6. GRAPHS WITH CONNECTED ANTIMEDIAN AND MEDIAN SETS

6.1 Introduction

This chapter begins with the computational complexity of the following problem: Given a graph $G$, a subset $X$ of its vertices and a positive integer $\lambda$, whether $X$ is the level set for some profile $\pi$ on $G$ such that $X = \{x : D_G(x, \pi) \geq \lambda\}$? Even though there is an exponential number of arbitrarily large profiles in a graph it is proved, in Section 6.2, that the problem is of polynomial time complexity. A similar result for antimedian and median sets are also obtained. Using this approach, it is obtained in Section 6.3, that Cocktail-party graphs are special instances of graphs with connected antimedian sets. Motivated by these results we look for more graphs with connected antimedian sets for arbitrary profiles. For that Cartesian product of graphs is considered and it is shown that the property of having connected antimedian sets, is invariant under Cartesian product operation. Thus we deduce that Hamming graphs and hence hypercubes have connected antimedian sets for arbitrary profiles. Then graphs which admit scale-2 embedding into hypercubes are considered, and it is shown that two special instances of such graphs, halfcubes and Johnson graphs, have connected antimedian sets for arbitrary profiles.

6.2 Polynomial Time Recognition of Level Sets

Let $G$ be a graph on $n$ vertices and let $P$ be the set of all profiles on $V(G)$. Consider a special subset $P'$ of $P$ consists of profiles with no repetition of vertices. Note that
$|P'| = 2^n - 1$. Hence even in this special case the direct approach to the problem (mentioned in Section 6.1) is exponential. However we have:

**Theorem 6.1:** Let $G$ be a connected graph and $S \subseteq V(G)$. Then it can be determined in polynomial time whether $S$ is the level set of some profile on $G$, i.e. whether $S = \{s : D(s, \pi) \geq \lambda\}$.

**Proof.** Let $V(G) = \{v_1, \ldots, v_n\}$ and assume without loss of generality that $S = \{v_1, \ldots, v_k\}$ for some $1 \leq k \leq n$. For $i = 1, \ldots, n$ set

$$f_i(x_1, \ldots, x_n) = \sum_{j=1}^{n} d_G(v_i, v_j)x_j.$$ 

Note that if $x_i$, $1 \leq i \leq n$, is interpreted as the number of times the vertex $v_i$ appears in a given profile $\pi$, then $f_i(x_1, \ldots, x_n) = D(v_i, \pi)$. Consider the following linear program $\text{LP}(f)$:

$$\min \{x_1 + \cdots + x_n\};$$

$$x_j \geq 0, \ j = 1, \ldots, n;$$

$$f_j(x_1, \ldots, x_n) \geq \lambda; \ j = 1, \ldots, k;$$

$$f_j(x_1, \ldots, x_n) < \lambda; \ j = k+1, \ldots, n.$$ 

If the linear program $\text{LP}(f)$ has an integer solution then $(v_i$ with $x_i$ repetition) is the required profile.

Suppose next that $\text{LP}(f)$ has no integer solution. Therefore, by the above interpretation of $\text{LP}(f)$ we conclude that, for the given $\lambda$, $S$ is not a level set for any profile on $G$.

In conclusion, $S$ is a level set for a given integer $\lambda$ if and only if $\text{LP}(f)$ has an integer solution. Now, the number of inequalities of the LP is $n$ and the size of the coefficients is bounded by $n - 1$, the largest possible distance in a connected graph on $n$ vertices. Therefore LP can be solved in polynomial time and the theorem is proved. \qed
Note that using the same approach, by just reversing the inequalities in the constraints of LP we get:

**Theorem 6.2:** Let $G$ be a connected graph and $S \subseteq V(G)$. Then it can be determined in polynomial time whether $S$ is the level set of some profile on $G$, such that $S = \{s : D(s, \pi) \leq \lambda\}$.

Similarly we have:

**Theorem 6.3:** Let $G$ be a connected graph and $S \subseteq V(G)$. Then it can be determined in polynomial time whether $S$ is the antимедиаn set of some profile on $G$.

**Proof.** Here also we follow the same assumptions as in the previous proof and we consider the following linear program $LP(f)$:

\[
\begin{align*}
\min & \{x_1 + \cdots + x_n\}; \\
x_j & \geq 0, \ j = 1, \ldots, n; \\
f_i(x_1, \ldots, x_n) &= f_j(x_1, \ldots, x_n); \ j = 2, \ldots, k; \\
f_i(x_1, \ldots, x_n) &\geq f_j(x_1, \ldots, x_n) + 1; \ j = k+1, \ldots, n.
\end{align*}
\]

Suppose first that the linear program $LP(f)$ has a solution. Since the coefficients of the linear program are all integers, the solution $x_i^*$, $1 \leq j \leq n$, must be rational. Let $r$ be an integer such that $rx_i^*$ are integers. Then

\[
\begin{align*}
f_i(rx_1, \ldots, rx_n) &= \sum_{j=1}^{n} d_G(v_i, v_j)rx_j = r \sum_{j=1}^{n} d_G(v_i, v_j)x_j = rf_i(x_1, \ldots, x_n).
\end{align*}
\]

It follows that $(rx_1, \ldots, rx_n)$ is a profile for which $S$ is the antимедиаn set.

Suppose next that $LP(f)$ has no solution. In other words, the constraint conditions have no feasible solution. Therefore, by the above interpretation of $LP(f)$ we conclude that $S$ is not an antимедиаn set for any profile on $G$. 

In similar lines as in the proof of Theorem 6.2 we conclude that, $S$ is an antimedian set if and only if $LP(f)$ has a solution and that the $LP$ can be solved in polynomial time, which complete the proof.

Observe that the same approach, by just reversing the inequalities in the constraints of the LP, also works for median sets. We therefore also have:

**Theorem 6.4:** Let $G$ be a connected graph and $S \subseteq V(G)$. Then it can be determined in polynomial time whether $S$ is the median set of some profile on $G$.

### 6.3 Graphs with Connected Antimedian Sets

In this section we look for graphs with connected antimedian sets for arbitrary profiles. Note that this is clearly the case for complete graphs. It is shown in this section that cocktail-party graphs, hypercubes, Hamming graphs, halfcubes and Johnson graphs are other special instances of such classes of graphs. For cocktail-party graphs, an LP programming approach as in the previous section is used. Then the property of connectedness of antimedian sets are considered under the Cartesian product operation, from which the case of hypercubes and Hamming graphs are deduced. Using scale embedding into the hypercubes the case of halfcubes and Johnson graphs are also examined.

First, let us consider the cocktail-party graphs;

**Proposition 6.5:** Let $G$ be cocktail-party graph. Then any antimedian set of $G$ is connected.

**Proof.** We use the LP from the proof of Theorem 6.3. Let $V(G) = \{v_1, \ldots, v_n\}$ and assume without loss of generality that $M = \{v_1v_2, v_3v_4, \ldots, v_{2k-1}v_{2k}\}$ is the set of non-edges of $G$.

First observe that if $S \subset V(G)$ induces a disconnected subgraph, then $S = \{v_p, v_{p+1}\}$ for some fixed odd $p$, $1 \leq p \leq 2k - 1$. Suppose $S$ is the antimedian set of
some profile \( \pi \) on \( G \). Then the following LP

\[
\begin{align*}
\min \{ x_1 + \cdots + x_n \}; \\
x_i \geq 0, \ i = 1, \ldots, n; \\
f_p(x_1, \ldots, x_n) = f_{p+1}(x_1, \ldots, x_n); \\
j \in \{1, \ldots, n\}, j \neq p, p + 1,
\end{align*}
\]

has a solution, where

\[
f_j(x_1, \ldots, x_n) = 2x_{j+1} + \sum_{\ell \not\in \{j, j+1\}} x_\ell; \quad j = 1, 3, \ldots, 2k - 1,
\]

(6.1)

\[
f_j(x_1, \ldots, x_n) = 2x_{j-1} + \sum_{\ell \not\in \{j-1, j\}} x_\ell; \quad j = 2, 4, \ldots, 2k,
\]

(6.2)

and

\[
f_j(x_1, \ldots, x_n) = \sum_{\ell = 1 \atop \ell \neq j}^n x_\ell; \quad j = 2k + 1, 2k + 2, \ldots, n.
\]

(6.3)

Since

\[
f_p(x_1, \ldots, x_n) = 2x_{p+1} + \sum_{\ell \not\in \{p, p+1\}} x_\ell
\]

and

\[
f_{p+1}(x_1, \ldots, x_n) = 2x_p + \sum_{\ell \not\in \{p, p+1\}} x_\ell
\]

we infer that \( x_p = x_{p+1} \).

Moreover, because

\[
f_q(x_1, \ldots, x_n) = 2x_{q+1} + \sum_{\ell \not\in \{q, q+1\}} x_\ell, \quad x_p = x_{p+1}, \text{ and } f_q(x_1, \ldots, x_n) < f_p(x_1, \ldots, x_n),
\]

we also find that

\[
2x_{q+1} + \sum_{\ell \not\in \{q, q+1\}} x_\ell < x_p + x_{p+1} + \sum_{\ell \not\in \{p, p+1\}} x_\ell = \sum x_\ell .
\]

This in turn implies that \( x_{q+1} > x_q \).
Analogously, because
\[ f_{q+1}(x_1, \ldots, x_n) = 2x_q + \sum_{t \notin \{q, q+1\}} x_t, \quad x_p = x_{p+1}, \text{ and } f_{q+1}(x_1, \ldots, x_n) < f_p(x_1, \ldots, x_n) \]
we obtain that \( x_q > x_{q+1} \) which is not possible. Hence the LP has no solution and therefore \( S \) cannot be an antimedian set. \( \square \)

An alternate proof using the structural property of a cocktail party graph is described below as a remark.

**Remark 6.6:** Let \( G \) be a cocktail party graph and suppose that the antimedian of \( G \) of some profile is not connected. Then necessarily it consists of the end vertices \( a, b \) of some edge removed from \( K_n \). Since \( a \) and \( b \) are adjacent in \( G \) to all the remaining vertices and \( D(a) = D(b) > D(c) \) for any other vertex \( c \), they both must be in the profile (with the same multiplicity). Take any other nonadjacent pair \( (x, y) \) of vertices in \( G \). If one of \( x \) or \( y \) is not in the profile, then it will be in the antimedian. Analogously, if both \( x \) and \( y \) are in the profile but with different multiplicities, that with smaller multiplicity will be an antimedian. Finally, if both have the same multiplicity, then all \( a, b, x, y \) will be in the antimedian, which is contrary to the assumption.

Note that cocktail-party graphs are special instances of graphs with connected antimedian sets.

To obtain more graphs with connected antimedian sets it is useful to consider Cartesian products of graphs.

Let \( \pi \) be a profile on a Cartesian product \( G \square H \). Then by the projection of \( \pi \) on \( G, \text{proj}_G \pi \), we mean the sequence \( (g \in G) \) such that \((g, h) \in \pi\), taking into account the multiplicities of projected vertices.

**Proposition 6.7:** Let \( \pi \) be a profile on \( G \square H \). Let \( \pi_G = \text{proj}_G \pi \) and \( \pi_H = \text{proj}_H \pi \). Then
\[ AM(\pi, G \square H) = AM(\pi_G, G) \times AM(\pi_H, H). \]
Proof. Let \((g, h)\) be an arbitrary vertex of \(G \square H\). Then, using the Distance Lemma, we compute the remoteness as follows:

\[
D_{G \square H}((g, h), \pi) = \sum_{(g', h') \in \pi} d_{G \square H}((g, h), (g', h'))
\]

\[
= \sum_{(g', h') \in \pi} (d_G(g, g') + d_H(h, h'))
\]

\[
= \sum_{g' \in \pi_G} d_G(g, g') + \sum_{h' \in \pi_H} d_H(g, g')
\]

\[
= D_G(g, \pi_G) + D_H(h, \pi_H).
\]

We conclude that \((g, h) \in AM(\pi, G \square H)\) if and only if \(g \in AM(\pi_G, G)\) and \(h \in AM(\pi_H, H)\).

\(\square\)

Corollary 6.8: Let \(G\) and \(H\) be connected graphs with connected antimedian sets. Then \(G \square H\) has connected antimedian sets. In particular, Hamming graphs and hence hypercubes have connected antimedian sets.

Next, we consider halfcubes.

Theorem 6.9: Let \(G\) be a halfcube, then \(M(\pi, G) (AM(\pi, G))\) is connected for any profile \(\pi\) in \(G\).

Proof. Let \(Q_d\) be the hypercube of dimension \(d\) in which \(G\) is scale-2 embedded. Let \(\pi\) be an arbitrary profile in \(G\) and \(|\pi| = k\). Note that by applying the Majority (Minority) rule for the given profile \(\pi\) of the halfcube embedded into hypercube \(Q_d\) (looking as the vertices of a hypercube), we get the median(antimedian) of \(\pi\) in \(Q_d\) which will be a sub-hypercube, say \(Q_r\). We analyze the property of \(M(\pi, G)(AM(\pi, G))\) by considering the following two cases separately.

Case 1: \(M(\pi, Q_d)(AM(\pi, Q_d))\) is a sub hypercube \(Q_r\) of cardinality greater than one. Clearly \(Q_r\) has half vertices in the corresponding halfcube - call this set \(X\). Set \(X\) forms a subhalfcube in \(G\), hence \(X\) is connected. Since the graph \(G\) is scale-2
embedded the remoteness in $G$ is obtained by dividing the corresponding remoteness in $Q_d$ by 2, we get $M(\pi, G) = X$, as we follow the Majority (Minority) rule on $\pi$.

Case 2: $M(\pi, Q_d)(AM(\pi, Q_d))$ in $Q_d$ contains exactly one vertex say $x$.

In this case if $x$ belongs to $G$, then clearly $M(\pi, G)(AM(\pi, G)) = \{x\}$ as the case may be and hence we are done.

In this case let the vertex $x$ is outside $G$. Note that $x = (x^1, \ldots, x^d)$ can be obtained from the Majority(Minority) rule among co-ordinates of the profile $\pi$. Let each

$$n_{x^i}(\pi) = m_i, \ 1 \leq m_i \leq n$$

Let $m = \text{minimum}(m_1, \ldots, m_d)$ (or maximum$(m_1, \ldots, m_d)$). Clearly if for any vertex $y$ obtained by changing any single $i^{th}$ coordinate of $x$, the remoteness changes by $2m_i - k$, where $|\pi| = k$. This change in remoteness is minimum for coordinates having Majority (Minority) value $m$.

Hence $M(\pi, G)(AM(\pi, G))$ is precisely the set of vertices obtained from $G$ by changing any coordinate of $x$, having minimum(maximum) Majority (Minority) $m_i$. These vertices are all adjacent to $x$, and hence forms a clique in $G$, which completes the proof. $\square$

From the proof of the above theorem, we have the following corollaries.

**Corollary 6.10:** Let $G$ be a halfcube, then $M(\pi, G)(AM(\pi, G))$ is a subhalfcube of $G$ or a clique for any profile $\pi$ in $G$.

**Corollary 6.11:** Let $Q_d$ be a hypercube of dimension $d$. Then for any profile $\pi$ on $G$, if $x = (x_1, \ldots, x_d)$ is a median(antimedian) vertex, with remoteness $R$ then for any other vertex $y = (y_1, \ldots, y_d)$, the remoteness of $y$ is given by $R + (-) \sum(|x_i - y_i|2 \times m_i - k)$
Similar approach can be used to find $AM(\pi, G)/(M(\pi, G))$ of a Johnson graph $J_{n,k}$.

**Theorem 6.12:** Let $G$ be a Johnson graph $J_{n,k}$ then $AM(\pi, G)/(M(\pi, G))$ is a sub Johnson graph.

**Proof.** Consider a scale-2 embedding of $G$ in to a hypercube $Q_d$ of dimension $d$. Let $\pi$ be any profile in $G$ and let $M(\pi, Q_d)$ is isomorphic to $Q_r$. Consider the vertex labelling $(x_1, x_2, \ldots, x_r, \ldots, x_d)$ of the vertices in $M(\pi, H)$. Without loss of generality we can assume that for all these vertices, co-ordinates at positions $r + 1$ up to $d$ are same, and the remaining positions, i.e., positions $1, \ldots, r$ constitute any vertex label corresponding to $Q_r$. Let $m$ is the total number of 1s, in positions $x_{r+1}, \ldots, x_d$. We analyze the property of median sets in $G$ by considering the following two cases separately.

Case 1: $M(\pi, Q_d) \cap G \neq \emptyset$.

Clearly $M(\pi, G)$ induce a sub Johnson graph isomorphic to $J_{r, (k-m)}$.

Case 2: $M(\pi, Q_d) \cap G = \emptyset$.

Note that in this case, either $m < k - r$ or $m > k$. Clearly if $m < k - r$ we get a point in $G$ by changing a minimum number of coordinates, say $p$, from the vertex in $M(\pi, Q_d)$ having 1s in positions $1, \ldots, r$. Similarly when $m > k$, we get a vertex in $G$ with a minimum number of changes, by selecting the vertex with 0s in positions $1, \ldots, r$. Since we are looking for median set in $G$, we select the positions, in such a way that the change in remoteness is minimum. Thus we select $p$ coordinate positions with smaller Majority values. If the Majority values are distinct we get a single vertex in $G$. Otherwise we make a selection among, say $p'$ positions. In this case the subgraph induced by the vertices of $G$ thus obtained will be isomorphic to $J_{p', p'}$.

Since the remoteness is same for all vertices in the median set we will get the same result independent of the vertex selected.
Hence we get a sub Johnson graph as the median set.

With similar arguments by taking the Minority values in coordinate positions, we can prove that antimedian sets also induce a sub Johnson graph, which completes the proof. □

From the above theorem we have the following remark:

**Remark 6.13:** Let $G$ be a Johnson graph, then $M(\pi, G)(AM(\pi, G))$ is connected for any profile $\pi$ in $G$.

### 6.4 Concluding Remarks

In this chapter, we obtained selected and specific instances of graphs with connected antimedian sets for arbitrary profiles. It is possible to have more such graphs due to Proposition 6.7. At this stage quite naturally the following question arises:

**Question 6.14:** Is it possible to characterize graphs with connected antimedians for arbitrary profiles?

An attempt to solve this problem is carried out in the next chapter.