Chapter 3

Solution of Matrix Games with Fuzzy Payoffs using Duality in Linear Programming

3.1 Introduction

One of the most useful results in the matrix game theory asserts that every two person zero-sum matrix game is equivalent to two linear programming problems which are dual to each other. The earliest study of two person zero-sum matrix game with fuzzy payoffs is due to Campos (Campos 1989) which remains the most basic reference on this topic. Later Nishizaki and Sakawa (Nishizaki and Sakawa 2001) extended the ideas of Campos (Campos 1989) to multi-objective matrix games. Campos (Campos 1989) introduced a number of different types of linear programming (LP) models to solve zero-sum fuzzy normal form games. In
his formulation, each player’s strategy set is a crisp set, but players have imprecise knowledge about the payoffs. He considered five different ways of ranking fuzzy numbers, and for each case he formulated the constraints using fuzzy triangular numbers. Two of these are based on the work of Yager (Yager 1981) and involve the use of a ranking function that maps the fuzzy numbers on to $\mathbb{R}$. A third approach involves the use of $\alpha$-cuts and is based on the work of Adamo (Adamo 1980). The last two approaches rank fuzzy numbers using possibility theory. This stems from the work of Dubois and Prade (Dubois and Prade 1983). Finally, the five different parametric LP models obtained through this transformation process are solved using conventional LP techniques to identify their fuzzy solutions. This exercise is performed with different numerical examples. Later Bector (Bector et al. 2004) presented a defuzzification function method to solve matrix games in fuzzy environment using linear programming method. In this chapter we study a more general type of matrix games with trapezoidal fuzzy numbers as payoffs as in (Bector and Chandra 2005), we use a suitable defuzzification function to convert them as primal-dual pairs in linear programming problem. We study the model which deals with the same problem but from totally different approaches. We expound our approach to solve a fuzzy matrix game by prime-dual method and a numerical example is also given.
3.2 Preliminaries

Let $\mathbb{R}^n$ denote the $n$-dimensional Euclidean space and $\mathbb{R}^{+n}$ be its non-negative orthant set. Let $A \in \mathbb{R}^{m \times n}$ be an $m \times n$ matrix and $e$ be the column matrix with all entries equal 1. A two person zero-sum matrix game $G$ is defined as $G = (S^m, S^n, A)$, where $S^m = \{x \in \mathbb{R}_+^m : e^T x = 1\}$ and $S^n = \{y \in \mathbb{R}_+^n : e^T y = 1\}$ are the strategy space for the player I and player II respectively. $A$ is called the payoff matrix. The quantity $K(x, y) = x^T A y$ is called the expected payoff of player I by player II.

Now the concept of double fuzzy constraints means constraints which are expressed as fuzzy inequalities involving fuzzy numbers. Let $N(\mathbb{R})$ be the set of all fuzzy numbers. Let $A, b, c$ respectively be $m \times n$ matrix, $m \times 1$ and $n \times 1$ vectors having entries from $N(\mathbb{R})$ and the double fuzzy constraints under consideration be given by $A x \preceq \bar{b}$ and $A^T y \succeq \bar{c}$ with adequacies $\bar{p}$ and $\bar{q}$ respectively. Based on a resolution method proposed in (Yager 1981) the constraint $A x \preceq \bar{b}$ is expressed as $\tilde{A} x \preceq \tilde{b}$ with adequacy $\tilde{p}$, namely $\tilde{p}_i$ measures the adequacy between the fuzzy numbers $\tilde{A}_i x$ and $\tilde{b}_i$ which are the $i$th component of fuzzy vectors $A x$ and $\bar{b}$ respectively.

Similarly the constraint $A^T y \succeq \bar{c}$ is expressed as $\tilde{A}^T y \succeq \tilde{c}$ with adequacy $\tilde{q}$, namely $\tilde{q}_j$ measures the adequacy between the fuzzy numbers $\tilde{A}_j^T y$ and $\tilde{c}_j$ which are the $j$th component of fuzzy vectors $A^T y$ and $\bar{c}$ respectively. Here $(\preceq)$, and $(\succeq)$ are relations between fuzzy numbers which preserve the ranking when fuzzy numbers are multiplied by positive scalars.
This could be with respect to any ranking function $F : N(\mathbb{R}) \rightarrow \mathbb{R}$ taken in Campos (Campos 1989) such that $\tilde{a}(\leq) \tilde{b}$ implies $F(\tilde{a}) \leq F(\tilde{b})$. Since in subsequent sections the function $F$ is used to defuzzify the given fuzzy linear programming problems. Therefore the double constraints of the type $\tilde{A}x \preceq_{\rho} \tilde{b}$ and $\tilde{A}y^T \succeq_{\eta} \tilde{c}$ are to be understood as $\tilde{A}_i x(\leq) \tilde{b}_i + \tilde{p}_i(1 - \lambda)$, for $0 \leq \lambda \leq 1$ and $i = 1, 2, \ldots, m$ and $\tilde{A}_j^T y(\geq) \tilde{c}_j - \tilde{q}_j(1 - \eta)$; for $0 \leq \eta \leq 1$ and $j = 1, 2, \ldots, n$ which are in turn means,

$$F(\tilde{A}_i x) \leq F(\tilde{b}_i) + (1 - \lambda)F(\tilde{p})$$

and

$$F(\tilde{A}_j^T y) \geq F(\tilde{c}_j) - (1 - \eta)F(\tilde{q}).$$

Now let $\tilde{a}_{ij}$, $\tilde{b}_i$, $\tilde{p}_i$, $\tilde{c}_j$ and $\tilde{q}_j$ are trapezoidal fuzzy numbers (TrFNs) and $F$ is the Yagers (Yager 1981) first index given by

$$F(D) = \frac{\int x \mu_D(x) dx}{\int \mu_D(x) dx} ;$$

where $dl$ and $du$ are the lower and upper limits of the support of the fuzzy number $D$.

The constraints $\tilde{A}x \preceq_{\rho} \tilde{b}$ and $\tilde{A}^T \succeq_{\eta} \tilde{c}$ respectively means
\[ \sum_{j=1}^{n} \left[ (a_{ij})_l + \frac{a_{ij} + \tilde{a}_{ij}}{2} + (a_{ij})_u \right] x_j \leq \left[ (b_i)_l + \frac{b_i + \tilde{b}_i}{2} + (b_i)_u \right] + (1 - \lambda)(p_i)_l + \frac{p_i + \tilde{p}_i}{2} + (p_i)_u \]

and

\[ \sum_{i=1}^{m} \left[ (a_{ij})_l + \frac{a_{ij} + \tilde{a}_{ij}}{2} + (a_{ij})_u \right] y_i \geq \left[ (c_j)_l + \frac{c_j + \tilde{c}_j}{2} + (c_j)_u \right] + (1 - \eta)(q_j)_l + \frac{q_j + \tilde{q}_j}{2} + (q_j)_u \]

for \( \lambda \in [0, 1], \eta \in [0, 1], i = 1, 2, \ldots, m \) and \( j = 1, 2, \ldots, n \).

Here, \( \tilde{a}_{ij} = ((a_{ij})_l, \frac{a_{ij} + \tilde{a}_{ij}}{2}, (a_{ij})_u), \ b_{ij} = ((b_i)_l, \frac{b_i + \tilde{b}_i}{2}, (b_i)_u), \ p_i = ((p_i)_l, \frac{p_i + \tilde{p}_i}{2}, (p_i)_u) \) and \( \tilde{q}_j = ((q_j)_l, \frac{q_j + \tilde{q}_j}{2}, (q_j)_u) \) are Trapezoidal Fuzzy numbers.

### 3.2.1 Duality in Linear Programming

In the crisp theory of primal-dual linear programming we have the primal as

\[
\begin{align*}
\text{max} \quad & c^T x \\
\text{subject to} \quad & Ax \leq b; \\
& x \geq 0
\end{align*}
\]

and its dual pair is
Theorem 3.1. A is bounded by quadratice $A = \text{t.s. a. f.}.$

\[
\begin{align*}
D_1 & \quad \min \ b^T y \\
\text{subject to} & \quad A^T y \geq c; \\
& \quad y \geq 0
\end{align*}
\]

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and $A$ is an $m \times n$ real matrix.

**Definition 3.1. Fuzzy Number**

A fuzzy set $A$ in $\mathbb{R}$ is called a fuzzy number if it satisfies the following condition

(i) $A$ is normal

(ii) $\alpha A$ is a closed interval for every $\alpha \in (0, 1]$

(iii) The support of $A$ is bounded

**Definition 3.2. (Trapezoidal fuzzy number) (TrFN)**

A fuzzy number $A$ is called a trapezoidal fuzzy number if its membership function is given by

\[
\mu_A(x) = \begin{cases} 
0 & \text{if } x < a_l, x > a_u \\
\frac{x - a_l}{a - a_l} & \text{if } a_l \leq x \leq a \\
1 & \text{if } a \leq x \leq \bar{a} \\
\frac{a_u - x}{a_u - \bar{a}} & \text{if } \bar{a} < x \leq a_u
\end{cases}
\]
3.3 Primal-dual fuzzy linear programming

The TrFN \( A \) is denoted by quadruplet \( A = (a_l, a, a_u) \).

Theorem 3.1.

Let \((x, \lambda)\) and \((y, \eta)\) are feasible fuzzy pair of primal-dual linear programming problem then

\[
F(c^T x) - F(b^T y) \leq (1 - \lambda)F(p^T y) + (1 - \eta)F(q^T x).
\]

3.3 Primal-Dual Fuzzy Linear Programming

In the crisp pair of primal-dual linear programming problems, the fuzzy version of the usual primal and dual problem as

\[P_2\]

\[
\begin{align*}
\text{max} & \quad c^T x \\
\text{subject to} & \quad Ax \leq b; \\
& \quad x \geq 0
\end{align*}
\]

and

\[D_2\]

\[
\begin{align*}
\text{min} & \quad b^T y \\
\text{subject to} & \quad A^T y \geq c; \\
& \quad y \geq 0
\end{align*}
\]
3.3 Primal-dual fuzzy linear programming

Here $\tilde{A}$ is an $m \times n$ matrix of fuzzy numbers and $\tilde{b}$ and $\tilde{c}$ respectively are $m \times 1$ and $n \times 1$ vectors of fuzzy numbers. The symbols $\preceq$ and $\succeq$ are fuzzy versions of the symbols $\leq$ and $\geq$ respectively and have the interpretation "essentially less than or equal to" and "essentially greater than or equal to" as explained in Zimmerman (Zimmerman 1978).

**Theorem 3.2.**

The triplet $(\tilde{x}, \tilde{y}, \tilde{v}) \in S^m \times S^n \times \mathbb{R}$ is a solution of the game $G$ if and only if $\tilde{x}$ is optimal to $P_1$, $\tilde{y}$ is optimal to $D_1$ and $\tilde{v}$ is the common value of $P_1$ and its dual $D_1$. The double fuzzy constraint $\tilde{A}x \preceq \tilde{b}$ and $\tilde{A}^Ty \succeq \tilde{c}$ are to be in mind with respect to a suitable defuzzification function $F$ and adequacies $\tilde{p}$ and $\tilde{q}$. Again, the defuzzification function $F$ once chosen is to be kept fixed. If $F : N(\mathbb{R}) \rightarrow \mathbb{R}$ is the chosen defuzzification function fuzzy numbers for constraints in $P_2$ and $D_2$ then the same defuzzification function $F$ for the objective function in $P_2$ and $D_2$ we get $P_3$ and $D_3$ as

$$P_3$$

$$\max \quad F(\tilde{c}^Tx)$$

subject to

$$F(\tilde{A}x) \leq F(\tilde{b}) + (1 - \lambda)F(\tilde{p});$$

$$\lambda \leq 1;$$

$$x, \lambda \geq 0$$

and
3.3 Primal-dual fuzzy linear programming

$$D_3$$

$$\begin{align*}
\min & \quad F(b^Ty) \\
\text{subject to} & \quad F(\bar{A}^Ty) \geq F(\bar{c}) - (1 - \eta)F(\bar{q}) \\
& \quad \eta \leq 1; \\
& \quad y, \eta \geq 0
\end{align*}$$

Here $\bar{p}$ and $\bar{q}$ measures the adequacies in the primal and dual constraints. The pair $P_3$ and $D_3$ are termed as fuzzy pair of primal-dual linear programming problems.

In case $\bar{A}$, $\bar{c}$ and $\bar{b}$ are crisp and $\lambda = 1$ and $\eta = 1$, then the pair $P_3$ and $D_3$ reduce to the usual crisp primal-dual pair and the theorem 3.2 becomes the usual weak duality theorem.

**Definition 3.3.**

Let $\bar{v}, \bar{w} \in N(\mathbb{R})$. Then, $(\bar{v}, \bar{w})$ is called a reasonable solution of the fuzzy matrix game $FG$ if there exists $x^* \in S^m, y^* \in S^n$ satisfying

(i) $$(x^*)^T\bar{A}y \geq \bar{v} \quad \forall \ y \in S^n$$

(ii) $$x^T\bar{A}y \leq \bar{w} \quad \forall \ x \in S^m$$

If $(\bar{v}, \bar{w})$ is a reasonable solution of $FG$ then $\bar{v}$ (respectively $\bar{w}$) is called a reasonable value for player I (respectively player II)
Definition 3.4. (Bector and Chandra 2005)

Let $T_1$ and $T_2$ be the set of all reasonable values $\tilde{v}$ and $\tilde{w}$ for player I and player II respectively where $\tilde{v}, \tilde{w} \in \mathbb{R}^n$. Let there exists $\tilde{v}^* \in T_1$, $\tilde{w}^* \in T_2$ such that $F(\tilde{v}^*) \geq F(\tilde{v}) \forall \tilde{v} \in T_1$ and $F(\tilde{w}^*) \leq F(\tilde{w}) \forall \tilde{w} \in T_2$. Then $(x^*, y^*, \tilde{v}^*, \tilde{w}^*)$ is called the solution of the game $FG$, where $\tilde{v}^*$ (respectively $\tilde{w}^*$) is the value of the game $FG$ for player I (respectively player II) and $x^*$ (respectively $y^*$) is called an optimal strategy for player I (respectively player II).

From the above definition for the game $FG$ we construct the pair of fuzzy linear programming problems for player I and player II as

\begin{align*}
\text{P}_4 & \quad \text{max} \quad F(\tilde{v}) \\
\text{subject to} & \quad x^T \tilde{A} y \gtrsim_{\tilde{b}} \tilde{v}; \\
& \quad \forall \ y \in S^n, \ x \in S^m
\end{align*}

and

\begin{align*}
\text{D}_4 & \quad \text{min} \quad F(\tilde{w}) \\
\text{subject to} & \quad x^T \tilde{A} y \lesssim_{\tilde{b}} \tilde{w}; \\
& \quad \forall \ y \in S^n, \ x \in S^m
\end{align*}

Now, using the double fuzzy constraints and applying the relations $(\leq)$ and $(\geq)$ preserve the ranking when fuzzy numbers are multiplied by
positive scalars, we have to consider only the extreme points of sets $S^m$ and $S^n$ in the constraints of $P_4$ and $D_4$. Then the problem $P_4$ and $D_4$ will be converted into

\[ P_5 \]

\[
\begin{align*}
  & \text{max } F(\bar{v}) \\
  & \text{subject to} \\
  & x^T \tilde{A}_j \geq \bar{v}; j = 1, 2, \ldots, n. \\
  & e^T x = 1; \\
  & x \geq 0
\end{align*}
\]

and

\[ D_5 \]

\[
\begin{align*}
  & \text{min } F(\bar{w}) \\
  & \text{subject to} \\
  & \tilde{A}_i y \leq \bar{w}; i = 1, 2, \ldots, m. \\
  & e^T y = 1; \\
  & y \geq 0
\end{align*}
\]

Here $\tilde{A}_i$ (respectively $\tilde{A}_j$) denotes the $i$th row (respectively $j$th column) of $\tilde{A}$ ($i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$) using the resolution procedure for the double fuzzy constraints in $P_5$ and $D_5$, we obtain
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\[ P_6 \]

\[ \max \ F(\tilde{v}) \]

subject to

\[ \sum_{i=1}^{m} \tilde{a}_{ij} x_i (\geq) \tilde{v} - (1 - \lambda) \tilde{p}; j = 1, 2, \ldots, n. \]

\[ e^T x = 1; \]

\[ \lambda \leq 1; \]

\[ x, \lambda \geq 0 \]

where \( \tilde{a}_{ij} = ((a_{ij})_l, \frac{a_{ij} + \bar{a}_{ij}}{2}, (a_{ij})_u), \tilde{v} = (v)_l, \frac{v + \bar{v}}{2}, (v)_u), \]

\[ \tilde{p} = (p)_l, \frac{p + \bar{p}}{2}, (p)_u \]

and

\[ D_6 \]

\[ \min \ F(\tilde{w}) \]

subject to

\[ \sum_{j=1}^{n} \tilde{a}_{ij} y_j (\leq) \tilde{w} + (1 - \eta) \tilde{q}; i = 1, 2, \ldots, m. \]

\[ e^T y = 1; \]

\[ \eta \leq 1; \]

\[ y, \eta \geq 0 \]

where \( \tilde{w} = (w)_l, \frac{w + \bar{w}}{2}, (w)_u), \tilde{q} = (q)_l, \frac{q + \bar{q}}{2}, (q)_u) \)

By applying the defuzzification function \( F : \mathcal{N}(\mathbb{R}) \rightarrow \mathbb{R} \) for the constraints \( P_6 \) and \( D_6 \) these problems can further be written as
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\( P_7 \)

\[ \begin{align*} & \max F(\tilde{v}) \\ & \text{subject to} \\ & \quad \sum_{i=1}^{m} F(\tilde{a}_{ij})x_i \geq F(\tilde{v}) - (1 - \lambda)F(\tilde{p}); j = 1, 2, \ldots, n. \\ & \quad e^T x = 1; \\ & \quad \lambda \leq 1; \\ & \quad x, \lambda \geq 0 \end{align*} \]

and

\( D_7 \)

\[ \begin{align*} & \min F(\tilde{w}) \\ & \text{subject to} \\ & \quad \sum_{j=1}^{m} F(\tilde{a}_{ij})y_j \leq F(\tilde{w}) + (1 - \eta)F(\tilde{q}); i = 1, 2, \ldots, m. \\ & \quad e^T y = 1; \\ & \quad \eta \leq 1; \\ & \quad y, \eta \geq 0 \end{align*} \]

From the above we observe that for solving the fuzzy matrix game FG we have to solve the crisp linear programming problems \( P_7 \) and \( D_7 \) for player I and player II respectively. Also \((x^*, \lambda^*, \tilde{v}^*)\) is an optimal solution of \( P_7 \) then for player I, \( x^* \) is an optimal strategy, \( \tilde{v}^* \) is the fuzzy value and \((1 - \lambda^*)\tilde{p}\) is the measure of the adequacy level for the double fuzzy constraints \( P_7 \). Similar interpretation can also be given to optimal solutions
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\((y^*, \eta^*, \tilde{w}^*)\) of the problem \(D_7\).

**Theorem 3.3.**

The pair \(P_7 - D_7\) constitutes a fuzzy optimal-dual pair in the sense of the theorem [3.1]

**Theorem 3.4.** (Bector and Chandra 2005)

The fuzzy matrix game \(FG\) determined by \(FG = (S^m, S^n, \tilde{A})\) is equivalent to two crisp linear programming problems \(P_7\) and \(D_7\) which constitutes a primal-dual pair in the sense of duality for linear programming with fuzzy parameters.

It is important that the crisp problems \(P_7\) and \(D_7\) which do not constitutes a primal-dual pair in the sense of duality in linear programming but are dual in "fuzzy" sense. Therefore if \((x^*, \lambda^*, \tilde{v}^*)\) is optimal to \(P_7\) and \((y^*, \eta^*, \tilde{w}^*)\) is optimal to \(D_7\), then in general we should not expect that \(F(\tilde{v}^*) = F(\tilde{w}^*)\).

If all the fuzzy numbers are to be taken as crisp numbers that is, \(\tilde{a}_{ij} = a_{ij}, \tilde{b}_i = b_i, \tilde{c}_j = c_j\) and in the optimal solutions of \(P_7\) and \(D_7\), \(\lambda^* = \eta^* = 1\), then the fuzzy game \(FG\) reduces to the crisp two person zero-sum game \(G\). Thus if \(\tilde{A}, \tilde{b}, \tilde{c}\) are crisp numbers and \(\lambda^* = \eta^* = 1\) \(FG\) reduces to \(G\), the pair \(P_7\) and \(D_7\) reduces to the pair primal-dual and the theorem 3.3 reduces to theorem 1.5

It is generally difficult to obtain exact membership functions for fuzzy values \(\tilde{v}^*\) and \(\tilde{w}^*\) because of the large number of parameters. For example if \(\tilde{v}\) is a TrFN \((\nu_l, \bar{v}, \underline{v}, v_u)\) then to determine \(\tilde{v}\) completely we need all of the four variables. Therefore from the computational point of view
it becomes easier to take \( F(\bar{\nu}) \) and \( F(\bar{\omega}) \) as real variables \( V \) and \( W \) respectively and modify \( P_7 \) and \( D_7 \) as

\[
P_8 \quad \max \ V
\]

subject to

\[
\sum_{i=1}^{m} F(\tilde{a}_{ij}) x_i \geq V - (1 - \lambda) F(\tilde{p}); \quad j = 1, 2, \ldots, n.
\]

\[
e^T x = 1;
\]

\[
\lambda \leq 1;
\]

\[
x, \lambda \geq 0
\]

and

\[
D_8 \quad \min \ W
\]

subject to

\[
\sum_{j=1}^{n} F(\tilde{a}_{ij}) y_j \leq W + (1 - \eta) F(\tilde{q}); \quad i = 1, 2, \ldots, m.
\]

\[
e^T y = 1;
\]

\[
\eta \leq 1;
\]

\[
y, \eta \geq 0
\]

Thus, although we know that value for player I (respectively player II) is fuzzy with certain membership function we shall get only numerical values \( v^* \) (respectively \( w^* \)) for player I (respectively player II) is fuzzy.
with certain membership function we shall get only numerical values $v^*$ (respectively $w^*$) for player I (respectively player II) and the actual fuzzy value for player I and player II will be “close to” $v^*$ and $w^*$ respectively. Thus, it can be concluded that we cannot get exact membership function for the fuzzy values of player I and player II, even though these are very much desirable. When $F$ is Yagers first index (Yager 1981) the numerical values $v^*$ (respectively $w^*$) will represent the “centroid” or “average” value for player I (respectively player II). The following illustration makes the procedure more clear. The matrix $A$ has been taken from (Bector and Chandra 2005) and the TrFN is defined accordingly to avoid tedious steps of calculations.

**Illustration**

Consider the fuzzy game defined by the matrix of fuzzy numbers

$$A = \begin{bmatrix} 180 & 156 \\ 90 & 180 \end{bmatrix}$$

where,

$$180 = (170, 175, 185, 195)$$

$$156 = (150, 154, 156, 159)$$

$$90 = (80, 85, 95, 100)$$

Assuming that player I and player II have margins

$$\tilde{p}_1 = \tilde{p}_2 = (0.08, 0.05, 0.15, 0.11)$$

$$\tilde{q}_1 = \tilde{q}_2 = (0.14, 0.05, 0.25, 0.17)$$

$$\tilde{v} = (155, 160, 170, 175)$$

$$\tilde{w} = (170, 175, 185, 190)$$
According to theorem (3.4) to solve this game we have to solve the following two crisp linear programming problems \((P_1)\) and \((D_1)\) for player I and player II respectively.

\[ P_1 \]

\[
\max \frac{v_I + \frac{v_U + v_v}{2} + v_u}{3} \\
\text{subject to} \\
545x_1 + 270x_2 \geq 495 - (1 - \lambda)(0.29) \\
464x_1 + 545x_2 \geq 495 - (1 - \lambda)(0.29) \\
x_1 + x_2 = 1, \\
\lambda \leq 1; \\
x_1, x_2, \lambda \geq 0
\]

and

\[ D_1 \]

\[
\max \frac{w_I + \frac{w + w}{2} + w_u}{3} \\
\text{subject to} \\
545y_1 + 464y_2 \leq 540 + (1 - \eta)(0.46) \\
270y_1 + 545y_2 \leq 540 + (1 - \eta)(0.46) \\
y_1 + y_2 = 1, \\
\eta \leq 1; \\
y_1, y_2, \eta \geq 0
\]
Now to get the full membership representations of the value for player I (respectively player II) one needs that in the optimal solution of \((P_1)\) respectively \((D_1)\) all variables \((v_1, \bar{v}, v_u)\) (respectively \((w_1, \bar{w}, w_u)\)) come out to be non zero that is they are basic variables. This seems to be most unlikely as there are much less number of constraints and therefore many of the variables are going to be nonbasic and hence take zero values only.

This observation motivates us to take 
\[
V = \frac{v_1 + \frac{v + \bar{v}}{2} + v_u}{3},
\]
\[
W = \frac{w_1 + \frac{w + \bar{w}}{2} + w_u}{3}
\]
and consider the following problem of \(P_2\) and \(D_2\) for the variables \(V\) and \(W\)

\[
\begin{align*}
\max V \\
\text{subject to} \\
545x_1 + 270x_2 &\geq 495 - (1 - \lambda)(0.29) \\
464x_1 + 545x_2 &\geq 495 - (1 - \lambda)(0.29) \\
x_1 + x_2 &= 1, \\
\lambda &\leq 1; \\
x_1, x_2, \lambda &\geq 0
\end{align*}
\]
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\[ D_2 \]

\[ \text{max } W \]

subject to

\[ 545y_1 + 464y_2 \leq 540 + (1 - \eta)(0.46) \]
\[ 270y_1 + 545y_2 \leq 540 + (1 - \eta)(0.46) \]
\[ y_1 + y_2 = 1, \]
\[ \eta \leq 1; \]
\[ y_1, y_2, \eta \geq 0 \]

Solving the above linear programming problem, we obtain

\( (x_1^* = 0.7725, x_2^* = 0.2275, v = 160.91, \lambda^* = 0) \) and

\( (y_1^* = 0.2275, y_2^* = 0.7725, w = 160.65, \eta^* = 0). \)

Therefore we obtain optimal strategies for player I and player II as

\( (x_1^* = 0.7725, x_2^* = 0.2275) \) and

\( (y_1^* = 0.2275, y_2^* = 0.7725) \) respectively.

Also the fuzzy value of the game for player I is close to 160.91. In a similar manner, the fuzzy value of the game for player II is also close to 160.65.

Note 3.1.

This chapter presented a study of matrix games with fuzzy payoffs. We first established duality for linear programming problems with fuzzy parameters and then the same was employed to develop a solution procedure for solving two person zero-sum matrix game with fuzzy payoffs. Here we used the method of analyzing the system of double fuzzy inequalities of the type \( \tilde{A}x \leq \tilde{b} \). If a co-
incides with \( \bar{a} \), then the TrFN becomes TFN. So we can apply the same results for triangular fuzzy numbers also. The next chapter deals with a new concept solution in Bi-matrix games with fuzzy payoffs.