Chapter 4.

Orderings for convolution of random variables in terms of simultaneous majorization of some of the parameters

4.1 Introduction

Stochastic ordering, hazard rate ordering, and likelihood ratio ordering are useful orderings with applications in reliability theory and survival analysis and have been studied for convolutions of random variables.

The distribution function of convolution of non-identical random variables may be complicated and the ordering properties help in obtaining bounds and approximations for certain characteristics. For some distributions, ordering properties of the convolution of random variables with respect to majorization order of one parameter with the other parameters remaining fixed have been considered by many researchers. Korwar (2002) studied the stochastic orderings of convolutions of uniform random variables and gamma random variables with different scale parameters but with a common shape parameter. Fathimanesh and Khaledi (2008) have considered likelihood ratio order for convolution of independent generalized Rayleigh random variables with
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different scale parameter with the other parameters being the same. Zhao and Balakrishnan (2009, 2010) have studied ordering properties for heterogeneous exponential and geometric random variables and heterogeneous Erlang and Pascal random variables. For more references in stochastic orderings the reader is referred to Boland et al. (1994), Kochar and Ma (1999), Khaledi and Kochar (2002, 2004, and 2006) and Zhao (2011).

The above articles, except for Zhao and Balakrishnan (2010) and Zhao (2011), establish the likelihood ratio ordering property of convolution of random variables in terms of the majorization order of one parameter with the other parameters being the same. Some important statistical distribution functions have \( k \) (> 1) parameters, and we need to consider simultaneous majorization of some of the parameters. The purpose of this chapter is to show that to study likelihood ratio ordering of convolution of random variables in terms of the multi-parameters majorization order it is enough to consider the likelihood ratio ordering of convolution of random variables in terms of the one-parameter majorization order separately. The Theorem 4.2 and Theorem 4.4, given below, extend it to likelihood ratio ordering of convolution of random variables in terms of simultaneous majorization of some of the parameters.

Suppose that \( X_{\theta_i} \), \( i = 1, \ldots, n \) are independent random variables with distribution functions \( F(x, \theta_i) \), where \( \theta_i = (\theta_{i1}, \ldots, \theta_{in}) \), \( i = 1, \ldots, n \). In this section, we study the likelihood ratio ordering of the convolution of random variables in terms of the multi-parameters majorization order. We prove that if the likelihood ratio ordering of convolution of independent random variables in terms of majorization order of one parameter is satisfied for \( t \) \((t \leq k)\) parameters, then the likelihood ratio ordering of the convolution of the random variables in terms of the majorization order of the vector parameters of dimension \( t \) holds. In the general case suppose

\[
\begin{align*}
(\theta_{i1}^*, \ldots, \theta_{in}^*) & \mbox{ majorize } (\theta_{i1}, \ldots, \theta_{nj}) & \mbox{ for } j = 1, \ldots, t \\
(\theta_{i1}^*, \ldots, \theta_{nj}^*) & \mbox{ be majorized by } (\theta_{i1}, \ldots, \theta_{nj}) & \mbox{ for } j = t + 1, \ldots, s \\
\mbox{unknown} & & \mbox{for } j = s + 1, \ldots, k.
\end{align*}
\]
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Then if the likelihood ratio ordering of convolution of independent random variables in terms of majorization order of one parameter holds, we show that

$$\sum_{i=1}^{n} X_{\theta'_{i}, \theta'_{i(i+1)}, \ldots, \theta'_{i(i+n)}}$$

is larger than

$$\sum_{i=1}^{n} X_{\theta_{i}, \theta_{i(i+1)}, \ldots, \theta_{i(i+n)}}$$

according to the likelihood ratio ordering.

We give below the Remarks we use in this chapter.

**Remark 4.1.** Let $h(\omega; \theta_{1}, \theta_{2})$ be the density function of $X_{\theta_{1}} + X_{\theta_{2}}$, if $(\theta'_{1}, \theta'_{2}) > (\theta_{1}, \theta_{2})$, then $\theta_{1} + \theta_{2} = \theta'_{1} + \theta'_{2} = c$, $(\theta_{2} = c - \theta_{1}$ and $\theta'_{2} = c - \theta'_{1}$), for fixed $c$. In the following Remark and in the rest of this section we write $h(\omega; \theta_{1})$ instead of $h(\omega; \theta_{1}, c - \theta_{1})$.

**Remark 4.2.** Let $X_{\theta_{1}}, X_{\theta_{2}}$ be independent random variables from density functions $f(x; \theta_{i})$ $i = 1, 2$, and let $h(\omega; \theta_{1}, \theta_{2})$ be the density function of $X_{\theta_{1}} + X_{\theta_{2}}$.

Suppose $(\theta'_{1}, \theta'_{2}) > (\theta_{1}, \theta_{2})$, implies $X_{\theta'_{1}} + X_{\theta'_{2}} \geq_{u} X_{\theta_{1}} + X_{\theta_{2}}$, and let $\theta_{1} + \theta_{2} = \theta'_{1} + \theta'_{2} = c$.

Without loss of generality assume that $\theta_{1} \geq \theta_{2}$ and $\theta'_{1} \geq \theta'_{2}$, then $X_{\theta'_{1}} + X_{\theta'_{2}} \geq_{u} X_{\theta_{1}} + X_{\theta_{2}}$ is equivalent to $h(\omega; \theta'_{1})/ h(\omega; \theta_{1})$ is increasing in $\omega$ or $h(\omega; \theta_{1})$ is $TP_{2}(\omega, \theta_{1})$. 

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We call \( h(\omega; \theta_i) \), \( TN_2(\omega, \theta_i) \) if \((\theta_i^*,\theta_j^*) \succ (\theta_i,\theta_j)\), implies \( X_{\theta_i^*} + X_{\theta_j} \leq_{lr} X_{\theta_i} + X_{\theta_j} \) that is, \( h(\omega; \theta_i^*) / h(\omega; \theta_i) \) is decreasing in \( \omega \).

### 4.2 L.R.O of the convolution of random variables in terms of the multi-parameters majorization order

In next theorems we consider distribution functions with \( k \) parameters and the likelihood ratio ordering of convolution of random variables in terms of simultaneous majorization of some of the parameters.

Note that if

\[
(\theta_{i,j}^*,\theta_{j,j}^*) \succ (\theta_{i,j},\theta_{j,j}) , \ j = 1, ..., t ,
\]

then \( \theta_{i,j} + \theta_{j,j} = \theta_{i,j}^* + \theta_{j,j}^* = c_j \), for fixed \( c_j \). In the rest of this chapter we write

\[
h(\omega, \theta_1, ..., \theta_{i,j}, \theta_{i,(j+1)}, ..., \theta_{i,k}, \theta_{2(j+1)}, ..., \theta_{2k})
\]

instead of

\[
h(\omega, \theta_1, ..., \theta_{i,j}, \theta_{i,(j+1)}, ..., \theta_{i,k}, c_i - \theta_1, ..., c_i - \theta_{i,j}, \theta_{2(j+1)}, ..., \theta_{2k}) .
\]

**Theorem 4.1.** Let \( X_{\theta_i}, X_{\theta_j} \) be independent random variables from density functions \( f(x; \theta_i) \), where \( \theta_i = (\theta_{i,1}, ..., \theta_{i,k}) \), \( i = 1, 2 \). If \((\theta_{i,j}^*,\theta_{j,j}^*) \succ (\theta_{i,j},\theta_{j,j})\), implies

\[
\sum_{i=1}^2 X_{\theta_{i,1}, ..., \theta_{i,(j-1)}, \theta_{i,j}^*, \theta_{i,(j+1)}, ..., \theta_{i,k}} \geq_{lr} \sum_{i=1}^2 X_{\theta_{i,1}, ..., \theta_{i,(j-1)}, \theta_{i,j}, \theta_{i,(j+1)}, ..., \theta_{i,k}}
\]

for all \( j = 1, ..., t \), for \( t \) such that \( 1 \leq t \leq k \), then
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\[
\sum_{i=1}^{2} X_{\theta_{1}, \theta_{2}, \theta_{3}, \ldots, \theta_{n}} \geq \sum_{j=1}^{2} X_{\theta_{1}, \theta_{2}, \theta_{3}, \ldots, \theta_{n}}.
\]

(4.1)

Proof. We use mathematical induction to prove the theorem. First we show that (4.1) holds for \( t = 2 \).

Let \( X_{\theta_{i}} \) be independent random variables with density function \( f(x; \theta_{1}, \ldots, \theta_{n}) \), \( i = 1, 2 \).

Now it is given that \((\theta_{1}^{*}, \theta_{2}^{*}) \succ (\theta_{1}, \theta_{2})\), \( j = 1, 2 \), implies

\[
\sum_{j=1}^{2} X_{\theta_{1}, \theta_{2}, \theta_{3}, \ldots, \theta_{n}} \geq \sum_{j=1}^{2} X_{\theta_{1}, \theta_{2}, \theta_{3}, \ldots, \theta_{n}}
\]

and

\[
\sum_{j=1}^{2} X_{\theta_{1}, \theta_{2}, \theta_{3}, \ldots, \theta_{n}} \geq \sum_{j=1}^{2} X_{\theta_{1}, \theta_{2}, \theta_{3}, \ldots, \theta_{n}}
\]

then for \( \theta_{1} + \theta_{2} = \theta_{1}^{*} + \theta_{2}^{*} = c_{1} \) and \( \theta_{1} + \theta_{2} = \theta_{1}^{*} + \theta_{2}^{*} = c_{2} \),

\( h(\omega, \theta_{1}, \ldots, \theta_{n}, \theta_{21}, \ldots, \theta_{2k}) \) is \( TP_{2}(\omega, \theta_{1}) \) and \( TP_{2}(\omega, \theta_{1}) \) with the other parameters fixed. Using the definition of total positivity for \( \omega < \omega^{*} \) and \( \theta_{i} < \theta_{i}^{*} \) we have;

\[
\frac{h(\omega, \theta_{1}, \ldots, \theta_{n}, \theta_{21}, \ldots, \theta_{2k})}{h(\omega, \theta_{1}, \ldots, \theta_{n}, \theta_{21}, \ldots, \theta_{2k})} \leq \frac{h(\omega^{*}, \theta_{1}, \ldots, \theta_{n}, \theta_{21}, \ldots, \theta_{2k})}{h(\omega^{*}, \theta_{1}, \ldots, \theta_{n}, \theta_{21}, \ldots, \theta_{2k})}.
\]

Thus,

\[
\frac{h(\omega, \theta_{1}, \theta_{2}, \theta_{3}, \ldots, \theta_{k}, \theta_{21}, \ldots, \theta_{2k})}{h(\omega, \theta_{1}, \theta_{2}, \theta_{3}, \ldots, \theta_{k}, \theta_{21}, \ldots, \theta_{2k})} \leq \frac{h(\omega^{*}, \theta_{1}, \theta_{2}, \theta_{3}, \ldots, \theta_{k}, \theta_{21}, \ldots, \theta_{2k})}{h(\omega^{*}, \theta_{1}, \theta_{2}, \theta_{3}, \ldots, \theta_{k}, \theta_{21}, \ldots, \theta_{2k})}.
\]

(4.2)

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Similarly if \( h(\omega, \theta_{11}, \theta_{12}, \theta_{13}, \ldots, \theta_{1k}, \theta_{21}, \ldots, \theta_{2k}) \) is \( TP_2(\omega, \theta_{12}) \) for fixed \( \theta_{11}^* \) and \( \theta_{12} < \theta_{12}^* \), we have;

\[
\frac{h(\omega, \theta_{11}^*, \theta_{12}^*, \ldots, \theta_{1k}, \theta_{21}, \ldots, \theta_{2k})}{h(\omega, \theta_{11}, \theta_{12}, \ldots, \theta_{1k}, \theta_{21}, \ldots, \theta_{2k})} \leq \frac{h(\omega, \theta_{11}^*, \theta_{12}^*, \ldots, \theta_{1k}, \theta_{21}, \ldots, \theta_{2k})}{h(\omega, \theta_{11}, \theta_{12}, \ldots, \theta_{1k}, \theta_{21}, \ldots, \theta_{2k})},
\]

and thus,

\[
\frac{h(\omega, \theta_{11}^*, \theta_{12}^*, \ldots, \theta_{1k}, \theta_{21}, \ldots, \theta_{2k})}{h(\omega, \theta_{11}, \theta_{12}, \ldots, \theta_{1k}, \theta_{21}, \ldots, \theta_{2k})} \leq \frac{h(\omega, \theta_{11}^*, \theta_{12}^*, \ldots, \theta_{1k}, \theta_{21}, \ldots, \theta_{2k})}{h(\omega, \theta_{11}, \theta_{12}, \ldots, \theta_{1k}, \theta_{21}, \ldots, \theta_{2k})}.
\]

From (4.2) and (4.3) we have;

\[
\frac{h(\omega, \theta_{11}^*, \theta_{12}^*, \ldots, \theta_{1k}, \theta_{21}, \ldots, \theta_{2k})}{h(\omega, \theta_{11}, \theta_{12}, \ldots, \theta_{1k}, \theta_{21}, \ldots, \theta_{2k})} \leq \frac{h(\omega, \theta_{11}^*, \theta_{12}^*, \ldots, \theta_{1k}, \theta_{21}, \ldots, \theta_{2k})}{h(\omega, \theta_{11}, \theta_{12}, \ldots, \theta_{1k}, \theta_{21}, \ldots, \theta_{2k})}.
\]

or

\[
\frac{h(\omega, \theta_{11}^*, \theta_{12}^*, \ldots, \theta_{1k}, \theta_{21}, \ldots, \theta_{2k})}{h(\omega, \theta_{11}, \theta_{12}, \ldots, \theta_{1k}, \theta_{21}, \ldots, \theta_{2k})} \leq \frac{h(\omega, \theta_{11}^*, \theta_{12}^*, \ldots, \theta_{1k}, \theta_{21}, \ldots, \theta_{2k})}{h(\omega, \theta_{11}, \theta_{12}, \ldots, \theta_{1k}, \theta_{21}, \ldots, \theta_{2k})}.
\]

It is clear from (4.4), for \( j = 1, 2 \), if \( (\theta_{1j}^*, \theta_{2j}^*) \succ (\theta_{1j}, \theta_{2j}) \), and \( h(\omega, \theta_{11}, \ldots, \theta_{1k}, \theta_{21}, \ldots, \theta_{2k}) \) is \( TP_2(\omega, \theta_{1j}) \), then

\[
\sum_{i=1}^{2} X_{\theta_{ij}, \theta_{ij}, \ldots, \theta_{ij}} \geq \sum_{i=1}^{2} X_{\theta_{ij}, \theta_{ij}, \ldots, \theta_{ij}}.
\]
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Next we suppose that (4.1) holds for \( t = p \) and we show that it is holds for \( t = p + 1 \) parameters.

If \((\theta_1', \theta_2') \succ (\theta_1, \theta_2)\) then \(\theta_1' + \theta_2' = \theta_1' + \theta_2' = c_j\) for \( j = 1, 2, \ldots, p \), and if

\[
h(\omega, \theta_1, \ldots, \theta_p, \theta_{l(p+1)}, \ldots, \theta_{l_k}, \theta_{21}, \ldots, \theta_{2p}, \theta_{2(p+1)} \ldots, \theta_{2k})
\]
be \(\text{TP}_2(\omega, \theta_1')\), \( j = 1, 2, \ldots, p \) with the other parameters fixed, and suppose that

\[
\sum_{i=1}^{2} X_{\theta_1', \ldots, \theta_{p'}, \theta_{l(p+1)}, \ldots, \theta_{l_k}, \theta_{21}, \ldots, \theta_{2k}} \geq p \sum_{i=1}^{2} X_{\theta_1, \ldots, \theta_{p}, \theta_{l(p+1)}, \ldots, \theta_{l_k}, \theta_{21}, \ldots, \theta_{2k}}
\]

then we have for \( \omega \leq \omega^* \),

\[
\frac{h(\omega, \theta_1', \theta_1', \ldots, \theta_{p'}, \theta_{l(p+1)} \ldots, \theta_{l_k}, \theta_{2(p+1)} \ldots, \theta_{2k})}{h(\omega, \theta_1', \theta_1', \ldots, \theta_{p'}, \theta_{l(p+1)} \ldots, \theta_{l_k}, \theta_{2(p+1)} \ldots, \theta_{2k})} \leq \frac{h(\omega, \theta_1, \theta_1, \ldots, \theta_p, \theta_{l(p+1)}, \ldots, \theta_{l_k}, \theta_{2(p+1)}, \ldots, \theta_{2k})}{h(\omega, \theta_1, \theta_1, \ldots, \theta_p, \theta_{l(p+1)}, \ldots, \theta_{l_k}, \theta_{2(p+1)}, \ldots, \theta_{2k})}
\]

which is equivalent to,

\[
\frac{h(\omega, \theta_1', \theta_1', \ldots, \theta_{p'}, \theta_{l(p+1)} \ldots, \theta_{l_k}, \theta_{2(p+1)} \ldots, \theta_{2k})}{h(\omega, \theta_1', \theta_1', \ldots, \theta_{p'}, \theta_{l(p+1)} \ldots, \theta_{l_k}, \theta_{2(p+1)} \ldots, \theta_{2k})} \leq \frac{h(\omega, \theta_1, \theta_1, \ldots, \theta_p, \theta_{l(p+1)}, \ldots, \theta_{l_k}, \theta_{2(p+1)}, \ldots, \theta_{2k})}{h(\omega, \theta_1, \theta_1, \ldots, \theta_p, \theta_{l(p+1)}, \ldots, \theta_{l_k}, \theta_{2(p+1)}, \ldots, \theta_{2k})}
\]

(4.6)

Now, if \( h(\omega, \theta_1, \ldots, \theta_p, \theta_{l(p+1)}, \ldots, \theta_{l_k}, \theta_{21}, \ldots, \theta_{2p}, \theta_{2(p+1)} \ldots, \theta_{2k}) \) is \(\text{TP}_2(\omega, \theta_{l(p+1)})\) with the other parameters fixed, then
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\[
\frac{h(\omega, \theta_{11}', \theta_{12}', \ldots, \theta_{1p}', \theta_{1(p+1)}', \ldots, \theta_{1k}, \theta_{2(p+1)}', \ldots, \theta_{2k})}{h(\omega, \theta_{11}', \theta_{12}', \ldots, \theta_{1p}', \theta_{1(p+1)}', \ldots, \theta_{1k}, \theta_{2(p+1)}', \ldots, \theta_{2k})} \leq \frac{h(\omega', \theta_{11}', \theta_{12}', \ldots, \theta_{1p}', \theta_{1(p+1)}', \ldots, \theta_{1k}, \theta_{2(p+1)}', \ldots, \theta_{2k})}{h(\omega', \theta_{11}', \theta_{12}', \ldots, \theta_{1p}', \theta_{1(p+1)}', \ldots, \theta_{1k}, \theta_{2(p+1)}', \ldots, \theta_{2k})}
\]

or

\[
\frac{h(\omega, \theta_{11}', \theta_{12}', \ldots, \theta_{1p}', \theta_{1(p+1)}', \ldots, \theta_{1k}, \theta_{2(p+1)}', \ldots, \theta_{2k})}{h(\omega, \theta_{11}', \theta_{12}', \ldots, \theta_{1p}', \theta_{1(p+1)}', \ldots, \theta_{1k}, \theta_{2(p+1)}', \ldots, \theta_{2k})} \leq \frac{h(\omega, \theta_{11}', \theta_{12}', \ldots, \theta_{1p}', \theta_{1(p+1)}', \ldots, \theta_{1k}, \theta_{2(p+1)}', \ldots, \theta_{2k})}{h(\omega', \theta_{11}', \theta_{12}', \ldots, \theta_{1p}', \theta_{1(p+1)}', \ldots, \theta_{1k}, \theta_{2(p+1)}', \ldots, \theta_{2k})}.
\]

From (4.6) and (4.7) we have;

\[
\frac{h(\omega, \theta_{11}', \theta_{12}', \ldots, \theta_{1p}', \theta_{1(p+1)}', \ldots, \theta_{1k}, \theta_{2(p+1)}', \ldots, \theta_{2k})}{h(\omega', \theta_{11}', \theta_{12}', \ldots, \theta_{1p}', \theta_{1(p+1)}', \ldots, \theta_{1k}, \theta_{2(p+1)}', \ldots, \theta_{2k})} \leq \frac{h(\omega, \theta_{11}', \theta_{12}', \ldots, \theta_{1p}', \theta_{1(p+1)}', \ldots, \theta_{1k}, \theta_{2(p+1)}', \ldots, \theta_{2k})}{h(\omega', \theta_{11}', \theta_{12}', \ldots, \theta_{1p}', \theta_{1(p+1)}', \ldots, \theta_{1k}, \theta_{2(p+1)}', \ldots, \theta_{2k})}
\]

equivalently,

\[
\frac{h(\omega, \theta_{11}', \theta_{12}', \ldots, \theta_{1p}', \theta_{1(p+1)}', \ldots, \theta_{1k}, \theta_{2(p+1)}', \ldots, \theta_{2k})}{h(\omega, \theta_{11}', \theta_{12}', \ldots, \theta_{1p}', \theta_{1(p+1)}', \ldots, \theta_{1k}, \theta_{2(p+1)}', \ldots, \theta_{2k})} \leq \frac{h(\omega, \theta_{11}', \theta_{12}', \ldots, \theta_{1p}', \theta_{1(p+1)}', \ldots, \theta_{1k}, \theta_{2(p+1)}', \ldots, \theta_{2k})}{h(\omega', \theta_{11}', \theta_{12}', \ldots, \theta_{1p}', \theta_{1(p+1)}', \ldots, \theta_{1k}, \theta_{2(p+1)}', \ldots, \theta_{2k})}.
\]

It follows from (4.8)
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\[
\sum_{i=1}^{2} X_{\theta_i} \geq \sum_{i=1}^{2} X_{\theta_i^*} \cdot (4.9)
\]

From (4.5) and (4.9) the proof of the theorem is complete.

In next theorem we extend Theorem 4.1 for a sum of \( n \) independent random variables having log-concave density function.

**Theorem 4.2.** Let \( X_{\theta_1}, \ldots, X_{\theta_n} \) be independent random variables from log-concave density functions \( f(x; \theta_i) \), with \( \theta_i = (\theta_{i1}, \ldots, \theta_{ik}) \), \( i = 1, \ldots, n \), and let \( X_{\theta_1^*}, \ldots, X_{\theta_n^*} \) be another set of independent random variables, independent of the first set, from log-concave density function \( f(x; \theta_i^*) \) with \( \theta_i^* = (\theta_{i1}^*, \ldots, \theta_{ik}^*) \), \( i = 1, \ldots, n \).

If \( (\theta_{1j}^*, \ldots, \theta_{nj}^*) \succ (\theta_{1j}, \ldots, \theta_{nj}) \) for \( j = 1, \ldots, t \), \( (t \leq k) \) with the other parameters fixed, implies

\[
\sum_{i=1}^{n} X_{\theta_i} \geq \sum_{i=1}^{n} X_{\theta_i^*} \cdot (4.10)
\]

then we get the likelihood ratio ordering of convolution of \( n \) random variables in terms of the majorization order of \( t \) vector parameters, that is

\[
\sum_{i=1}^{n} X_{\theta_i} \geq \sum_{i=1}^{n} X_{\theta_i^*} \cdot (4.10)
\]

**Proof.** Let for \( j = 1, \ldots, t \); \( (\theta_{1j}^*, \theta_{2j}^*) \succ (\theta_{1j}, \theta_{2j}) \) and \( \theta_{ij}^* = \theta_{ij} \), for \( i = 3, \ldots, n \). It follows from (4.1) that
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\[
\sum_{i=1}^{2} X_{\theta_1, \ldots, \theta_{(i+1)}, \ldots, \theta_k} \geq tr \sum_{i=1}^{2} X_{\theta_1, \ldots, \theta_{(i+1)}, \ldots, \theta_k}.
\]

Let \( S_{n-2} = \sum_{i=3}^{n} X_{\theta_1, \ldots, \theta_{(i+1)}, \ldots, \theta_k} \). From Dharmadhikari and Joag-dev (1988, p.17) convolution of log concave densities is log-concave, thus \( S_{n-2} \) has a log-concave density. Due to the independence of \( X_{\theta_i} \), \( i = 1, \ldots, n \) we know that \( S_{n-2} \) is independent of \( X_{\theta_1, \ldots, \theta_{(i+1)}, \ldots, \theta_k} \), \( X_{\theta_2, \ldots, \theta_{(i+1)}, \ldots, \theta_k} \), \( X_{\theta_1, \ldots, \theta_2, \theta_{(i+1)}, \ldots, \theta_k} \) and \( X_{\theta_2, \ldots, \theta_2, \theta_{(i+1)}, \ldots, \theta_k} \).

Then using Lemma 1.2 we have;

\[
\sum_{i=1}^{2} X_{\theta_1, \ldots, \theta_{(i+1)}, \ldots, \theta_k} + S_{n-2} \geq tr \sum_{i=1}^{2} X_{\theta_1, \ldots, \theta_{(i+1)}, \ldots, \theta_k} + S_{n-2}
\]

or

\[
\sum_{i=1}^{n} X_{\theta_1, \ldots, \theta_{(i+1)}, \ldots, \theta_k} \geq tr \sum_{i=1}^{2} X_{\theta_1, \ldots, \theta_{(i+1)}, \ldots, \theta_k} + \sum_{i=3}^{n} X_{\theta_1, \ldots, \theta_{(i+1)}, \ldots, \theta_k}.
\]

Now, we use Lemma 1.1 to obtain that,

\[
\sum_{i=1}^{n} X_{\theta_1, \ldots, \theta_{(i+1)}, \ldots, \theta_k} \geq tr \sum_{i=1}^{n} X_{\theta_1, \ldots, \theta_{(i+1)}, \ldots, \theta_k}.
\]

**Theorem 4.3.** Let \( X_{\theta_1}, X_{\theta_2} \) be independent random variables from density functions \( f(x; \theta_i) \), where \( \theta_i = \theta_{i1}, \ldots, \theta_{ik}, i = 1, 2 \) and let the density function of \( \sum_{i=1}^{2} X_{\theta_1, \ldots, \theta_k} \) be

\[
h(\omega, \theta_{i1}, \ldots, \theta_{i(1+1)}, \ldots, \theta_{i(k+1)}, \ldots, \theta_{2,1}, \ldots, \theta_{2(1+1)}, \ldots, \theta_{2,k})
\]

If \( (\theta_{1j}, \theta_{2j}) \prec (\theta_{1j}, \theta_{2j}) \) for \( j = 1, \ldots, s, 1 \leq s \leq k \), implies
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\[ \sum_{i=1}^{2} X_{\theta_{ij}, \theta_{ij}^*} \geq_{tr} \sum_{i=1}^{2} X_{\theta_{ij}, \theta_{ij}^*} \]

that is, the corresponding density function is \( TP_{2} (\omega, \theta_{j}) \) for each \( j = 1, ..., t \), and

\[ \sum_{i=1}^{2} X_{\theta_{ij}, \theta_{ij}^*} \geq_{tr} \sum_{i=1}^{2} X_{\theta_{ij}, \theta_{ij}^*} \]

that is the density function is \( TN_{2} (\omega, \theta_{j}) \) for \( j = t + 1, ..., s \), with the other parameters fixed, then

\[ \sum_{i=1}^{2} X_{\theta_{ij}, \theta_{ij}^*} \geq_{tr} \sum_{i=1}^{2} X_{\theta_{ij}, \theta_{ij}^*} \]

\[ \text{(4.11)} \]

**Proof.** Suppose \((\theta_{1j}^*, \theta_{2j}^*) \succ (\theta_{1j}, \theta_{2j})\) for \( j = 1, ..., t \), implies that

\[ h(\omega, \theta_{11}, ..., \theta_{1t}, \theta_{1(t+1)}, ..., \theta_{1j}, \theta_{1(t+1)}, ..., \theta_{1k}, \theta_{21}, ..., \theta_{2t}, \theta_{2(t+1)}, ..., \theta_{2s}, \theta_{2(t+1)}, ..., \theta_{2k}) \]

is \( TP_{2} (\omega, \theta_{j}) \) \( j = 1, ..., t \) with the other parameters fixed, then from Theorem 2.1 we have;

\[ \sum_{i=1}^{2} X_{\theta_{ij}, \theta_{ij}^*} \geq_{tr} \sum_{i=1}^{2} X_{\theta_{ij}, \theta_{ij}^*} \]

\[ \text{(4.12)} \]

and hence;

\[ \frac{h(\omega, \theta_{11}^*, \theta_{12}^*, ..., \theta_{1j}^*, \theta_{1(t+1)}, ..., \theta_{1k}, \theta_{21}, ..., \theta_{2s}, \theta_{2(t+1)}, ..., \theta_{2k})}{h(\omega, \theta_{11}, \theta_{12}, ..., \theta_{1j}, \theta_{1(t+1)}, ..., \theta_{1k}, \theta_{21}, ..., \theta_{2s}, \theta_{2(t+1)}, ..., \theta_{2k})} \]

\[ \leq \frac{h(\omega, \theta_{11}^*, \theta_{12}^*, ..., \theta_{1j}^*, \theta_{1(t+1)}, ..., \theta_{1k}, \theta_{21}, ..., \theta_{2s}, \theta_{2(t+1)}, ..., \theta_{2k})}{h(\omega, \theta_{11}, \theta_{12}, ..., \theta_{1j}, \theta_{1(t+1)}, ..., \theta_{1k}, \theta_{21}, ..., \theta_{2s}, \theta_{2(t+1)}, ..., \theta_{2k})} \]

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\[
\begin{align*}
\frac{h(\omega, \theta_1^*, \theta_2^*, ..., \theta_t^*, \theta_{(t+1)}^*, ..., \theta_{1s}^*, \theta_{1s+1}^*, ..., \theta_{1k}, \theta_{2s+1}, ..., \theta_{2k})}{h(\omega, \theta_1, \theta_2, ..., \theta_t, \theta_{(t+1)}, ..., \theta_{1s}, \theta_{1s+1}, ..., \theta_{1k}, \theta_{2s+1}, ..., \theta_{2k})} & \leq \frac{h(\omega, \theta_1, \theta_2, ..., \theta_t, \theta_{(t+1)}, ..., \theta_{1s}, \theta_{1s+1}, ..., \theta_{1k}, \theta_{2s+1}, ..., \theta_{2k})}{h(\omega, \theta_1^*, \theta_2^*, ..., \theta_t^*, \theta_{(t+1)}^*, ..., \theta_{1s}^*, \theta_{1s+1}^*, ..., \theta_{1k}, \theta_{2s+1}, ..., \theta_{2k})} \quad \text{(4.13)}
\end{align*}
\]

Now \((\theta_j^*, \theta_{2j}) > (\theta_j, \theta_{2j})\) for \(j = 1, ..., t\), implies

\[
\frac{h(\omega, \theta_1, \theta_2, ..., \theta_t, \theta_{(t+1)}, ..., \theta_{1s}, \theta_{1s+1}, ..., \theta_{1k}, \theta_{2s+1}, ..., \theta_{2k})}{h(\omega, \theta_1^*, \theta_2^*, ..., \theta_t^*, \theta_{(t+1)}^*, ..., \theta_{1s}^*, \theta_{1s+1}^*, ..., \theta_{1k}, \theta_{2s+1}, ..., \theta_{2k})} \quad \text{is} \quad TN_2(\omega, \theta_y) \quad \text{for} \quad j = t + 1, ..., s.
\]

At first let \((\theta_{(t+1)}^*, \theta_{2(t+1)}^*) > (\theta_{(t+1)}, \theta_{2(t+1)})\), and let \(h(.)\) be \(TN_2(\omega, \theta_{(t+1)})\) with the other parameters fixed, then we have;

\[
\begin{align*}
\frac{h(\omega, \theta_1, \theta_2, ..., \theta_t, \theta_{(t+1)}, ..., \theta_{1s}, \theta_{1s+1}, ..., \theta_{1k}, \theta_{2s+1}, ..., \theta_{2k})}{h(\omega, \theta_1^*, \theta_2^*, ..., \theta_t^*, \theta_{(t+1)}^*, ..., \theta_{1s}^*, \theta_{1s+1}^*, ..., \theta_{1k}, \theta_{2s+1}, ..., \theta_{2k})} & \leq \frac{h(\omega, \theta_1, \theta_2, ..., \theta_t, \theta_{(t+1)}, ..., \theta_{1s}, \theta_{1s+1}, ..., \theta_{1k}, \theta_{2s+1}, ..., \theta_{2k})}{h(\omega, \theta_1^*, \theta_2^*, ..., \theta_t^*, \theta_{(t+1)}^*, ..., \theta_{1s}^*, \theta_{1s+1}^*, ..., \theta_{1k}, \theta_{2s+1}, ..., \theta_{2k})} \\
& \quad \text{(4.14)}
\end{align*}
\]

or

\[
\begin{align*}
\frac{h(\omega, \theta_1, \theta_2, ..., \theta_t, \theta_{(t+1)}, ..., \theta_{1s}, \theta_{1s+1}, ..., \theta_{1k}, \theta_{2s+1}, ..., \theta_{2k})}{h(\omega, \theta_1^*, \theta_2^*, ..., \theta_t^*, \theta_{(t+1)}^*, ..., \theta_{1s}^*, \theta_{1s+1}^*, ..., \theta_{1k}, \theta_{2s+1}, ..., \theta_{2k})} & \leq \frac{h(\omega, \theta_1, \theta_2, ..., \theta_t, \theta_{(t+1)}, ..., \theta_{1s}, \theta_{1s+1}, ..., \theta_{1k}, \theta_{2s+1}, ..., \theta_{2k})}{h(\omega, \theta_1^*, \theta_2^*, ..., \theta_t^*, \theta_{(t+1)}^*, ..., \theta_{1s}^*, \theta_{1s+1}^*, ..., \theta_{1k}, \theta_{2s+1}, ..., \theta_{2k})} \quad \text{(4.13)}
\end{align*}
\]

From (4.13) and (4.14) we have

\[
\begin{align*}
\frac{h(\omega, \theta_1^*, \theta_2^*, ..., \theta_t^*, \theta_{(t+1)}^*, ..., \theta_{1s}^*, \theta_{1s+1}^*, ..., \theta_{1k}, \theta_{2s+1}, ..., \theta_{2k})}{h(\omega, \theta_1, \theta_2, ..., \theta_t, \theta_{(t+1)}, ..., \theta_{1s}, \theta_{1s+1}, ..., \theta_{1k}, \theta_{2s+1}, ..., \theta_{2k})} & \leq \frac{h(\omega, \theta_1, \theta_2, ..., \theta_t, \theta_{(t+1)}, ..., \theta_{1s}, \theta_{1s+1}, ..., \theta_{1k}, \theta_{2s+1}, ..., \theta_{2k})}{h(\omega, \theta_1^*, \theta_2^*, ..., \theta_t^*, \theta_{(t+1)}^*, ..., \theta_{1s}^*, \theta_{1s+1}^*, ..., \theta_{1k}, \theta_{2s+1}, ..., \theta_{2k})} \quad \text{(4.14)}
\end{align*}
\]

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\[ h(\omega, \theta_1^*, \theta_2^*, \ldots, \theta_t^*, \theta_{t+1}, \ldots, \theta_{s+1}, \ldots, \theta_{s+1}, \ldots, \theta_k, \theta_{2k}) \]
\[ h(\omega, \theta_1^*, \theta_1, \ldots, \theta_t, \theta_{t+1}, \ldots, \theta_{s+1}, \ldots, \theta_k, \theta_{2k}) \]
\[ \leq h(\omega, \theta_1^*, \theta_2^*, \ldots, \theta_t^*, \theta_{t+1}, \ldots, \theta_{s+1}, \ldots, \theta_k, \theta_{2k}) \]

(4.15)

Continuing in this manner it is clear that if \((\theta_{ij}^*, \theta_{ij}) > (\theta_{ij}, \theta_{ij}), j = 1, \ldots, s\), and if

\[ h(\omega, \theta_1^*, \theta_2^*, \ldots, \theta_t^*, \theta_{t+1}, \ldots, \theta_k, \theta_{2k}) \]

is \(TN_2(\omega, \theta_g^*)\) for \(j = t+1, \ldots, s\) then;

\[ h(\omega, \theta_1^*, \theta_2^*, \ldots, \theta_t^*, \theta_{t+1}, \ldots, \theta_k, \theta_{2k}) \]
\[ h(\omega, \theta_1^*, \theta_2^*, \ldots, \theta_t^*, \theta_{t+1}, \ldots, \theta_k, \theta_{2k}) \]
\[ \leq h(\omega, \theta_1^*, \theta_2^*, \ldots, \theta_t^*, \theta_{t+1}, \ldots, \theta_k, \theta_{2k}) \]

(4.16)

From (4.16) it is clear that

\[ \sum_{i=1}^{2} X_{\theta_1^*, \theta_2^*, \ldots, \theta_t^*, \theta_{t+1}, \ldots, \theta_k} \geq_{tr} \sum_{i=1}^{2} X_{\theta_1^*, \theta_2^*, \ldots, \theta_t^*, \theta_{t+1}, \ldots, \theta_k} . \]

**Theorem 4.4.** Let \(X_{\theta_1}, \ldots, X_{\theta_n}\) be independent random variables from log-concave density function \(f(x, \theta)\), \(\theta = \theta_1, \ldots, \theta_k\), \(i = 1, \ldots, n\). Let \(\sum_{i=1}^{n} X_{\theta_1, \ldots, \theta_k}\) have density function

\[ h(\omega, \theta_1, \ldots, \theta_{t+1}, \ldots, \theta_{t+1}, \ldots, \theta_k, \theta_{2k}) \]

and \((\theta_{ij}^*, \theta_{ij}) > (\theta_{ij}, \theta_{ij}), j = 1, \ldots, s\), \(1 \leq s \leq k\) imply that,

\[ \sum_{i=1}^{n} X_{\theta_1, \ldots, \theta_k} \geq_{tr} \sum_{i=1}^{n} X_{\theta_1, \ldots, \theta_k} . \]
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for \( j = 1, \ldots, t \), \((t \leq s)\) and

\[
\sum_{i=1}^{n} X_{\theta_1, \ldots, \theta_{i+1}, \ldots, \theta_k} \geq_{pr} \sum_{i=1}^{n} X_{\theta_1, \ldots, \theta_{i+1}, \ldots, \theta_k},
\]

for \( j = t+1, \ldots, s \), \((s \leq k)\) with the other parameters fixed then;

\[
\sum_{i=1}^{n} X_{\theta_1, \ldots, \theta_{i+1}, \ldots, \theta_k} \geq_{pr} \sum_{i=1}^{n} X_{\theta_1, \ldots, \theta_{i+1}, \ldots, \theta_k}. \tag{4.17}
\]

The proof is omitted since it is similar to that of Theorem 4.2.

We use some result of this section in the chapter 5 for generalized Rayleigh random variables with parameters \( \nu \) and \( \lambda \).