CHAPTER 4

Inequalities on $k$-idempotent matrices

In this chapter, it is shown that all the standard partial orderings [2,7,24] such as Lowener, star and rank subtractivity are preserved under the fixed product of disjoint transpositions $k$. That is all the partial orderings are preserved under $k$-unitary similarity. Relation between $k$-hermitian matrix and $k$-idempotent matrix is derived here by means of Lowener partial order. It is proved that all the partial orderings are preserved for $k$-idempotent matrices when they are squared.
4.1 \( k \)-invariant partial orderings on matrices

In this section, it is to be proved that all the standard partial orderings are preserved under \( k \)-unitary similarity. The Lowener partial order, the star order and the minus order (rank subtractivity order) denoted by \( \succeq_{L} \), \( \succeq_{*} \) and \( \succeq_{rs} \) respectively are defined as follows:

**Definition 4.1.1**

For \( A, B \in \mathbb{C}^{n \times n} \),

\[ A \succeq_{L} B \iff A - B \geq 0. \]

\[ A \succeq_{*} B \iff B^*B = B^*A \text{ and } BB^* = AB^* \]

\[ A \succeq_{rs} B \iff \text{rank}(A - B) = \text{rank}(A) - \text{rank}(B). \]

**Remark 4.1.2**

The following useful result (cf.[25]) explains the relation between Lowener partial order and the spectral radius of the transformation. For \( A, B \in \mathbb{C}^{n \times n} \),

\[ A \succeq_{L} B \iff \rho(A^\dagger B) \leq 1 \text{ and } R(B) \subseteq R(A), \]

where \( \rho(A) = \max \{ |\lambda| : \lambda \text{-an eigen value of } A \} \) is the spectral radius of \( A \).

**Lemma 4.1.3**

If \( K \) is the associated permutation matrix of \( k \), then for \( A, B \in \mathbb{C}^{n \times n} \),

\[ A \succeq_{L} B \iff KA \succeq_{L} KB \iff AK \succeq_{L} BK \]

**Proof**

\[ A \succeq_{L} B \iff \rho(A^\dagger B) \leq 1 \text{ and } R(B) \subseteq R(A) \]

\[ \iff \rho(A^\dagger KKB) \leq 1 \text{ and } B = AA^\dagger B \] \[ \text{ [by remark 4.1.2] } \]

\[ \iff \rho(A^\dagger KKB) \leq 1 \text{ and } KB = KAA^\dagger KKB \] \[ \text{ [by theorem 1.2.9] } \]

\[ \iff \rho((KA)^\dagger KB) \leq 1 \text{ and } R(KB) \subseteq R(KA) \] \[ \text{ [by theorem 1.2.9] } \]
\[
\Leftrightarrow KA \succeq_l KB \quad \text{[by remark 4.1.2]}
\]

Similarly,
\[
A \succeq_l B \Leftrightarrow \rho(A^\dagger B) \leq 1 \text{ and } R(B) \subseteq R(A) \quad \text{[by remark 4.1.2]}
\]
\[
\Leftrightarrow \rho(KA^\dagger BK) \leq 1 \text{ and } B = AA^\dagger B \quad \text{[by theorem 1.2.9]}
\]
\[
\Leftrightarrow \rho[(AK)^\dagger BK] \leq 1 \text{ and } BK = AK(AK)^\dagger BK
\]
\[
\Leftrightarrow \rho[(AK)^\dagger BK] \leq 1 \text{ and } R(BK) \subseteq R(AK) \quad \text{[by theorem 1.2.9]}
\]
\[
\Leftrightarrow AK \succeq_l BK \quad \text{[by remark 4.1.2]}
\]

Hence the proof. ■

**Result 4.1.4**

It can be easily verified that Lowener ordering is preserved under unitary similarity. That is

\[
A \succeq_l B \Leftrightarrow P^*AP \succeq_l P^*BP
\]

**Theorem 4.1.5**

Lowener ordering is preserved for \(k\)-unitary similarity.

**Proof**
\[
A \succeq_l B \Leftrightarrow KA \succeq_l KB \quad \text{[by lemma 4.1.3]}
\]
\[
\Leftrightarrow P^*KAP \succeq_l P^*KBP \quad \text{[by result 4.1.4]}
\]
\[
\Leftrightarrow KP^*KAP \succeq_l KP^*KBP \quad \text{[by lemma 4.1.3]}
\]
\[
\Leftrightarrow KP^{-1}KAP \succeq_l KP^{-1}KBP
\]

If \(C = KP^{-1}KAP\) then \(C\) is unitarily \(k\)-similar to \(A\).

If \(D = KP^{-1}KBP\) then \(D\) is unitarily \(k\)-similar to \(B\).
Therefore $C \succeq_l D$ and hence Lowener ordering is preserved for $k$-unitary similarity.

**Note 4.1.6**

If $K = I$ in theorem 4.1.5, it reduces to result 4.1.4.

**Remark 4.1.7**

The following result (see[24]) establishes a relation between star ordering and rank subtractivity. For $A, B \in \mathbb{C}^{n \times n}$,

$$A \succeq \ast B \iff A \succeq \ast \frac{1}{r^2} \quad \text{and} \quad (A - B)^\dagger = A^\dagger - B^\dagger$$

**Lemma 4.1.8**

If $K$ is the associated permutation matrix of $k$, then for $A, B \in \mathbb{C}^{n \times n}$,

$$A \succeq \ast B \iff KA \succeq \ast KB \iff AK \succeq \ast BK$$

**Proof**

$$A \succeq \ast B \iff B^*B = B^*A \text{ and } BB^* = AB^*$$

$$\iff B^*KKB = B^*KKKA \text{ and } KBB^*K = KAB^*K$$

$$\iff (KB)^*KB = (KB)^*KA \text{ and } KB(KB)^* = KA(KB)^*$$

$$\iff KA \succeq \ast KB$$

Similarly,

$$A \succeq \ast B \iff B^*B = B^*A \text{ and } BB^* = AB^*$$

$$\iff KB^*BK = KB^*AK \text{ and } BKKB^* = AKKB^*$$

$$\iff (BK)^*BK = (BK)^*AK \text{ and } BK(BK)^* = AK(BK)^*$$

$$\iff AK \succeq \ast BK$$
Result 4.1.9

It can be easily verified that star ordering is preserved under unitary similarity.

That is

\[ A \succeq B \iff P^*AP \succeq P^*BP \]

Theorem 4.1.10

Star ordering is preserved for $k$-unitary similarity.

Proof

\[ A \succeq B \iff KA \succeq KB \quad \text{[by lemma 4.1.8]} \]

\[ \iff P^*KAP \succeq P^*KBP \quad \text{[by result 4.1.9]} \]

\[ \iff KP^*KAP \succeq KP^*KBP \quad \text{[by lemma 4.1.8]} \]

\[ \iff KP^{-1}KAP \succeq KP^{-1}KBP \]

If \( C = KP^{-1}KAP \) then \( C \) is unitarily $k$-similar to \( A \).

If \( D = KP^{-1}KBP \) then \( D \) is unitarily $k$-similar to \( B \).

Therefore \( C \succeq D \) and hence star ordering is preserved for $k$-unitary similarity. \[ \square \]

Note 4.1.11

If \( K = I \) in theorem 4.1.10, it reduces to result 4.1.9.

Remark 4.1.12

The following result exhibits an equivalent condition for minus ordering of two matrices \( A \) and \( B \). For \( A, B \in \mathbb{C}^{n \times n} \),

\[ A \succeq_{rs} B \iff B = BA^{(1)}B = BA^{(1)}A = AA^{(1)}B \]

Lemma 4.1.13

If \( K \) is the associated permutation matrix of \( k \), then for \( A, B \in \mathbb{C}^{n \times n} \),

\[ A \succeq_{rs} B \iff KA \succeq_{rs} KB \iff AK \succeq_{rs} BK \]
Proof

\[ A \succeq B \iff \text{rank}(A - B) = \text{rank}(A) - \text{rank}(B) \]

\[ \iff \text{rank}(K(A - B)) = \text{rank}(KA) - \text{rank}(KB) \]

\[ \iff \text{rank}(KA - KB) = \text{rank}(KA) - \text{rank}(KB) \]

\[ \iff KA \succeq KB \]

Similarly,

\[ A \succeq B \iff \text{rank}(A - B) = \text{rank}(A) - \text{rank}(B) \]

\[ \iff \text{rank}((A - B)K) = \text{rank}(AK) - \text{rank}(BK) \]

\[ \iff \text{rank}(AK - BK) = \text{rank}(AK) - \text{rank}(BK) \]

\[ \iff AK \succeq BK \]

Result 4.1.14

It can be easily verified that rank subtractivity ordering is preserved under unitary similarity. That is

\[ A \succeq B \iff P^*AP \succeq P^*BP \]

Theorem 4.1.15

Rank subtractivity ordering is preserved for \( k \)-unitary similarity.

Proof

\[ A \succeq B \iff KA \succeq KB \]

\[ \iff P^*KAP \succeq P^*KBP \]

\[ \iff KP^*KAP \succeq KP^*KBP \]

\[ \iff KP^{-1}KAP \succeq KP^{-1}KBP \]

If \( C = KP^{-1}KAP \) then \( C \) is unitarily \( k \)-similar to \( A \).

If \( D = KP^{-1}KBP \) then \( D \) is unitarily \( k \)-similar to \( B \).
Therefore $C \preceq_{rs} D$ and hence rank subactivity ordering is preserved for $k$-unitary similarity.

\textbf{Note 4.1.16}

If $K = I$ in \textit{theorem 4.1.15}, it reduces to \textit{result 4.1.14}.
4.2 Partial orderings on \( k \)-idempotent matrices

In this section, partial orderings on \( k \)-idempotent matrices is discussed and a relation between a square hermitian matrix and a \( k \)-idempotent matrix through Lowener partial order is discovered.

**Lemma 4.2.1**

If \( A \succeq_l B \) for matrices \( A \) and \( B \) then \( A - B \) is \( k \)-hermitian.

**Proof**

\[ A \succeq_l B \implies KA \succeq_l KB \]  
[by lemma 4.1.3]

\[ KA - KB \geq 0 \]

\[ K(A - B) \geq 0 \]

Hence \( A - B \) is \( k \)-hermitian positive semi definite.

That is \( (A - B)^* = K(A - B)K \)

\( A - B \) is \( k \)-hermitian.

**Theorem 4.2.2**

Let \( A \succeq_l B \) and \( A \succeq_r B \). If \( B \) is a \( k \)-idempotent matrix then \( B^{(1)} = KAK \).

**Proof**

\( A \succeq_r B \) implies that

\[ BB^* = AB^* \]

\[ B^*B = B^*A \]

Therefore \( B^*(A - B) = 0 \)

\[ [(A - B)^*B]^* = 0 \]

(4.1) \( (A - B)^*B = 0 \)

Since \( A \succeq_l B \), we have \( (A - B)^* = K(A - B)K \) by lemma 4.2.1.

Hence from (4.1), we have
\[ K(A - B)KB = 0 \]
\[ KAKB - KBKB = 0 \]
\[ KAKB - B^3 = 0 \]
\[ BKAKB - B = 0 \]

That is \( B(KAK)B = B \)

Therefore \( B^{(1)} = KAK \) \[ \square \]

**Theorem 4.2.3**

Let \( A \geq B \). If \( A \) is \( k \)-hermitian \( k \)-idempotent then \( B \) is \( k \)-hermitian and vice versa.

**Proof**

Since \( A \geq B \), we have \( A - B \) is \( k \)-hermitian. \[ \text{[ by lemma 4.2.1]} \]

That is \( (A - B)^* = K(A - B)K \)

(4.2) \[ A^* - B^* = KAK - KBK \]

If \( A \) is \( k \)-hermitian \( k \)-idempotent then \( A \) is square hermitian by *theorem 2.3.1*.

(4.2) becomes

\[ A^2 - B^* = A^2 - KBK \]

\[ B^* = KBK \]

Hence \( B \) is \( k \)-hermitian.

On the other hand, if \( B \) is \( k \)-hermitian \( k \)-idempotent then \( B \) is square hermitian by *theorem 2.3.1*.

(4.2) becomes

\[ A^* - B^2 = KAK - B^2 \]

\[ A^* = KAK \]
Hence $A$ is $k$-hermitian.

**Theorem 4.2.4**

Let $A$ and $B$ are $k$-idempotent matrices. Then

\[ A \geq_l B \text{ if and only if } A^2 \geq_l B^2 \]

**Proof**

Assuming that $A \geq_l B$

\[ KA \geq_l KB \]

\[ KAK \geq_l KBK \]

Therefore \[ A^2 \geq_l B^2 \]

Conversely, if we assume $A^2 \geq_l B^2$, then we have $A^4 \geq_l B^4$ by what we have proved above and theorem 2.1.7 (b).

That is $A \geq_l B$. \[ \text{[ by theorem 2.1.7 (c) ]} \]

Hence the theorem is proved.

**Theorem 4.2.5**

Let $A$ and $B$ are $k$-idempotent matrices. Then

\[ A \geq_v B \text{ if and only if } A^2 \geq_v B^2 \]

**Proof**

Assuming that $A \geq_v B$ then we have

(i) $B^*B = B^*A$

(ii) $BB^* = AB^*$

From (i), $KB^*BK = KB^*AK$

\[ (B^*)^2KKB^2 = (B^*)^2KKA^2 \] \[ \text{[ by theorem 2.1.7 (a) ]} \]

(4.3) \[ (B^2)^*B^2 = (B^2)^*A^2 \]
From (ii), \( KBB^*K = KAB^*K \)

\[
B^2KK(B^*)^2 = A^2KK(B^*)^2
\]

(4.4) \( B^2(B^2)^* = A^2(B^2)^* \)

From (4.3) and (4.4), we have \( A^2 \geq_r B^2 \)

Conversely, if we assume that \( A^2 \geq_r B^2 \) then we have \( A^4 \geq_r B^4 \) by what we have proved above and theorem 2.1.7 (b).

That is \( A \geq_r B \) \hspace{1cm} [ by theorem 2.1.7 (c) ]

Hence the theorem is proved.

\[ \]

**Theorem 4.2.6**

Let \( A \) and \( B \) are \( k \)-idempotent matrices. Then

\( A \geq_{rs} B \) if and only if \( A^2 \geq_{rs} B^2 \)

**Proof**

Assuming that \( A \geq_{rs} B \)

\[
B = BA^{(1)}A = AA^{(1)}B = BA^{(1)}B
\]

[ by remark 4.1.12]

\[
KBB = KBA^{(1)}AK = KAA^{(1)}BK = KBA^{(1)}BK
\]

\[
B^2 = B^2KA^{(1)}KA^2 = A^2KA^{(1)}KB^2 = B^2KA^{(1)}KB^2
\]

\[
\]

Therefore \( A^2 \geq_{rs} B^2 \) \hspace{1cm} [ by remark 4.1.12]

Conversely, if we assume that \( A^2 \geq_{rs} B^2 \) then we have \( A^4 \geq_{rs} B^4 \) by what we have proved above and theorem 2.1.7 (b).

That is \( A \geq_{rs} B \) \hspace{1cm} [ by theorem 2.1.7 (c) ]

Hence the theorem is proved.
**Theorem 4.2.7**

If $A$ and $B$ are two disjoint square hermitian and skew square hermitian matrices such that $A \gtrless_{l} B$ then $A - B$ is $k$-idempotent.

**Proof**

\[
(A - B)^2 = A^2 + B^2 \quad \text{[by } AB = BA = 0 \text{]}
\]

\[
= A^* - B^*
\]

(4.5) \quad (A - B)^2 = (A - B)^*

Since $A \gtrless_{l} B$, we have $A - B$ is $k$-hermitian. \text{[by lemma 4.2.1]}

That is $(A - B)^* = K(A - B)K$

(4.5) becomes,

\[
(A - B)^2 = K(A - B)K
\]

\[
K(A - B)^2K = (A - B)
\]

Hence $A - B$ is $k$-idempotent. □

**Note 4.2.8**

The following theorem can be considered to be somehow a reverse of the above theorem 4.2.7

**Theorem 4.2.9**

If $A$ and $B$ are two disjoint $k$-idempotent matrices such that $A \gtrless_{l} B$ then $A$ and $B$ are square hermitian matrices.

**Proof**

$A \gtrless_{l} B$ implies that $A - B$ is $k$-hermitian \text{[by lemma 4.2.1]}

(4.6) \quad (A - B)^* = K(A - B)K

\[
A^* - B^* = KAK - KBK
\]

(4.7) \quad A^* - B^* = A^2 - B^2
Squaring (4.6), we have

\[
\begin{align*}
[(A - B)^\ast]^2 &= K(A - B)^2K \\
[(A - B)^2]^\ast &= K(A^2 + B^2)K & \text{[ by } AB = BA = 0 \text{ ]}
\end{align*}
\]

\[
[(A^2 + B^2)]^\ast = KA^2K + KB^2K
\]

\[
[(A^2 + B^2)]^\ast = A + B
\]

\[
A^2 + B^2 = (A + B)^\ast
\]

(4.8) \quad A^2 + B^2 = A^\ast + B^\ast

From (4.7) and (4.8), solving for \( A^\ast \) and \( B^\ast \) we have \( A^\ast = A^2 \) and \( B^\ast = B^2 \).

That is \( A \) and \( B \) are square Hermitian matrices. \( \blacksquare \)