generated by intermingling of a countable infinite number of streams of Poisson demands. Patel (1976) defined this distribution as follows:

A scalar random variable $W_1$ is said to have a $(k$-tuple) stuttering Poisson distribution (SPD) with parameters $\lambda_1, \ldots, \lambda_k (k > 1)$ if

$$P_x = P(W_1 = x) = \exp \left( - \sum_{j=1}^{k} \lambda_j \right) \sum_{i_1, \ldots, i_k} \prod_{j=1}^{k} \frac{\lambda_j^{i_j}}{i_j!}$$

(4.1.1)

where $\lambda_j > 0$ for $j = 1, \ldots, k$; $x = 0, 1, \ldots$ and $\sum$ denote the $k$-tuple sum summed over the set $I_1 = \{(i_1, \ldots, i_k) : \sum_{j=1}^{k} j i_j = x\}$ of $k$-tuple of non-negative integers.

The SPD is also known in the literature as the extended Poisson distribution of order $k$, as per Aki (1985). The probability generating function (p.g.f.) of SPD defined in (4.1.1) is the following.

$$P_1(t) = \exp \left\{ \sum_{j=1}^{k} \lambda_j (t^j - 1) \right\}$$

(4.1.2)
If we set \( \lambda_1 = \lambda_2 = \cdots = \lambda_k = \lambda/k \), then we get the Poisson distribution of order \( k \), introduced by Philippou, Georghiou and Philippou (1983).

The SPD is an extension of Hermite distribution, introduced and studied by Kemp and Kemp (1965), which has found applications in physical sciences, biological sciences and operations research. Certain properties of Hermite distribution were studied by Kemp and Kemp (1966), Patel (1985) and Kumar (1990). Gupta and Jain (1974) considered another generalization of the Hermite distribution.

From (4.1.2) it is obvious that \( P_1(t) \) is the p.g.f. of \( \sum_{r=1}^{k} rV_r \), where \( V_r \) is distributed as Poisson with parameter \( \lambda_r \), \( r = 1, \ldots, k \) and \( V_1, \ldots, V_k \) are independent. By compounding different parameters of the SPD, Patil and Raghunandanan (1990) derived stuttering Poisson gamma, stuttering Poisson reciprocal gamma and stuttering Poisson negative binomial distributions. Certain applications of SPD in operations research problems are discussed in Adelson (1966) and in the area of statistical quality control are discussed in Maritz (1952). SPD has been found useful in many scientific areas including accident statistics and ecology.

The results presented in this chapter are mainly based on Moothatha and Kumar (1995). In Section 4.2 we establish that the SPD can be obtained as the distribution of a random sum of certain types of independent and identically distributed discrete random variables. The Poisson binomial distribution is shown to be a special case of SPD. In Sections 4.3 and 4.4 certain recurrence relations for probabilities and factorial moments are derived.
4.2 A MODEL LEADING TO SPD

For a fixed positive integer \( k \), let \( \nu_j = \lambda_j/d \), where \( \lambda_j > 0 \), \( j = 0, 1, \ldots, k \) and \( d = \sum_{j=0}^{k} \lambda_j \). Let \( N \) be a Poisson random variable with parameter \( d > 0 \), with the following p.g.f.

\[
P_N(t) = \exp[d(t-1)]
\]

Consider the sequence \( \{Y_n, n \geq 1\} \) of independent and identically distributed random variables, where \( Y_1 \) is assumed to have the following p.g.f.

\[
\eta(t) = \sum_{j=0}^{k} \nu_j t^j
\]

We assume that \( \{Y_n, n \geq 1\} \) and \( N \) are independent. Define \( S_0 = 0 \) and \( S_m = \sum_{k=1}^{m} Y_k \) for each \( m \geq 1 \). Set \( S_N = \sum_{n=0}^{\infty} S_n I_{[N=n]} \), where \( I_E \) denotes the indicator function of an event \( E \). Then the p.g.f. of \( S_N \) is given below.

\[
P_{S_N}(t) = \mathbb{E}(t^{S_N})
\]

\[
= \sum_{n=0}^{\infty} \mathbb{E}\left(t^{S_N} | N = n\right) P(N = n)
\]

\[
= \sum_{n=0}^{\infty} [\eta(t)]^n P(N = n)
\]

\[
= P_N[\eta(t)]
\]

\[
= \exp\left\{ \sum_{j=1}^{k} \lambda_j (t^j - 1) \right\}
\]

Obviously \( P_{S_N}(t) \) is the p.g.f. of a SPD.

**REMARK 4.2.1.** Consider the special case that

\[
\lambda_j = d \binom{k}{j} p^j (1-p)^{k-j} \quad \text{for } j = 0, 1, \ldots, k
\]
where \( d > 0, 0 < p < 1 \). Then \( Y_1 \) is distributed as binomial with parameters \( p \) and \( k \). Consequently \( S_N \) has the Poisson binomial distribution. Thus SPD is a generalization of Poisson binomial distribution.

### 4.3 Recurrence Relation for Probabilities

For \( k > 1 \) and \( x_m \geq 0, m = 1, \ldots, k-1 \) we consider the class of functions \( P_n(x) = P_n(x_1, \ldots, x_{k-1}), n = 0, 1, \ldots \), defined by the following generating function.

\[
g(x; t) = \exp \left[ 2 \sum_{m=1}^{k-1} x_m \lambda_m t^m + \lambda_k t^k \right]
= \sum_{r=0}^{\infty} P_r(x) t^r \tag{4.3.1}
\]

Our interest in the class of functions \( P_n(x), n \geq 0 \) arises because of the recurrence relations (4.3.3) to (4.3.5) and (4.4.2) we are able to establish in the light of the following. For a random variable \( W_1 \) with p.g.f. (4.1.2), \( P_n = P(W_1 = n) \) satisfies

\[
P_n = P_n \left( \frac{1}{2}, \ldots, \frac{1}{2} \right) \exp (-d). \tag{4.3.2}
\]

In deriving the following results, we have adapted the literature of Hermite polynomials, discussed for example in Lebedev (1965) and Saksena, Kazim and Pathan (1972). In what follows \( P_{-r}(x) \) stands for zero for a positive integer \( r \).

**Result 4.3.1.** The following is a simple recurrence relation for \( P_n(x) \), \( n \geq 0 \).

\[
(n + 1)P_{n+1}(x) = 2 \sum_{m=1}^{k-1} mx_m \lambda_m P_{n-m+1}(x) + k \lambda_k P_{n-k+1}(x) \tag{4.3.3}
\]
Proof. Consider the following obtainable from (4.3.1), on differentia-
tion with respect to \( t \).

\[
\frac{\partial g(x,t)}{\partial t} = \left[ 2 \sum_{m=1}^{k-1} mx_m \lambda_m t^{m-1} + k \lambda k t^{k-1} \right] g(x,t)
\]

\[
= \sum_{r=0}^{\infty} (r+1) P_{r+1}(x) t^r.
\]

By using (4.3.1) this leads to the following:

\[
\sum_{r=0}^{\infty} (r+1) P_{r+1}(x) t^r = 2 \sum_{r=0}^{\infty} \sum_{m=1}^{k-1} mx_m \lambda_m P_r(x) t^{m+r-1}
\]

\[
+ k \lambda k \sum_{r=0}^{\infty} P_r(x) t^{r+k-1}
\]

Equating coefficients of \( t^n \) on both sides, we get the relation (4.3.3).

\[ \square \]

**COROLLARY 4.3.1.** We have the following simple recurrence relation for
probabilities \( P_n \) of SPD.

\[
(n + 1) P_{n+1} = \sum_{m=1}^{k} m \lambda_m P_{n-m+1}, \tag{4.3.4}
\]

in which for any positive integer \( r \), \( P_{-r} = 0 \).

Proof. In (4.3.3) put \( x_m = \frac{1}{z} \) for \( m = 1, \ldots, k - 1 \), multiply throughout
by \( \exp \left( - \sum_{j=1}^{k} \lambda_j \right) \) and use the relation (4.3.2). \[ \square \]

Patel (1976) has given the recurrence relation (4.3.4) without an
explicit proof of the result. In the light of (4.3.2) and (4.3.4) we obtain
successively the following.

\[ P_0 = e^{-d} \]
\[ P_1 = \lambda_1 e^{-d} \]
\[ P_2 = e^{-d} (\lambda_1^2 + 2\lambda_2)/2 \]
\[ P_3 = e^{-d} [\lambda_1^3 + 6(\lambda_1 \lambda_2 + \lambda_3)]/6 \]
\[ P_4 = e^{-d} [\lambda_1^4 + 12(\lambda_1^2 \lambda_2 + \lambda_2^2) + 24(\lambda_1 \lambda_3 + \lambda_4)]/24 \]
\[ P_5 = e^{-d} [\lambda_1^5 + 20\lambda_1^3 \lambda_2 + 60(\lambda_1^2 \lambda_3 + \lambda_1 \lambda_2^2) + 120(\lambda_1 \lambda_4 + \lambda_2 \lambda_3 + \lambda_5)]/120 \]

**RESULT 4.3.2.** Let \( \mathbf{x}^* \) denote a \( 1 \times (k-1) \) vector with each co-ordinate a real positive number \( x \). Let \( P'_n(\mathbf{x}^*) \) denote the differential coefficient of \( P_n(\mathbf{x}^*) \) with respect to \( x \). Then we have the following differential recurrence relation for \( n \geq 1 \).

\[
P'_n(\mathbf{x}^*) = 2 \sum_{m=1}^{k-1} \lambda_m P_{n-m}(\mathbf{x}^*) \tag{4.3.5}
\]

**Proof.** In (4.3.1) replace \( x \) by \( \mathbf{x}^* \) and differentiate with respect to \( x \) to obtain the following.

\[
\sum_{r=0}^{\infty} P'_r(\mathbf{x}^*) t^r = \left[ 2 \sum_{m=1}^{k-1} \lambda_m t^m \right] \cdot g(\mathbf{x}^*; t)
\]

\[
= 2 \sum_{r=0}^{\infty} \sum_{m=1}^{k-1} \lambda_m P_r(\mathbf{x}^*) t^{r+m}
\]

Equating coefficients of \( t^n \) on both sides we get (4.3.5). \( \square \)
4.4 Recurrence Relation for Factorial Moments

The factorial moment generating function $F(t)$ of $k$-tuple SPD with p.g.f. $P_I(t)$ given in (4.1.2) is the following:

$$F(t) = P_I(1 + t)$$

$$= \exp \left\{ \sum_{j=1}^{k} \sum_{m=1}^{j} \lambda_j \binom{j}{m} t^m \right\}$$

$$= \exp \left\{ \sum_{m=1}^{k} \sum_{j=m}^{k} \lambda_j \binom{j}{m} t^m \right\}$$

(4.4.1)

Then we have the following.

RESULT 4.4.1. Let $\mu_{[r]}$ denote the $r$-th factorial moment of $SPD$ with p.g.f. (4.1.2). Then the following is a recurrence relation for factorial moments of $SPD$ for $n \geq k - 1$.

$$\mu_{[n+1]} = \sum_{m=1}^{k} \sum_{j=m}^{k} m \lambda_j \binom{j}{m} \left\{ \frac{n!}{(n-m+1)!} \right\} \mu_{[n-m+1]}$$

(4.4.2)

Proof. From (4.3.1) we have the following:

$$\exp \left\{ 2 \sum_{m=1}^{k-1} x_m \lambda_m t^m + \lambda_k t^k \right\} = \sum_{r=0}^{\infty} P_r(x) t^r$$

(4.4.3)

Let us define, $y_m = (2a_m)^{-1} \sum_{j=m}^{k} \lambda_j \binom{j}{m}$ for $m = 1, \ldots, k - 1$ and $y = (y_1, \ldots, y_{k-1})$. Then from (4.4.1) and (4.4.3) we obtain

$$F(t) = \exp \left\{ \sum_{m=1}^{k} \sum_{j=m}^{k} \lambda_j \binom{j}{m} t^m \right\} = \sum_{r=0}^{\infty} P_r(y) t^r.$$
Then equating the coefficients of \( r^r / r! \) on both sides of (4.4.4) we get

\[
\mu[r] = r!P_r(y). 
\] (4.4.5)

Multiplying relation (4.3.3) by \( n! \) and putting \( x_m = y_m \) for \( m = 1, \ldots, k-1 \) we obtain the following:

\[
(n + 1)!P_{n+1}(y) = \sum_{m=1}^{k-1} \sum_{j=m}^{k} m\lambda_j \binom{j}{m} n!P_{n-m+1}(y) + n!k\lambda_k P_{n-k+1}(y)
\]

This relation together with (4.4.5) implies (4.4.2).

**Remark 4.4.1.** In the light of Remark 4.2.1 if we put \( \lambda_j = d \binom{k}{j} p^j (1-p)^{k-j} \) for \( j = 0, 1, \ldots, k \) in (4.3.4) and (4.4.2), then we obtain the recurrence relations for probabilities and factorial moments respectively of the Poisson binomial distribution with parameters \( d, k \) and \( p \).