CHAPTER 6

Fingero-imbibition in double phase flow through porous media with magnetic fluid
6.1 Introduction

This chapter discusses the phenomena of finger-imbibition in double phase flow through homogeneous, slightly dipping porous media involving magnetic fluid. This phenomenon arises on account of simultaneous occurrence of two important phenomena viz. Imbibition and fingering [chapter 3, 4]. We have assumed that injection of preferentially wetting, less viscous fluid (viz. injected fluid) into porous medium saturated with resident fluid (viz. native fluid) is initiated under imbibition and in consequence, the resident fluid is pushed by pushed by drive in secondary recovery process. Verma called this conjoint phenomenon as fingero-imbibition.

It is well-known that when a porous medium filled with some resident fluid is brought into contact with another fluid which preferentially wets the medium, there is a spontaneous flow of the wetting fluid into the medium and a counter flow of the resident fluid from the medium. Such a phenomena arising due to difference in wetting abilities is called counter-current imbibitions. Similarly, when a fluid contained in a porous medium is displaced by another fluid of velocity, instead of regular displacement of whole front, protuberances (fingers) may occur which shoot through the porous medium at relatively great speeds. This phenomenon is called fingering or instabilities. The phenomena of fingering and imbibitions occurring simultaneously in displacement process, have gained much current importance due to their frequent occurrence in the problem of petroleum technology and many authors have discussed them from different point of view.

In this chapter, the underlying assumptions are that the two fluids are immiscible and the injected fluid is less viscous as well as preferentially wetting with respect to porous materials and with capillary pressure.

The mathematical formulation of basic equations yields a non-linear differential equation governing with finger imbition in the investigated liquid-displacement problem. A numerical solution has been obtained by ADM.
PROBLEM 1: FINGERO- IMBIBITION IN A HOMOGENEOUS MEDIUM INVOLVING MAGNETIC FLUID

6.2 Statement of the problem

We consider here a finite cylindrical mass of porous matrix of length \(L (=1)\) saturated with native liquid \((o)\), is completely surrounded by an impermeable surface except for one end of the cylinder which is labelled as the imbibitions phase \((x=0)\) and this end is exposed to an adjacent formation of ‘injected liquid’ \((w)\) which involves a thin layer of suitable magnetic fluid. It is assumed that the later fluid is preferentially wetting and less viscous. This arrangement gives rise to a displacement process in which the injection of the fluid \((w)\) is initiated by imbibitions and the consequent displacement of native liquid \((o)\) produces protuberances (fingers). This arrangement describes a one-dimensional phenomenon of fingero- imbibition.

6.3 Formulation of the problem

Assuming that the flow of two immiscible phase is governed by Darcy’s law, we may write the seepage velocity of injected and native fluid as,

\[
V_w = - \left( \frac{K_w}{\delta_w} \right) K \left[ \frac{\partial p_w}{\partial x} + \gamma H \frac{\partial H}{\partial x} \right] \quad (6.3.1)
\]

\[
V_o = - \left( \frac{K_o}{\delta_o} \right) K \left[ \frac{\partial p_o}{\partial x} \right] \quad (6.3.2)
\]

Where \(\gamma = \mu_o \lambda + \frac{16\mu_o \lambda^2 t^3}{g (t+2)^3}\)

\(K=\) the permeability of the homogeneous medium

\(K_w = \) relative permeability of injected fluid, which is function of \(S_w\)

\(K_o = \) relative permeability of injected fluid, which is function of \(S_o\)

\(S_w = \) the saturation of injected fluid

\(S_o = \) the saturation of native fluid

\(P_w = \) pressure of injected fluid

\(P_o = \) pressure of native fluid

\(\delta_w, \delta_o = \) Constants kinematics viscosities
\( g \) = acceleration due to gravity.

Neglecting the variation in phase densities, the equation of continuity for injected fluid can be written as:

\[
p \left( \frac{\partial S_w}{\partial t} \right) + \left( \frac{\partial V_w}{\partial x} \right) = 0 \tag{6.3.3}
\]

where \( p \) is porosity of the medium.

From the definition of phase saturation it is obvious that \( S_w + S_o = 1 \). The analytical condition (Scheidegger, 1960) governing imbibitions phenomenon is

\[
V_w + V_o = 0 \tag{6.3.4}
\]

From the definition of capillary pressure \( P_c \) as the pressure discontinuity between two phases yields \( P_c = P_o - P_w \) \tag{6.3.5}

Substituting the values of \( V_o \) and \( V_w \) from equations (6.3.1) and (6.3.2) into equation (6.3.4), we get,

\[
\left( \frac{K_w}{\delta_w} \right) K \left[ \frac{\partial P_w}{\partial x} + \gamma H \frac{\partial H}{\partial x} \right] + \left( \frac{K_o}{\delta_o} \right) K \left[ \frac{\partial P_o}{\partial x} \right] = 0 \tag{6.3.6}
\]

Thus, equation (6.3.6) reduces to the form,

\[
\frac{\partial P_w}{\partial x} = \left( \frac{\left( \frac{K_o}{\delta_o} \right) \frac{\partial P_c}{\partial x} + \left( \frac{K_w}{K_o} \right) \frac{\partial P_w}{\partial x} + \gamma H \frac{\partial H}{\partial x}}{\left( \frac{K_o}{\delta_o} + \frac{K_w}{\delta_w} \right)} \right) \tag{6.3.7}
\]

Equation (6.3.7) together with (6.3.7) yields

\[
V_w = \left( \frac{K_w K_o}{K_o \delta_w + K_w \delta_o} \right) K \left[ \frac{\partial P_c}{\partial x} - \gamma H \frac{\partial H}{\partial x} \right] \tag{6.3.8}
\]

Substituting equation (6.3.8) into (6.3.3) we get,

\[
p \left( \frac{\partial S_w}{\partial t} \right) + \frac{\partial}{\partial x} \left[ \frac{K_w K_o}{K_o \delta_w + K_w \delta_o} K \left[ \frac{\partial P_c}{\partial x} - \gamma H \frac{\partial H}{\partial x} \right] \right] = 0 \tag{6.3.9}
\]

This is the desired non-linear differential equation describing the finger-imbibition phenomenon for the flow of two immiscible phases through porous media.
Since the present investigation involves injected fluid and a viscous native fluid, therefore according to Scheidegger (1960) approximation, we may write equation (6.3.9) in the form

\[ p \left( \frac{\partial S_w}{\partial t} \right) + \frac{\partial}{\partial x} \left[ \left( \frac{K_o}{\delta_o} \right) K \left[ \frac{\partial P_c}{\partial x} - \gamma H \frac{\partial H}{\partial x} \right] \right] = 0 \]  \hspace{1cm} (6.3.10)

as

\[ \frac{K_w K_o}{K_o \delta_w + K_w \delta_o} \approx \frac{K_o}{\delta_o} \]

As this stat, for definiteness of the mathematical analysis, we assume standard relationship due to Scheidegger and Johnson [9], Muskat [25], between phase saturation and relative permeability as

\[ K_w = S_w, K_o = S_o = 1 - S_w \text{ and } P_c = -\beta_o S_w \]  \hspace{1cm} (6.3.11)

Where \( \beta_o \) is capillary pressure coefficient.

Substituting the values from equation (6.3.11) into (6.3.10) we get,

\[ p \left( \frac{\partial S_w}{\partial t} \right) + \frac{K}{\delta_o} \frac{\partial}{\partial x} \left( (1 - S_w) \left[ -\beta_o \frac{\partial S_w}{\partial x} - \gamma H \frac{\partial H}{\partial x} \right] \right) = 0 \]  \hspace{1cm} (6.3.12)

Considering the magnetic fluid \( H \) in the x-direction only, we may write [27], \( H = \frac{\Lambda}{x^n} \) where \( \Lambda \) is a constant parameter and \( n \) is an integer. Using the value of \( H \) for \( n=-1 \) in equation (6.3.12), we get,

\[ p \left( \frac{\partial S_w}{\partial t} \right) + \frac{K}{\delta_o} \frac{\partial}{\partial x} \left( (1 - S_w) \left[ -\beta_o \frac{\partial S_w}{\partial x} - \gamma x \lambda^2 \right] \right) = 0 \]  \hspace{1cm} (6.3.13)

A set of suitable initial and boundary conditions associated to problem (6.3.13) are

\[ S_w (x, 0) = S_c \text{ for all } x > 0 \]  \hspace{1cm} (6.3.14)

\[ S_w (0, t) = S_o; \quad S_w (L, t) = S_1 \text{ for all } t \geq 0 \]  \hspace{1cm} (6.3.15)

Equation (6.3.13) is reduced to dimensionless form by setting

\[ X = \frac{x}{L}, \quad T = \frac{K \beta t}{\delta_o L^2 p}, \quad S^*_w (x, t) = 1 - S_w (x, t) \]
and then equation (6.3.13) takes the form

$$\frac{\partial s_w}{\partial t} = \frac{\partial}{\partial x} \left( S_w \frac{\partial s_w}{\partial x} \right) - C_0 \frac{\partial}{\partial x} (S_w x)$$  \hspace{1cm} (6.3.16)

Where $C_0 = \frac{\gamma \Lambda^2 L^2}{\beta_0}$

With auxiliary conditions

$$S_w(x, 0) = 1 - S_c \hspace{1cm} 0 < x \leq L \hspace{1cm} (6.3.16(a))$$

$$S_w(0, t) = 1 - S_o \hspace{1cm} \text{for all } t \hspace{1cm} (6.3.16(b))$$

$$S_w(L, t) = 1 - S_1 \hspace{1cm} \text{for all } t \hspace{1cm} (6.3.16(c))$$

In equation (6.3.16), the asterisk are dropped for simplicity. *Equation (6.3.16) is desired nonlinear differential equation of motion for the flow of two immiscible liquids in homogeneous medium with effect of magnetic fluid.*

The problem is solved by using Adomian Decomposition Method. The numerical values are shown by table. Curves indicate the behaviour of saturation of water corresponding to various time periods.

### 6.4 Solution of the problem using Adomian Decomposition Method:

$$\frac{\partial s_w}{\partial t} = \frac{\partial}{\partial x} \left( S_w \frac{\partial s_w}{\partial x} \right) - C_0 \frac{\partial}{\partial x} (S_w x)$$  \hspace{1cm} (6.4.1)

$$\therefore (s_w)_t = (s_w(s_w)_x)_x - c_0 (s_w x)_x$$

Taking the initial condition $s_w(x, 0) = s_{w0} = f(x)$  \hspace{1cm} (6.4.2)

Applying the operator J on both the sides of equation (6.4.1) using initial condition (6.4.2),

$$s_w(x, t) = f(x) + \int \left[ \phi_1(s_w(x, t)) \right] - c_0 \int \left[ \phi_2(s_w(x, t)) \right]$$  \hspace{1cm} (6.4.3)

Where $\phi_1(s_w(x, t)) = s_w(s_w)_x + (s_w)_x^2$ and $\phi_2(s_w(x, t)) = s_w + xs_w_x$

Following Adomian decomposition method, the solution is represented as infinite series like,

$$s_w(x, t) = \sum_{n=0}^{\infty} s_{wn}(x, t)$$  \hspace{1cm} (6.4.4)
The nonlinear operator $\phi_1(s_w)$ & $\phi_2(s_w)$ are decomposed in these forms,

$$
\phi_1(s_w(x, t)) = \sum_{n=0}^{\infty} A_n, \quad \phi_1(s_w(x, t)) = \sum_{n=0}^{\infty} B_n
$$

(6.4.5)

Where $A_n$ and $B_n$ are so called Adomian polynomials and have the form,

$$
A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[ \phi_1 \left( \sum_{k=0}^{\infty} \lambda^k s_{w,k} \right) \right]_{\lambda=0}
$$

$$
= \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[ \left( \sum_{k=0}^{\infty} \lambda^k s_{w,k} \right) \left( \sum_{k=0}^{\infty} \lambda^k s_{w,0} \right) + \left( \sum_{k=0}^{\infty} \lambda^k (s_{w,k})^2 \right) \right]_{\lambda=0}
$$

$$
B_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[ \phi_2 \left( \sum_{k=0}^{\infty} \lambda^k s_{w,k} \right) \right]_{\lambda=0}
$$

$$
= \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[ \left( \sum_{k=0}^{\infty} \lambda^k s_{w,k} \right) + \left( \sum_{k=0}^{\infty} \lambda^k (kx) s_{w,kx} \right) \right]_{\lambda=0}
$$

The first three components of these polynomials are,

$$
A_0 = s_{w,0} (s_{w,0})_{xx} + (s_{w,0x})^2
$$

$$
A_1 = s_{w,0} s_{w,1} + s_{w,1} (s_{w,0})_{xx} + (s_{w,1x})^2
$$

$$
A_2 = s_{w,0} s_{w,2} + s_{w,1} s_{w,1xx} + s_{w,2} (s_{w,0})_{xxx} + (s_{w,2x})^2
$$

$$
A_3 = s_{w,0} s_{w,3} + s_{w,1} s_{w,2} + s_{w,2} s_{w,1xx} + s_{w,3} (s_{w,0})_{xxx} + (s_{w,3x})^2
$$

Similarly, $B_0 = s_{w,0}$

$$
B_1 = s_{w,1} + x s_{w,1x}
$$

$$
B_2 = s_{w,2} + (2x) s_{w,2x}
$$

$$
B_3 = s_{w,3} + (3x) s_{w,3x}
$$
\( \text{B}_4 = s_{w_4} + (4x)s_{w_4} \)

Other polynomials can be generated in like manner, substituting the decomposition series (6.4.4) and (6.4.5) into equation (6.4.3) yields the following recursive formula,

\[
S_w(x, t) = f(x)
\]

\[
s_{w_{n+1}}(x, t) = J(A_n) - c_0 J(B_n); \quad n \geq 0
\]

Let \( S_w(x, t) = f(x) = \frac{e^x - 1}{e - 1} \) and \( c_0 = 1 \)

\[
S_w_1 = J(A_0) - J(B_0)
\]

\[
= J \left( s_{w_0} (s_{w_0})_x + (s_{w_0})^2 \right) - J(s_{w_0})
\]

\[
= f_1 T \quad \text{Where } f_1 = ff_{xx} + f_x^2 - f = \frac{2e^{2x} - e^{x+1} + e - 1}{(e-1)^2}
\]

\[
S_w_2 = J(A_1) - J(B_1)
\]

\[
= J \left( s_{w_0} s_{w_1 xx} + s_{w_1} s_{w_0 xx} + (s_{w_1})^2 \right) - J(s_{w_1} + xs_{w_1 x})
\]

\[
= f_2 T \quad \text{Where } f_2 = ff_{1 xx} + f_1 f_{xx} + f_1 x^2 - f_1 - x f_1 x
\]

\[
= \frac{2e^{3x+1} + (8x-2)e^{2x+1} - (4x+3)e^{2(x+1)} - 10e^{3x} + (6-4x)e^{2x} + (x-2)e^{x+1} + e^x + 16e^{4x} + (x+1)e^{x+3} - e^3 + 3e^2 - 3e + 1 - 2xe^x + 2}{(e-1)^4}
\]

\[
\therefore \quad S_w(x, t) = \frac{e^x - 1}{e - 1} + \frac{2e^{2x} - e^{x+1} + e - 1}{(e-1)^2} \cdot t +
\]

\[
\frac{2e^{3x+1} + (8x-2)e^{2x+1} - (4x+3)e^{2(x+1)} - 10e^{3x} + (6-4x)e^{2x}}{(e-1)^4} \cdot t^2 + \frac{(x-2)e^{x+1} + e^x + 16e^{4x} + (x+1)e^{x+3} - e^3 + 3e^2 - 3e + 1 - 2xe^x + 2}{(e-1)^4} \cdot \frac{t^2}{2} + \ldots
\]
6.5 Results:

The following table shows the approximate solution for saturation of injected liquid for different values of x at different time using adomian decomposition method.

<table>
<thead>
<tr>
<th>x</th>
<th>t=0</th>
<th>t=0.1</th>
<th>t=0.2</th>
<th>t=0.3</th>
<th>t=0.4</th>
<th>t=0.5</th>
<th>t=0.6</th>
<th>t=0.7</th>
<th>t=0.8</th>
<th>t=0.9</th>
<th>t=1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.1</td>
<td>0.061207</td>
<td>0.007029</td>
<td>0.006803</td>
<td>0.006249</td>
<td>0.006075</td>
<td>0.005945</td>
<td>0.005847</td>
<td>0.005769</td>
<td>0.005707</td>
<td>0.005657</td>
<td>0</td>
</tr>
<tr>
<td>0.2</td>
<td>0.128851</td>
<td>0.023344</td>
<td>0.019957</td>
<td>0.017941</td>
<td>0.016758</td>
<td>0.016013</td>
<td>0.015511</td>
<td>0.015153</td>
<td>0.014887</td>
<td>0.014683</td>
<td>0.014521</td>
</tr>
<tr>
<td>0.3</td>
<td>0.203610</td>
<td>0.051125</td>
<td>0.041561</td>
<td>0.033515</td>
<td>0.031774</td>
<td>0.030634</td>
<td>0.029841</td>
<td>0.029264</td>
<td>0.028826</td>
<td>0.028485</td>
<td>0.028145</td>
</tr>
<tr>
<td>0.4</td>
<td>0.286230</td>
<td>0.093212</td>
<td>0.074629</td>
<td>0.064902</td>
<td>0.059609</td>
<td>0.056451</td>
<td>0.054409</td>
<td>0.053004</td>
<td>0.051988</td>
<td>0.051225</td>
<td>0.050634</td>
</tr>
<tr>
<td>0.5</td>
<td>0.377541</td>
<td>0.153387</td>
<td>0.123574</td>
<td>0.108177</td>
<td>0.099874</td>
<td>0.094952</td>
<td>0.091788</td>
<td>0.089621</td>
<td>0.088061</td>
<td>0.086893</td>
<td>0.085991</td>
</tr>
<tr>
<td>0.6</td>
<td>0.478454</td>
<td>0.236789</td>
<td>0.194964</td>
<td>0.173472</td>
<td>0.161919</td>
<td>0.155090</td>
<td>0.150709</td>
<td>0.147714</td>
<td>0.145561</td>
<td>0.143952</td>
<td>0.142711</td>
</tr>
<tr>
<td>0.7</td>
<td>0.589980</td>
<td>0.350557</td>
<td>0.298705</td>
<td>0.272075</td>
<td>0.257766</td>
<td>0.249312</td>
<td>0.243886</td>
<td>0.240179</td>
<td>0.237516</td>
<td>0.235525</td>
<td>0.233989</td>
</tr>
<tr>
<td>0.8</td>
<td>0.713236</td>
<td>0.504811</td>
<td>0.449894</td>
<td>0.421633</td>
<td>0.406427</td>
<td>0.397430</td>
<td>0.391656</td>
<td>0.387706</td>
<td>0.384866</td>
<td>0.382742</td>
<td>0.381103</td>
</tr>
<tr>
<td>0.9</td>
<td>0.849455</td>
<td>0.714154</td>
<td>0.671719</td>
<td>0.649803</td>
<td>0.637982</td>
<td>0.630975</td>
<td>0.626471</td>
<td>0.623386</td>
<td>0.621165</td>
<td>0.619503</td>
<td>0.618219</td>
</tr>
</tbody>
</table>

6.6 Interpretation
In graph X-axis represents the values of x and Y-axis represents the saturation of injected liquid \( (s_w) \) of porous media of length one. Solution is obtained for \( c_0 = 1 \).

Form given boundary conditions that the saturation of injected liquid i.e. \( (S_w=S_0=1) \) at imbibition face \( x=0 \) & there is no flow across other end \( (x=1) \) insensitive of time.

It is clear from graph that, for each value of \( t \), saturation of injected liquid involving magnetic fluid decreases with increase in values of \( x \) (or as we move ahead) and at \( x=1 \), saturation is decreased to zero and at particular point \( x \) of observed region, saturation of injected fluid increases with increase in time but the rate of increase of the saturation slows down at each point as time increases.

**PROBLEM 2: FINGER-O-IMBIBITION IN A SLIGHTLY DIPPING POROUS MEDIA INVOLVING MAGNETIC FLUID**

**6.7 Formulation of the problem**

In above model, the cylindrical of porous matrix is inclined at small angle \( \theta \) with the horizontal only and remaining process of fingero – imbibition goes as it is. Therefore, equation of seepage velocity of injected liquid \( (w) \) and native liquid \( (V_o) \) becomes

\[
V_w = -\left(\frac{K_w}{\delta_w}\right)K\left(\frac{\partial \rho_w}{\partial x} + \gamma H \frac{\partial H}{\partial x} + \rho_w g \sin \theta\right) \tag{6.7.1}
\]

Where \( \gamma = \mu_0 \lambda + \frac{16 \mu_0 r^3}{9(t+2)^3} \)

\[
V_o = -\left(\frac{K_o}{\delta_o}\right)K\left(\frac{\partial \rho_o}{\partial x} + \rho_o g \sin \theta\right) \tag{6.7.2}
\]

Where the phase densities \( \rho_w \) and \( \rho_o \) are regarded as constant, \( \delta_w \) and \( \delta_o \) are assumed to be invariant to magnetic field, \( g \) is the acceleration due to gravity and \( \theta \) is inclination.

At the imbibitions phase, we have,
\[ V_o = -V_w \quad (6.7.3) \]

Also,
\[ P_c = P_o - P_w \quad (6.7.4) \]

by the definition of capillary pressure.

On simplifying equation (6.7.1) and equation (6.7.2) by using equation (6.7.3) and equation (6.7.4), we obtain

\[ \frac{\partial P_o}{\partial x} = -K_w \delta_o \left[ \gamma H \frac{\partial H}{\partial x} + \rho_w g \sin \theta \right] - K_o \delta_w \left[ \frac{\partial P_o}{\partial x} + \rho_o g \sin \theta \right] \left( \frac{\partial P_c}{\partial x} \right) \]

Substituting above equation into equation (6.7.1), we obtain

\[ V_o = \frac{K K_o K_w}{(K_o/\delta_o + K_w/\delta_w)} \left[ \frac{\partial P_c}{\partial x} - \gamma H \frac{\partial H}{\partial x} + (\rho_o - \rho_w)g \sin \theta \right] \]

Since \( \frac{K_o K_w}{(K_o/\delta_o + K_w/\delta_w)} \approx \frac{K_o}{\delta_o} \), Above equation reduces to the form

\[ V_o = \frac{K K_o}{\delta_o} \left[ \frac{\partial P_c}{\partial x} - \gamma H \frac{\partial H}{\partial x} + (\rho_o - \rho_w)g \sin \theta \right] \quad (6.7.5) \]

The equations of continuity of the two phases remain same as problem1. From equation (6.7.5) and (6.3.3), we get,

\[ p \left( \frac{\partial S_w}{\partial t} \right) + \frac{\partial}{\partial x} \left( \frac{K K_o}{\delta_o} \left[ \frac{\partial P_c}{\partial x} - \gamma H \frac{\partial H}{\partial x} + (\rho_o - \rho_w)g \sin \theta \right] \right) = 0 \quad (6.7.6) \]

At this state, for definiteness of the mathematical analysis, we assume standard relationship between phase saturation and relative permeability as in problem1 [eq.(6.3.11)] and considering the magnetic fluid \( H \) in the \( x \)-direction only. We may write [16] \( H = \frac{\Lambda}{x^n} \) where \( \Lambda \) is a constant parameter and \( n \) is an integer.

Using the value of \( H \) for \( n=-1 \) in equation (6.7.6), we get, Therefore by equation (6.3.11), we have

\[ p \left( \frac{\partial S_w}{\partial t} \right) + K \frac{\partial}{\delta_o \partial x} \left( (1 - S_w) \left[ -\beta_0 \frac{\partial S_w}{\partial x} - \gamma \Lambda^2 x + (\rho_o - \rho_w)g \sin \theta \right] \right) = 0 \quad (6.7.7) \]

A set of suitable boundary and initial conditions associated to problem (6.7.7) are
Equation (6.7.7) together with (6.7.8) and (6.7.9) constitute the desired differential system. Equation (6.7.7) is reduced to dimensionless form by setting
\[ X = \frac{x}{L}, \quad T = \frac{t}{L^2 (C_1/C_2)}, \quad (1 - S_w(x,t)) = S_w(x,t) \]

So that \[ \frac{\partial s_w}{\partial T} - \frac{\partial}{\partial x} s_w \frac{\partial s_w}{\partial x} + c_0 \frac{\partial}{\partial x} (s_w x) - c_1 \left( \frac{\partial s_w}{\partial x} \right) = 0 \] (6.7.10)

Where \[ C_o = \frac{\Lambda^2 L^2 \alpha}{\beta_o}, \quad C_1 = \frac{L}{\beta_o} (\rho_o - \rho_w) g \sin \theta \] Asterisks are dropped for simplicity.

The initial and boundary conditions (6.7.8) & (6.7.9) now becomes,
\[ S_w(x,0) = 1 - S_1 \quad \text{for all } x > 0 \] (6.7.11)
\[ S_w(0,t) = 1 - S_{w_o}, \quad S_w(L, t) = 1 - S_{w_1} \quad \text{for all } t \geq 0 \] (6.7.12)

Equation (6.7.12) is desired nonlinear differential equation of motion for the flow of two immiscible liquids in homogeneous medium with effect of magnetic fluid.

The problem is solved by using Adomian Decomposition Method. The numerical values are shown by table. Curves indicate the behaviour of saturation of water corresponding to various time periods.

6.8 Solution of the problem using Adomian Decomposition Method:

\[ \frac{\partial s_w}{\partial T} = \frac{\partial}{\partial x} \left( s_w \frac{\partial s_w}{\partial x} \right) - c_0 \frac{\partial}{\partial x} (s_w x) + c_1 \left( \frac{\partial s_w}{\partial x} \right) \]

\[ \therefore (s_w)_T = (s_w (s_w)_x)_x - c_0 (s_w x)_x + c_1 (s_w)_x \] (6.8.1)

Taking the initial condition \[ s_w(x,0) = s_{w_0} = f(x) \] (6.8.2)

Applying the operator \( J \) on both the sides of equation (6.8.1) using initial condition (6.8.2),
\[ s_w(x, T) = f(x) + J \left( \phi_1(s_w(x, T)) \right) - c_0 \left[ J \left( \phi_2(s_w(x, T)) \right) \right] + c_1 \left[ J \left( \phi_2(s_w(x, T)) \right) \right] \]  
\[(6.8.3)\]

Where \( \phi_1(s_w(x, T)) = s_w(s_w)_{xx} + (s_w)_x^2 \), \( \phi_2(s_w(x, T)) = s_w + xs_w_x \) and \( \phi_3(s_w(x, T)) = s_w_x \)

Following Adomian decomposition method, the solution is represented as infinite series like,

\[ s_w(x, T) = \sum_{n=0}^{\infty} s_{wn}(x, T) \]  
\[(6.8.4)\]

The nonlinear operator \( \phi_1(s_w) \& \phi_2(s_w) \) are decomposed in these forms,

\[ \phi_1(s_w(x, T)) = \sum_{n=0}^{\infty} A_n \ , \ \phi_1(s_w(x, T)) = \sum_{n=0}^{\infty} B_n \]  
\[(6.8.5)\]

Where \( A_n \) and \( B_n \) are so called Adomian polynomials and have the form,

\[ A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[ \phi_1 \left( \sum_{k=0}^{\infty} \lambda^k s_{wk} \right) \right]_{\lambda=0} \]

\[ = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[ \left( \sum_{k=0}^{\infty} \lambda^k s_{wk} \right) \left( \sum_{k=0}^{\infty} \lambda^k s_{wk_{xx}} \right) + \left( \sum_{k=0}^{\infty} \lambda^k (s_{wk_{xx}})^2 \right) \right]_{\lambda=0} \]

\[ B_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[ \phi_2 \left( \sum_{k=0}^{\infty} \lambda^k s_{wk} \right) \right]_{\lambda=0} \]

\[ = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[ \left( \sum_{k=0}^{\infty} \lambda^k s_{wk} \right) + \left( \sum_{k=0}^{\infty} \lambda^k (kx) s_{wk_{xx}} \right) \right]_{\lambda=0} \]

\[ C_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[ \phi_2 \left( \sum_{k=0}^{\infty} \lambda^k s_{wk} \right) \right]_{\lambda=0} \]

\[ = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[ \sum_{k=0}^{\infty} \lambda^k s_{wk_{xx}} \right]_{\lambda=0} \]
The first three components of these polynomials are,

\[ A_0 = s_{w0}(s_{w0})_{xx} + (s_{w0x})^2 \]
\[ A_1 = s_{w0}s_{w1xx} + s_{w1}s_{w0xx} + (s_{w1x})^2 \]
\[ A_2 = s_{w0}s_{w2xx} + s_{w1}s_{w1xx} + s_{w2}s_{w0xx} + (s_{w2x})^2 \]
\[ A_3 = s_{w0}s_{w3xx} + s_{w1}s_{w2xx} + s_{w2}s_{w1xx} + s_{w3}s_{w0xx} + (s_{w3x})^2 \]

Similarly, \( B_0 = s_{w0} \)

\[ B_1 = s_{w1} + xs_{w1x} \]
\[ B_2 = s_{w2} + (2x)s_{w2x} \]
\[ B_3 = s_{w3} + (3x)s_{w3x} \]

Similarly, \( C_0 = s_{w0x}, C_1 = s_{w1x}, C_2 = s_{w2x}, C_3 = s_{w3x} \)

Other polynomials can be generated in like manner, substituting the decomposition series (6.8.4) and (6.8.5) into equation (6.8.3) yields the following recursive formula,

\[ S_{w0}(x, T) = f(x) \]
\[ s_{wn+1}(x, T) = J(A_n) - c_0 J(B_n) + c_1 J(C_n); \quad n \geq 0 \]

Let \( S_{w0}(x, T) = f(x) = \frac{e^x - 1}{e - 1} \) and \( c_0 = 1, c_1 = 1 \)

\[ S_{w1} = J(A_0) - J(B_0) + J(C_0) \]
\[ = J(s_{w0}(s_{w0})_{xx} + (s_{w0x})^2) - J(s_{w0}) + J(s_{w0x}) \]
\[ = f_1 T \quad \text{Where} \quad f_1 = ff_{xx} + f_x^2 - f + f_x = \frac{2e^{2x} - e^x + e - 1}{(e - 1)^2} \]
\[ S_{w_2} = J(A_1) - J(B_1) + J(C_1) \]

\[ = J\left(s_{w_0} s_{w_1 x} + s_{w_0} s_{w_0 x} + \left(s_{w_1 x}\right)^2\right) - J\left(s_{w_1} + x s_{w_1 x}\right) + J\left(s_{w_1 x}\right) \]

\[ = f_2 T \quad \text{Where} \quad f_2 = \frac{x f_{1xx} + f_{1xx} + f_{1x}^2}{x + x f_{1x} + f_{1x}} \]

\[ = \frac{10e^{3x+1} - (14-8x)e^{2x+1} + (2-4x)e^{2(x+1)} - 18e^{3x} + (13-4x)e^{2x} - (1+2x)e^{x+1} + 16e^{4x} + e^3 - 3e^2 + 3e - 1 - (1+x)e^{x+2}}{(e-1)^4} \]

\[ : s_w(x, t) = \frac{e^{x-1}}{e-1} + \frac{2e^{2x-3e^x + 2e^{x+1} - e+1}}{(e-1)^2} T + \]

\[ + \frac{10e^{3x+1} - (14-8x)e^{2x+1} + (2-4x)e^{2(x+1)} - 18e^{3x} + (13-4x)e^{2x} - (1+2x)e^{x+1}}{(e-1)^4} \cdot \frac{T^2}{2} \]

\[ + \frac{xe^x + 16e^{4x} + e^3 - 3e^2 + 3e - 1 - (1+x)e^{x+2}}{(e-1)^6} \cdot \frac{T^2}{2} \]

6.9 Results:

The following table shows the approximate solution for saturation of injected liquid for different values of x at different time using adomian decomposition method.

<table>
<thead>
<tr>
<th>x</th>
<th>T=0</th>
<th>T=0.1</th>
<th>T=0.2</th>
<th>T=0.3</th>
<th>T=0.4</th>
<th>T=0.5</th>
<th>T=0.6</th>
<th>T=0.7</th>
<th>T=0.8</th>
<th>T=0.9</th>
<th>T=1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.1</td>
<td>0.061207</td>
<td>0.00889</td>
<td>0.00783</td>
<td>0.006656</td>
<td>0.005782</td>
<td>0.005148</td>
<td>0.004677</td>
<td>0.004316</td>
<td>0.004033</td>
<td>0.003805</td>
<td>0.003618</td>
</tr>
<tr>
<td>0.2</td>
<td>0.128851</td>
<td>0.026697</td>
<td>0.021659</td>
<td>0.017897</td>
<td>0.015399</td>
<td>0.013688</td>
<td>0.012461</td>
<td>0.011545</td>
<td>0.010839</td>
<td>0.010278</td>
<td>0.009823</td>
</tr>
<tr>
<td>0.3</td>
<td>0.20361</td>
<td>0.055553</td>
<td>0.043564</td>
<td>0.035763</td>
<td>0.030866</td>
<td>0.027618</td>
<td>0.02534</td>
<td>0.023666</td>
<td>0.022389</td>
<td>0.021385</td>
<td>0.020577</td>
</tr>
<tr>
<td>0.4</td>
<td>0.28623</td>
<td>0.098257</td>
<td>0.076569</td>
<td>0.06338</td>
<td>0.055362</td>
<td>0.050149</td>
<td>0.046543</td>
<td>0.04392</td>
<td>0.041936</td>
<td>0.040386</td>
<td>0.039145</td>
</tr>
<tr>
<td>0.5</td>
<td>0.377541</td>
<td>0.158568</td>
<td>0.125132</td>
<td>0.105566</td>
<td>0.093905</td>
<td>0.086419</td>
<td>0.081288</td>
<td>0.077581</td>
<td>0.074792</td>
<td>0.072624</td>
<td>0.070894</td>
</tr>
<tr>
<td>0.6</td>
<td>0.478454</td>
<td>0.241627</td>
<td>0.19592</td>
<td>0.169791</td>
<td>0.154411</td>
<td>0.144621</td>
<td>0.137951</td>
<td>0.13155</td>
<td>0.12956</td>
<td>0.126774</td>
<td>0.124556</td>
</tr>
<tr>
<td>0.7</td>
<td>0.58998</td>
<td>0.354609</td>
<td>0.298995</td>
<td>0.267661</td>
<td>0.249365</td>
<td>0.237781</td>
<td>0.22992</td>
<td>0.224286</td>
<td>0.220072</td>
<td>0.216814</td>
<td>0.214225</td>
</tr>
<tr>
<td>0.8</td>
<td>0.713236</td>
<td>0.50771</td>
<td>0.449666</td>
<td>0.417251</td>
<td>0.398419</td>
<td>0.386537</td>
<td>0.378494</td>
<td>0.37274</td>
<td>0.368445</td>
<td>0.365127</td>
<td>0.362494</td>
</tr>
<tr>
<td>0.9</td>
<td>0.849455</td>
<td>0.715654</td>
<td>0.671346</td>
<td>0.64673</td>
<td>0.632471</td>
<td>0.623492</td>
<td>0.617423</td>
<td>0.61087</td>
<td>0.609853</td>
<td>0.607357</td>
<td>0.605377</td>
</tr>
</tbody>
</table>
6.10 Interpretation

In the graph X-axis represents the values of x and Y-axis represents the saturation of injected liquid involving magnetic fluid ($s_{w}$) in porous media of length one.

It is clear from graph that, at particular time, saturation of injected liquid involving magnetic fluid decrease with increase in value of x (or as we move ahead) and at x=1, saturation is decreased to zero and as time increases, rate of increase of the saturation of injected liquid decreases at each layer.

It is clear from graphs of both the problem 1 and 2 saturation of injected fluid decrease at little bit faster rate in comparison with problem 1 for corresponding values of constants and step length in x- and y- direction respectively.