CHAPTER III
$g^*-$CLOSED FUZZY SETS, MAPS AND OTHER RELATED CONCEPTS IN FTS

3.1 INTRODUCTION

The concepts of $g^*$-closed sets and $T^{*}_{1/2}$, $T^{*}_{1/2}$ - spaces were introduced and studied by Veerakumar [54] in the year 2000 for general topological spaces. Also $g$-continuous maps were introduced and studied in [54].

In section 2 of this chapter, $g^*$-closed fuzzy sets have been introduced and studied and as an application of these fuzzy sets, two new spaces namely fuzzy-$T^{*}_{1/2}$ and fuzzy-$T_{1/2}$ have been introduced and studied for fuzzy topological spaces. It is observed that every $g^*$-closed fuzzy set is $g$-closed fuzzy set and a $b$-closed fuzzy set but not conversely. Characterisation of $g^*$-closed fuzzy sets is obtained. Further fuzzy $g^*$-closure and fuzzy $g^*$-interior concepts have been investigated. It is observed that every fuzzy-$T_{1/2}$ space is fuzzy-$T^{*}_{1/2}$ and also fuzzy-$T_{1/2}$.

In section 3, fuzzy $g^*$-continuous, $g^*$-closed and related concepts in fts have been introduced and studied. It is observed that every $fg^*$-continuous function is $fg$-continuous and also $fb$-continuous but not conversely. A characterisation of $g^*$-continuous maps is obtained. Results on composition of $g^*$-continuous, $g^*$-closed and $g^*$-homeomorphism maps are obtained. It is observed that every closed map is $g^*$-closed but not conversely. It is also observed that the image of $g^*$-closed fuzzy set is $g^*$-closed under fuzzy $gc$-irresolute, closed map. Also the image of normal fts is normal under an $f$-continuous, $fg^*$-closed map. It is proved that every strongly $gf$-continuous and also strongly $fb$-continuous map is
strongly \( f_{g^*} \)-continuous and the image of regular fts is regular under \( f \)-continuous \( f \)-open, \( f_{g^*} \)-closed map.

In the fourth and final section, \( g^* \)-compactness, countable \( g^* \)-compactness and \( g^* \)-Lindelof property have been introduced and studied. It is proved that every \( f_g \)-compact fts is \( f_{g^*} \)-compact but not conversely. A characterisation of \( g^* \)-compact and its related concepts in fts have been obtained. Further it is observed that the image of \( g^* \)-compact fts under \( f_{g^*} \)-continuous function is fuzzy compact. In the last part of this section, \( g^* \)-regular and \( g^* \)-normal fts have been introduced and studied. Characterisations of \( g^* \)-normal and \( g^* \)-regular fts are obtained. It is observed that every \( g^* \)-regular fts is regular. The image of \( g^* \)-regular fts is \( g^* \)-regular under open, \( f_{g^*} \)-irresolute bijective map. It is observed that every \( b \)-regular (resp. \( b \)-normal) fts is \( g^* \)-regular (resp. \( g^* \)-normal) but not conversely.

This chapter contains several counterexamples.

3.2 \( g^* \) - CLOSED FUZZY SETS IN FTS

3.2.1 Definition: A fuzzy set \( A \) of a fts \( X \) is called \( g^* \)-closed fuzzy set if \( \text{cl} (A) \leq U \) whenever \( A \leq U \) and \( U \) is \( g \)-open fuzzy set in \( X \).

3.2.2 Theorem: Every closed fuzzy set is \( g^* \)-closed fuzzy set in fts \( X \).

Proof: Let \( A \) be a closed fuzzy set in a fts \( X \). Let \( A \leq U \) where \( U \) is \( g \)-open fuzzy set in \( X \). Since \( A \) is closed, we have \( \text{cl} (A) = A \leq U \). That is \( \text{cl} (A) \leq U \). Hence \( A \) is \( g^* \)-closed fuzzy set.

The converse of the above theorem need not be true as seen from the following example.

3.2.3 Example: Let \( X = \{a,b,c\} \). The fuzzy sets \( A \) and \( B \) be defined as follows: \( A = \{(a, .4), (b, .5), (c, .7)\} \), \( B = \{(a, 1), (b, .9), (c, .8)\} \). Let
T = \{0, 1, A\}. Then (X, T) is fts. Note that fuzzy subset B is g*-closed but not closed in (X, T).

**3.2.4. Theorem:** In any fts X, every g*-closed fuzzy set is g-affilled.

**Proof:** Let A be a g*-closed fuzzy set in X. Let A \leq U where U is open and so g-open fuzzy set. Since A is g*-closed, we have cl(A) \leq U. And hence A is g-affilled fuzzy set.

The converse of the above theorem need not be true as seen from the following example.

**3.2.5 Example:** Let X = \{a, b, c\}. The fuzzy sets A and B be defined as follows: A = \{(a, .2), (b, .5), (c, .3)\}, B = \{(a, .5), (b, .2), (c, .3)\}. Let T = \{0, 1, A\}. Then (X, T) is fts. Here the fuzzy set B is g-affilled but not g*-affilled in (X, T).

**3.2.6 Theorem:** Every g* - closed fuzzy set is a b - closed fuzzy set in any fts X.

**Proof:** Let A be a g* - closed fuzzy set in a fts X.

Let A \leq U where U be open fuzzy set and so is g - open fuzzy set. Since A is g* - closed, we have cl(A) \leq U. Since bd(A) \leq clA \leq U, it follows that bd(A) \leq U. Hence A is b - closed in X.

The converse of the above theorem need not be true as seen from the following example.

**3.2.7 Example:** Let X = \{a, b, c\}. Fuzzy sets A and B be defined as follows: A = \{(a, .2), (b, .5), (c, .3)\}, B = \{(a, .5), (b, .2), (c, .3)\}. Then (X, T) is fts with fuzzy topology T = \{0, 1, A\}. Here the fuzzy set B is b - closed but not g* - closed.

**3.2.8 Theorem:** If a fuzzy set A of a fts X is both open and g* - closed fuzzy set then it is closed.

**Proof:** Suppose a fuzzy set A of a fts X is both open and g*-closed.
Now $A \leq A$, $A$ is open and so is $g$-open. This implies that $\text{cl}(A) \leq A$ since $A$ is $g^*$-closed. Also, we have $A \leq \text{cl}(A)$, which implies $\text{cl}(A) = A$. Hence $A$ is closed fuzzy set.

**3.2.9 Theorem:** If a fuzzy set $A$ is both open and $g^*$-closed then it is both regular open and regular closed in fts $X$.

**Proof:** Since $A$ is open fuzzy set, we have $A = \text{int}A = \text{int}(\text{cl} A)$, since $A$ is closed by theorem 3.2.8. So, $A = \text{int}(\text{cl} A)$. Hence $A$ is regular open.

Again $\text{int}A = A$ then $\text{cl}(\text{int} A) = \text{cl}(A) = A$, since $A$ is closed. That is $\text{cl}(\text{int} A) = A$. Hence $A$ is regular - closed fuzzy set.

**3.2.10 Theorem:** In a fts $X$, if a fuzzy set $A$ is both open and $g$-closed then $A$ is $g^*$-closed fuzzy set.

**Proof:** Let $A$ be open and $g$-closed fuzzy set in $X$. Now $A \leq A$ where $A$ is open and so is $g$-open. Since $A$ is $g$-closed, $\text{cl}(A) \leq A$. Also $A \leq \text{cl}(A)$. Therefore $\text{cl}(A) = A$. So $A$ is closed fuzzy set and hence $A$ is $g^*$-closed.

**3.2.11 Theorem:** In a fts $X$, if a fuzzy set $A$ is both open and $b$-closed then $A$ is $g^*$-closed.

**Proof:** We have $A \leq A$ where $A$ is open. Since $A$ is $b$-closed, we have $\text{bd} (A) \leq A$. We know that a fuzzy set $A$ is closed iff $\text{bd}(A) \leq A$, from [57]. Hence $A$ is closed fuzzy set and therefore $A$ is $g^*$-closed in $X$.

**3.2.12 Theorem:** If $A$ is a $g^*$-closed fuzzy set and $\text{cl}(A) \wedge (1 - \text{cl}A) = 0$ then there is no non-zero $g$-closed fuzzy set $F$ such that $F \leq \text{cl} (A) \wedge (1 - A)$.

**Proof:** Suppose $F$ is any $g$-closed fuzzy set of $X$ such that $F \leq \text{cl} (A) \wedge (1 - A)$. Now $F \leq 1 - A$ implies that $A \leq 1 - F$, $1 - F$ is $g$-open. Since $A$ is $g^*$-closed, $\text{cl}(A) \leq 1 - F$ which implies $F \leq 1 - \text{cl} (A)$. Thus $F \leq \text{cl}(A)$ and $F \leq 1 - \text{cl}(A)$. Therefore $F \leq \text{cl} (A) \wedge (1 - \text{cl} (A)) = 0$ which implies $F = 0$. Hence the result.
3.2.13 **Theorem:** If a fuzzy set $A$ is $g^*$-closed in $X$ such that $A \leq B \leq \text{cl}(A)$, then $B$ is also a $g^*$-closed fuzzy set of $X$.

**Proof:** Let $U$ be $g$-open fuzzy set such that $B \leq U$, then $A \leq U$. Since $A$ is $g^*$-closed, $\text{cl}(A) \leq U$. Now $\text{cl}(B) \leq \text{cl}(\text{cl}(A)) = \text{cl}(A) \leq U$. That is $\text{cl}(B) \leq U$. Hence $B$ is a $g^*$-closed fuzzy set.

3.2.14 **Theorem:** A finite union of $g^*$-closed fuzzy sets is a $g^*$-closed fuzzy set.

**Proof:** Let $A$ and $B$ be $g^*$-closed fuzzy sets in a fts $X$. To prove that $A \vee B$ is $g^*$-closed. Let $A \vee B \leq U$ where $U$ is $g$-open fuzzy set. Then $A \leq U$ and $B \leq U$ and so $\text{cl}(A) \leq U$ and $\text{cl}(B) \leq U$ since $A$ and $B$ are $g^*$-closed fuzzy sets. This implies that $\text{cl}(A) \vee \text{cl}(B) \leq U$, and so $\text{cl}(A \vee B) \leq U$. Hence $A \vee B$ is $g^*$-closed.

Thus a finite union of $g^*$-closed fuzzy sets is $g^*$-closed.

The following is due to S.R. Malghan and S.S. Benchalli [31].

3.2.15 **Theorem** [31]: $X$ is a normal fts iff for any two closed fuzzy sets $a$ and $b$ in $X$ such that $a \leq 1 - b$ there exist open fuzzy sets $c$, $d$ such that $a \leq c$, $b \leq d$ and $c \leq 1 - d$.

3.2.16 **Theorem:** If $(X,T)$ is normal fts and $F$ is closed and $A$ is $g^*$-closed such that $A \leq 1 - F$ then there exists open fuzzy sets $U$ and $V$ such that $F \leq U$, $A \leq V$ and $U \leq 1 - V$.

**Proof:** Given $A \leq 1 - F$, $1 - F$ is open and so $g$-open fuzzy set. Since $A$ is $g^*$-closed, $\text{cl}(A) \leq 1 - F$. Since $(X, T)$ is normal from 3.2.15, there exists two open fuzzy sets $U$ and $V$ such that $F \leq U$, $\text{cl}(A) \leq V$ and $U \leq 1 - V$ we have $F \leq U$, $A \leq V$ [$A \leq \text{cl}(A)$].

3.2.17 **Theorem:** In a fts $(X, T)$, $T = \text{the family of all closed fuzzy sets and } X$ is fuzzy-$T_{1/2}$ iff every fuzzy subset of $X$ is a $g^*$-closed fuzzy set.

**Proof:** Suppose that every fuzzy set of $X$ is $g^*$-closed fuzzy set. Let $A \in T$. Then, since $A \leq A$, $A$ is open and so is $g$-open and $A$ is
g*-closed, we have cl(A) ≤ A. But A ≤ cl(A). Therefore cl(A) = A. That is A is a closed fuzzy set.

Also if B is closed, then 1 − B ∈ T and therefore closed by hypothesis, and hence B is an open fuzzy set.

Conversely, let T = family of all closed fuzzy sets and X be fuzzy-$T_{1/2}$ space. Let A be any fuzzy subset of X. Let A ≤ O where O is g - open. Then O is open in X since X is fuzzy - $T_{1/2}$. And so O ∈ T and therefore is closed. That is A is closed. Therefore cl(A) ≤ cl(O) = O. Hence A is g* - closed in X.

We introduce g*-open fuzzy sets.

3.2.18 Definition: A fuzzy set A of a fts X is called g* - open if its complement 1 − A is g* - closed.

We have the following characterization.

3.2.19 Theorem: A fuzzy set A of a fts X is g* - open iff F < int(A) whenever F is g - closed and F ≤ A.

Proof: Suppose A is g*- open fuzzy set. Then 1 − A is g* - closed. Let F be g - closed fuzzy set in X and F ≤ A. Then 1 − A ≤ 1 − F, 1 − F is g - open. Since 1 − A is g* - closed, we have cl(1 − A) ≤ 1 − F. Which implies F ≤ int A as cl(1 − A) = 1 − intA.

Conversely, assume that F ≤ int (A), whenever F ≤ A and F is g - closed fuzzy set in a fts X. Let 1 − A ≤ G where G is g-open fuzzy set in X. Then 1 − G ≤ A, where 1 − G is g-closed which implies that 1 − G ≤ int(A) implies that 1 − int(A) ≤ 1 − (1 − G). That is cl(1 − A ) ≤ G. Hence 1 − A is g*- closed and so A is g*- open fuzzy set.

3.2.20 Theorem: Every open fuzzy set is g*- open.

Proof: Let A be a open fuzzy set in a fts X. Then 1 − A is closed. And so 1 − A is g* - closed. Hence A is g* - open in X.
The converse of the above theorem need not be true as seen from the following example.

**3.2.21 Example:** Let \( X = \{ a, b, c \} \). Define the fuzzy sets \( A \) and \( B \) as follows: \( A = \{(a, .4), (b, .5), (c, .7)\} \), \( B = \{(a, 0), (b, .1), (c, .2)\} \). Then \((X, T)\) is a fts with the fuzzy topology \( T = \{ 0, 1, A \} \). Here the fuzzy set \( B \) is \( g^* \) - open but not open fuzzy set in \( X \).

**3.2.22 Theorem:** In a fts \( X \), every \( g^* \) - open fuzzy set is \( g \) - open.

**Proof:** Let \( A \) be \( g^* \) - open in a fts \( X \). Then \( 1 - A \) is \( g^* \) - closed in \( X \). And so \( 1 - A \) is \( g \)-closed. That is \( A \) is \( g \)-open in \( X \).

The converse of the above theorem need not be true as seen from the following example.

**3.2.23 Example:** In the example 3.2.5, the fuzzy subset \( 1 - B = \{(a, .5), (b, .8), (c, .7)\} \) is \( g \)-open but not \( g^* \)-open in \( X \).

**3.2.24 Theorem:** Every \( g^* \) - open fuzzy set is \( b \)-open fuzzy set.

**Proof:** Let \( A \) be \( g^* \)-open fuzzy set in a fts \( X \). Then \( 1 - A \) is \( g^* \)-closed. And therefore \( 1 - A \) is \( b \)-closed. Hence \( A \) is \( b \)-open fuzzy set.

The converse of the above theorem need not be true as seen from the following example.

**3.2.25 Example:** In the example 3.2.5 the fuzzy set \( 1 - B \) is \( b \)-open but not \( g^* \)-open in \( X \).

**3.2.26 Theorem:** If \( \text{int}A \leq B \leq A \) and if \( A \) is \( g^* \) - open then \( B \) is \( g^* \) - open in a fts \( X \).

**Proof:** We have \( \text{int}(A) \leq B \leq A \). Then \( (1 - A) \leq (1 - B) \leq \text{cl} (1 - A) \) and since \( 1 - A \) is \( g^* \) - closed, by the theorem 3.2.13, \( 1 - B \) is \( g^* \) - closed in \( X \). And hence \( B \) is \( g^* \) - open.

**3.2.27 Theorem:** If \( A \leq B \leq X \) where \( A \) is \( g^* \) - open relative to \( B \) and \( B \) is \( g^* \) - open relative to \( X \), then \( A \) is \( g^* \) - open relative to fts \( X \).
**Proof:** Let $F$ be $g$-closed fuzzy set and $F \leq A$. Then $F$ is $g$-closed relative to $B$ and hence $F \leq \text{int}_{I_p}(A)$ by the theorem 3.2.19. Hence $F \leq \text{int}(A) \land B$, which implies that $F \leq \text{int}A$. Again by theorem 3.2.19. $A$ is $g^*$-open in $X$.

**3.2.28 Theorem:** Finite intersection of $g^*$-open fuzzy sets is a $g^*$-open fuzzy set.

**Proof:** Let $A$ and $B$ be $g^*$-open fuzzy sets in a fts $X$. To prove that $A \land B$ is $g^*$-open. Let $F \leq A \land B$ where $F$ be $g$-closed. Then $F \leq A$, $F \leq B$. Then $F \leq \text{int}(A)$, $F \leq \text{int}B$ as $A$ and $B$ are $g^*$-open. Then $F \leq \text{int}(A) \land \text{int}(B) = \text{int}(A \land B)$. That is $F \leq \text{int}(A \land B)$. Hence $A \land B$ is $g^*$-open.

Thus finite intersection of $g^*$-open fuzzy sets is $g^*$-open.

**3.2.29 Theorem:** A fuzzy set $A$ is $g^*$-closed and $\text{cl}(A) \land (1 - \text{cl} A) = 0$ then $\text{cl}(A) \land (1 - A)$ is $g^*$-open in $X$.

**Proof:** Let $A$ be $g^*$-closed fuzzy set in a fts $X$. Let $F \leq \text{cl}(A) \land (1 - A)$, $F$ is $g$-closed in $X$. By the theorem 3.2.12, $F$ is zero and so $F \leq \text{int} (\text{cl}(A) \land (1 - A))$. By the theorem 3.2.19, $\text{cl}(A) \land (1 - A)$ is $g^*$-open in fts $X$.

Fuzzy $g^*$-closure ($g^* \text{cl}$) and fuzzy $g^*$-interior ($g^* \text{ - int}$) of a fuzzy set are defined as follows.

**3.2.30 Definition:** For any fuzzy set $A$ in any fts, $g^* \text{cl}(A) = \land \{ U : U \text{ is } g^* \text{- closed fuzzy set and } A \leq U \}$

$g^* \text{ int}(A) = \lor \{ V : V \text{ is } g^* \text{- open and } A \geq V \}$

**3.2.31 Theorem:** Let $A$ be any fuzzy set in a fts $(X, T)$. Then $g^* \text{cl}(A') = g^* \text{ cl}(1 - A) = 1 - g^* \text{ int}(A)$

$g^* \text{ int}(1 - A) = 1 - g^* \text{ cl}(A)$

**Proof:** We see that a $g^*$-open fuzzy set $U \leq A$ is precisely the complement of a $g^*$-closed set $V \geq 1 - A$, thus $g^* \text{ int}(A) = \lor \{ 1 - V : V \text{ is } g^* \text{- closed fuzzy set and } V \geq 1 - A \}$
\[ g^* \text{ int}(A) = 1 - g^* \text{ cl}(1 - A) \] 
That is \( g^* \text{ cl}(1 - A) = 1 - g^* \text{ int}(A) \)

Let \( g \) be any \( g^* \)-open fuzzy set. Then for any \( g^* \)-closed fuzzy set \( f \geq A, \ g = 1 - f \leq 1 - A \).

Now \( g^* \text{ cl}(A) = \bigwedge \{ 1 - g : g \text{ is } g^* \text{-open fuzzy set and } g \leq 1 - A \} \)
\[ = 1 - \bigvee \{ g : g \text{ is } g^*- \text{open and } g \leq 1 - A \} \]
\[ = 1 - g^* \text{ int} (1 - A) \]
Therefore \( g^* \text{ int}(1 - A) = 1 - g^* \text{ cl}(A) \).

3.2.32 **Theorem:** In a fts \((X, T)\) a fuzzy set \( A \) is \( g^* \)-closed iff \( A = g^* \text{ cl}(A) \).

**Proof:** Let \( A \) be a \( g^* \)-closed fuzzy set in fts \((X, T)\). Since \( A \leq A \) and \( A \) is \( g^* \)-closed, \( A \in \{ f : f \text{ is a } g^* \text{-closed fuzzy set and } A \leq f \} \) and \( A \leq f \) implies that \( A = \bigwedge \{ f : f \text{ is } g^*- \text{closed and } A \leq f \} \) that is \( A = g^* \text{ cl}(A) \).

Conversely, suppose that \( A = g^* \text{ cl}(A) \), that is \( A = \bigwedge \{ f : f \text{ is a } g^*- \text{closed fuzzy set and } A \leq f \} \). This implies that \( A \in \{ f : f \text{ is a } g^*- \text{closed fuzzy set and } A \leq f \} \). Hence \( A \) is \( g^* \)-closed fuzzy set.

3.2.33 **Theorem:** In a fts \( X \) the following results hold for \( g^* \)-closure.

1) \( g^* \text{ cl}(0) = 0 \).
2) \( g^* \text{ cl}(A) \) is \( g^* \)-closed fuzzy set in \( X \).
3) \( g^* \text{ cl}(A) \leq g^* \text{ cl}(B) \) if \( A \leq B \).
4) \( g^* \text{ cl}(g^* \text{ cl}(A)) = g^* \text{ cl}(A) \).
5) \( g^* \text{ cl}(A \lor B) \geq g^* \text{ cl}(A) \lor g^* \text{ cl}(B) \).
6) \( g^* \text{ cl}(A \land B) \leq g^* \text{ cl}(A) \land g^* \text{ cl}(B) \).

**Proof:** The easy verification is omitted.

3.2.34 **Theorem:** In a fts \( X \), a fuzzy set \( A \) is \( g^* \)-open iff \( A = g^* \text{ int}(A) \).
Proof: Let $A$ be $g^*$-open fuzzy set in $X$. Since $A \leq A$ and $A$ is $g^*$-open and $A \in \{ f : f \text{ is a } g^*\text{-open fuzzy set and } A \geq f \}$ and $A \geq f$ implies that $A = \bigvee \{ f : f \text{ is } g^*\text{-open and } A \geq f \}$. That is $A = g^* \text{ int} (A)$.

Conversely, suppose that $A = g^* \text{ int}(A)$, that is $A = \bigvee \{ f : f \text{ is } g^*\text{-open and } A \geq f \}$. This implies that $A \in \{ f : f \text{ is } g^*\text{-open and } A \geq f \}$. Hence $A$ is $g^*$-open fuzzy set.

3.2.35 Theorem: In a fts $X$, the following hold for $g^*$-interior.
1) $g^* \text{ int} (0) = 0$
2) $g^* \text{ int} (A) \leq g^* \text{ int} (B)$ if $A \leq B$.
3) $g^* \text{ int} (A)$ is $g^*$-open in $X$.
4) $g^* \text{ int} (g^* \text{ int} (A)) = g^* \text{ int}(A)$.
5) $g^* \text{ int} (A \vee B) \geq g^* \text{ int}(A) \vee g^* \text{ int}(B)$.
6) $g^* \text{ int} (A \wedge B) \leq g^* \text{ int}(A) \wedge g^* \text{ int}(B)$.
Proof: The routine proof is omitted.

3.2.36 Definition: A fts $X$ is called a fuzzy - $T^*_{1/2}$ if every $g^*$-closed fuzzy set is a closed fuzzy set.

3.2.37 Theorem: A fts $X$ is fuzzy - $T^*_{1/2}$ iff every $g^*$-open fuzzy set is open in $X$.
Proof: Suppose $X$ is fuzzy - $T^*_{1/2}$. Let $V$ be $g^*$-open fuzzy set in $X$. Then $1 - V$ is $g^*$-closed. Since $X$ is fuzzy - $T^*_{1/2}$, $1 - V$ is closed in $X$. Therefore $V$ is open in $X$.

Conversely, assume that every $g^*$-open fuzzy set in $X$ is open in $X$. Let $F$ be $g^*$-closed in $X$, then $1 - F$ is $g^*$-open in $X$. By hypothesis, $1 - F$ is open in $X$. Therefore $F$ is closed in $X$. Hence $X$ is fuzzy - $T^*_{1/2}$.

3.2.38 Theorem: Every fuzzy - $T_{1/2}$ space is fuzzy - $T^*_{1/2}$.
Proof: Let $X$ be a fuzzy - $T_{1/2}$ space. Let $F$ be $g^*$-closed fuzzy set in $X$. Then $F$ is $g^*$-closed in $X$. Since $X$ is fuzzy - $T_{1/2}$, $F$ is closed in $X$. Hence $X$ is fuzzy - $T^*_{1/2}$.
The converse of the above theorem need not be true as seen from the following example.

**3.2.39 Example:** Let $X = \{a, b, c\}$. The fuzzy sets $A$, $B$ and $C$ defined as follows: $A = \{(a, 1), (b, 0), (c, 0)\}$, $B = \{(a, 0), (b, 1), (c, 1)\}$ and $C = \{(a, 0), (b, 1), (c, 0)\}$. Then $(X, T)$ is a fts with $T = \{0, 1, A\}$. Then $(X, T)$ is fuzzy-$T^{*}_{1/2}$ as $g^{*}$-closed fuzzy set $B$ is closed in $X$. But $(X, T)$ is not fuzzy-$T_{1/2}$ since $g$-closed fuzzy set $C$ is not closed in $X$.

**3.2.40 Definition:** A fts $X$ is called fuzzy-$T_{1/2}$ if every $g$-closed fuzzy set of $X$ is a $g^{*}$-closed fuzzy set.

**3.2.41 Theorem:** Every fuzzy-$T_{1/2}$ space is fuzzy-$T^{*}_{1/2}$ space.

**Proof:** Let $X$ be a fuzzy-$T_{1/2}$ space. Let $A$ be $g$-closed fuzzy set of $X$. Since $X$ is a fuzzy-$T_{1/2}$ space, the set $A$ is closed and so $A$ is $g^{*}$-closed in $X$. Hence $X$ is fuzzy-$T_{1/2}$ space.

The converse of the above theorem need not be true as shown in the following example.

**3.2.42 Example:** Let $X = \{a, b, c\}$. Fuzzy sets $A$, $B$ and $C$ be defined as follows: $A = \{(a, 1), (b, 0), (c, 0)\}$, $B = \{(a, 0), (b, 1), (c, 1)\}$ and $C = \{(a, 0), (b, 1), (c, 0)\}$. Let $(X, T)$ be fts with $T = \{0, 1, A, B\}$. Then $X$ is fuzzy-$T^{*}_{1/2}$ fts but not fuzzy-$T_{1/2}$ as the fuzzy set $C$ is $g$-closed and is $g^{*}$-closed but not closed.

**3.2.43 Theorem:** A fts $X$ is fuzzy-$T_{1/2}$ iff it is fuzzy-$T^{*}_{1/2}$ and fuzzy-$T^{*}_{1/2}$.

**Proof:** Suppose $X$ is fuzzy-$T_{1/2}$. By the theorem 3.2.38 and 3.2.41, it follows that $X$ is fuzzy-$T^{*}_{1/2}$ and fuzzy-$T^{*}_{1/2}$.

Conversely, suppose $X$ is both fuzzy-$T^{*}_{1/2}$ and fuzzy-$T^{*}_{1/2}$. Let $A$ be $g$-closed in $X$. Since $X$ is fuzzy-$T^{*}_{1/2}$, $A$ is $g^{*}$-closed. Also, since $X$ is fuzzy-$T^{*}_{1/2}$, $A$ is closed fuzzy set. Thus $X$ is fuzzy-$T_{1/2}$ fts.

**3.2.44 Theorem:** A fts $X$ is fuzzy-$T_{1/2}$ iff every $g$-open fuzzy set in $X$ is $g^{*}$-open in $X$.
Proof: Assume that $X$ is fuzzy *$T_{1/2}$. Let $v$ be $g$-open fuzzy set in $X$. Then $1 - v$ is $g^*$-closed in $X$ since $X$ is fuzzy *$T_{1/2}$. Therefore $V$ is $g^*$-open in $X$.

Conversely, assume that every $g$-open fuzzy set in $X$ is $g^*$-open in $X$. Let $F$ be $g$-closed in $X$. Then $1 - F$ is $g$-open in $X$. By hypothesis, $1 - F$ open in $X$. Therefore $F$ is closed fuzzy set in $X$.

3.2.45 Remark: The two concepts fuzzy *$T_{1/2}$ and fuzzy-$T_{1/2}$ are independent of each other as seen from the next two examples.

3.2.46 Example: Let $X = \{a, b, c\}$. Define the fuzzy sets $A$, $B$ and $C$ as follows: $A = \{(a, 1), (b, 0), (c, 0)\}$, $B = \{(a, 0), (b, 1), (c, 1)\}$ and $C = \{(a, 0), (b, 1), (c, 0)\}$. Let $T = \{0, 1, A, B\}$. Then $T$ is fuzzy topology on $X$. Then $X$ is not fuzzy *$T_{1/2}$ as the fuzzy set $C$ is $g$-closed but not $g^*$-closed. And $X$ is fuzzy *$T^*_{1/2}$.

3.2.47 Example: In the example 3.2.42, $X$ is fuzzy-$T_{1/2}$ but not fuzzy-$T^*_{1/2}$ as the set $C$ is $g^*$-closed in $X$ but not closed fuzzy set in $X$.

3.3 $g^*$-CONTINUOUS AND OTHER RELATED MAPS IN FTS

In this section the concepts of fuzzy $g^*$-continuous, $g^*$-irresolute functions and $g^*$-homeomorphism have been introduced and studied. We also introduce the concepts of $g^*$-open and $g^*$-closed mappings in fuzzy topological spaces.

3.3.1 Definition: Let $X$ and $Y$ be two fts. A function $f : X \to Y$ is said to be fuzzy $g^*$-continuous (f $g^*$-continuous) if the inverse image of every open fuzzy set in $Y$ is $g^*$-open in $X$.

3.3.2 Theorem: A function $f : X \to Y$ is $f$ $g^*$-continuous iff the inverse image of every closed fuzzy set in $Y$ is $g^*$-closed in $X$.

Proof: Suppose the function $f : X \to Y$ is $fg^*$-continuous. Let $F$ be closed fuzzy set in $Y$. Then $1 - F$ is open fuzzy set in $Y$. Since $f$ is
fg* - continuous, \( f^{-1}(1 - F) \) is \( g^* \)-open in \( X \). But \( f^{-1}(1 - F) = 1 - f^{-1}(F) \) and so \( f^{-1}(F) \) is \( g^* \)-closed in \( X \).

Conversely, assume that the inverse image of every closed fuzzy set in \( Y \) is \( g^* \)-closed in \( X \). Let \( V \) be open fuzzy set in \( Y \). Then \( 1 - V \) is closed in \( Y \). By hypothesis, \( f^{-1}(1 - V) \) is \( g^* \)-closed fuzzy set in \( X \). But \( f^{-1}(1 - V) = 1 - f^{-1}(V) \) and so \( f^{-1}(V) \) is \( g^* \)-open fuzzy set in \( X \). Hence \( f \) is \( fg^* \)-continuous.

3.3.3 **Theorem:** Every fuzzy-continuous function is \( fg^* \)-continuous.

**Proof:** Let \( f : X \rightarrow Y \) be \( f \)-continuous. Let \( F \) be closed fuzzy set in \( Y \). Then \( f^{-1}(F) \) is closed fuzzy set in \( X \) since \( f \) is fuzzy continuous. And therefore \( f^{-1}(F) \) is \( g^* \)-closed in \( X \). Hence \( f \) is \( fg^* \)-continuous.

The converse of the above theorem need not be true as seen from the following example.

3.3.4 **Example:** Let \( X = Y = \{a, b, c\} \). Fuzzy sets \( A, B \) and \( C \) be defined as follows: \( A = \{(a, 0), (b, .1), (c, .2)\} \), \( B = \{(a, .4), (b, .5), (c, .7)\} \) and \( C = \{(a, 1), (b, .9), (c, .8)\} \). Consider \( T = \{0, 1, B\} \), \( \sigma = \{0, 1, A\} \). Then \( (X, T) \) and \( (Y, \sigma) \) are fts. Define \( f : X \rightarrow Y \) by \( f(a) = a \), \( f(b) = b \) and \( f(c) = c \). Then \( f \) is \( fg^* \)-continuous but not \( f \)-continuous as the fuzzy set \( C \) is closed in \( Y \) and \( f^{-1}(C) = C \) is not closed in \( X \) but \( g^* \)-closed in \( X \).

3.3.5 **Theorem:** Every \( fg^* \)-continuous function is \( fg \)-continuous.

**Proof:** Let \( f : X \rightarrow Y \) be \( fg^* \)-continuous. Let \( F \) be a closed fuzzy set in \( Y \). Since \( f \) is \( fg^* \)-continuous, \( f^{-1}(F) \) is \( g^* \)-closed in \( X \). And therefore \( f^{-1}(F) \) is \( g \)-closed in \( X \) as every \( g^* \)-closed fuzzy set is \( g \)-closed. Hence \( f \) is \( fg \)-continuous.

The converse of the above theorem need not be true as seen from the following example.

3.3.6 **Example:** Let \( X = \{a, b, c\} \). Fuzzy sets \( A, B \), and \( C \) be defined as follows: \( A = \{(a, 1), (b, 0), (c, 0)\} \), \( B = \{(a, 0), (b, 1), (c, 1)\} \), and
C = {(a, 0), (b, 1), (c, 0)}. Consider T = {0, 1, A, B} then (X, T) is fts. Define \( f : X \to Y \) by \( f(a) = b, f(b) = a \) and \( f(c) = c \). Then \( f \) is \( fg \)-continuous but not \( fg^* \)-continuous as the inverse image of closed fuzzy set \( A \) is \( f^{-1}(A) = C \) which is not \( g^* \)-closed.

**3.3.7 Theorem:** Every \( fg^* \)-continuous function is \( fb \)-continuous function.

**Proof:** Let \( f : X \to Y \) be \( fg^* \)-continuous function. Let \( F \) be closed fuzzy set in \( Y \). Then \( f^{-1}(F) \) is \( g^* \)-closed in \( X \). And so \( f^{-1}(F) \) is \( b \)-closed in \( X \) as every \( g^* \)-closed fuzzy set is \( b \)-closed. Hence \( f : X \to Y \) is \( fb \)-continuous.

The converse of the above theorem need not be true as seen from the following example.

**3.3.8 Example:** Let \( X = Y = \{a, b, c\} \). Fuzzy sets \( A, B, C \) and \( D \) be defined as follows: \( A = \{(a, .2), (b, .5), (c, .3)\} \), \( B = \{(a, .8), (b, .5), (c, .7)\} \), \( C = \{(a, .5), (b, .2), (c, .3)\} \) and \( D = \{(a, .5), (b, .8), (c, .7)\} \). Consider \( T = \{0, 1, A\}, \sigma = \{0, 1, A, B\} \). Then (\( X, T \)) and (\( Y, \sigma \)) are fts. Define \( f : X \to Y \) by \( f(a) = b, f(b) = a, f(c) = c \). Then \( f \) is \( fb \)-continuous but not \( fg^* \)-continuous as \( A \) is closed in \( Y \), but \( f^{-1}(A) = C \) is not \( g^* \)-closed in \( X \).

**3.3.9 Theorem:** If \( f : X \to Y \) is \( fg^* \)-continuous and \( X \) is fuzzy -\( T'_{1/2} \) fts. Then \( f \) is fuzzy -continuous.

**Proof:** Let \( f : X \to Y \) be \( fg^* \)-continuous. Let \( F \) be closed fuzzy set in \( Y \). Then \( f^{-1}(F) \) is \( g^* \)-closed in \( X \) since \( f \) is \( fg^* \)-continuous. Also since \( X \) is fuzzy -\( T'_{1/2} \), \( f^{-1}(F) \) is closed in \( X \). Hence \( f \) is \( f \)-continuous.

**3.3.10 Theorem:** If \( f : X \to Y \) is \( fg \)-continuous and \( X \) is fuzzy -\( T_{1/2} \) fts. Then \( f \) is \( fg^* \)-continuous.

**Proof:** Let \( f : X \to Y \) be \( fg \)-continuous. Let \( F \) be closed fuzzy set in \( Y \), then \( f^{-1}(F) \) is \( g \)-closed in \( X \). Since \( X \) is fuzzy -\( T_{1/2} \), \( f^{-1}(F) \) is \( g^* \)-closed in \( X \). Hence \( f \) is \( fg^* \)-continuous.
3.3.11 **Theorem:** If a function $f : X \rightarrow Y$ is $fb$ - continuous and $X$ is a $fb$ - space, then $f$ is $fg^*$ - continuous function.

**Proof:** Let $F$ be closed fuzzy set in $Y$. Then $f^{-1}(F)$ is $b$-closed in $X$, since $f$ is $fb$-continuous. And so $f^{-1}(F)$ is closed fuzzy set in $X$ as $X$ is $fb$ - space. Therefore $f^{-1}(F)$ is $g^*$- closed in $X$. Hence $f$ is $fg^*$- continuous.

3.3.12 **Theorem:** Every strongly $f$-continuous function is $fg^*$- continuous.

**Proof:** Let $f : X \rightarrow Y$ be a strongly fuzzy continuous. Let $F$ be closed fuzzy set in $Y$. Then $f^{-1}(F)$ is open and closed in $X$. And therefore $f^{-1}(F)$ is $g^*$- closed in $X$. Hence $f$ is $fg^*$ - continuous.

The converse of the above theorem need not be true as seen from the following example.

3.3.13 **Example:** In the example 3.3.4, the function $f$ is $fg^*$ - continuous but not strongly $f$- continuous, for the fuzzy set $C$ in $Y$, $f^{-1}(C) = C$ is not both open and closed in $X$.

3.3.14 **Theorem:** Every perfectly $f$-continuous function is $fg^*$-continuous.

**Proof:** Let $f : X \rightarrow Y$ be a perfectly $f$ - continuous. Let $V$ be open fuzzy set in $Y$. Then $f^{-1}(V)$ is both open and closed fuzzy set in $X$. And therefore $f^{-1}(V)$ is $g^*$ - open in $X$. Hence $f$ is $fg^*$ - continuous.

The converse of the above theorem need not be true as seen from the following example.

3.3.15 **Example:** In the example 3.3.4, the function $f$ is $fg^*$- continuous but not perfectly $f$ - continuous as the fuzzy set $A$ is open in $Y$ and $f^{-1}(A) = A$ is not both open and closed in $X$.

3.3.16 **Theorem:** Every completely $f$ - continuous function is $fg^*$- continuous.
Proof: Let $f: X \to Y$ be completely $f$-continuous. Let $V$ be open fuzzy set in $Y$. Then $f^{-1}(V)$ is regular-open fuzzy set in $X$. And therefore $f^{-1}(V)$ is open and so $g^*-\text{open}$ in $X$. Hence $f$ is $fg^*$-continuous.

The converse of the above theorem need not be true as shown from the following example.

3.3.17 Example: In the example 3.3.4, the function $f$ is $fg^*$-continuous but not completely $f$-continuous as the fuzzy set $A$ is open in $Y$ and $f^{-1}(A) = A$ is not regular-open in $X$.

3.3.18 Theorem: If $f : X \to Y$ is $fg^*$-continuous and $g : Y \to Z$ is $f$-continuous then $gof : X \to Z$ is $fg^*$-continuous.

Proof: Let $F$ be closed fuzzy set in $Z$. Then $g^{-1}(F)$ is closed in $Y$ since $g$ is $f$-continuous. And then $f^{-1}(g^{-1}(F))$ is $g^*$-closed in $X$ since $f$ is $fg^*$-continuous. Now $(gof)^{-1}(F) = f^{-1}(g^{-1}(F))$ is $g^*$-closed in $X$. Hence $gof : X \to Z$ is $fg^*$-continuous.

3.3.19 Theorem: If $f : X \to Y$ is $fg^*$-continuous and $g : Y \to Z$ is $fg^*$-continuous and $Y$ is fuzzy $T_{1/2}^*$ space. Then $gof : X \to Z$ is $fg^*$-continuous.

Proof: Let $F$ be closed fuzzy set in $Z$. Then $g^{-1}(F)$ is $g^*$-closed in $Y$ since $g$ is $fg^*$-continuous. Since $Y$ is fuzzy $T_{1/2}^*$, $g^{-1}(F)$ is closed in $Y$. And then $f^{-1}(g^{-1}(F))$ is $g^*$-closed in $X$ as $f$ is $fg^*$-continuous. Now $(gof)^{-1}(F) = f^{-1}(g^{-1}(F))$ is $g^*$-closed in $X$. Hence $gof$ is $fg^*$-continuous.

3.3.20 Theorem: If $f$ is $f$-continuous, $f$-closed mapping of $X$ onto $Y$ and $g$ is a mapping from $Y$ to $Z$ and $X$, $Y$ are fuzzy $T_{1/2}^*$ spaces then $gof$ is $fg^*$-continuous iff $g$ is $fg^*$-continuous.

Proof: Suppose $gof$ is $fg^*$-continuous. Let $F$ be a closed fuzzy set of $Z$. Then $(gof)^{-1}(F) = f^{-1}(g^{-1}(F))$ is $g^*$-closed in $X$. Since $X$ is fuzzy $T_{1/2}^*$,
f^{-1}(g^{-1}(F)) \text{ is closed. Also since } f \text{ is closed map, } f(f^{-1}(g^{-1}(F)) = g^{-1}(F) \text{ is closed in } Y \text{ and so } g^{-1}(F) \text{ is } g^*\text{-closed in } Y. \text{ Hence } g \text{ is } fg^*\text{-continuous.}

Conversely, suppose } g \text{ is } fg^*\text{-continuous. Let } F \text{ be closed fuzzy set of } Z. \text{ Then } g^{-1}(F) \text{ is } g^*\text{-closed in } Y. \text{ Since } Y \text{ is fuzzy } T^{*\frac{1}{2}}, g^{-1}(F) \text{ is closed in } Y. \text{ Also since } f \text{ is } f\text{-continuous, } f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F) \text{ is closed and so } g^*\text{-closed in } X. \text{ Hence } gof \text{ is } fg^*\text{-continuous.}

3.3.21 Theorem: Let } X, X_1 \text{ and } X_2 \text{ be fts and } p_i : X_i \times X_2 \to X_i (i = 1, 2) \text{ be the projection mappings. If } f : X \to X_1 \times X_2 \text{ is } fg^*\text{-continuous then the composition } p_i \circ f \text{ is } fg^*\text{-continuous.}

Proof: Let } V \text{ be open fuzzy set in } X_i (i = 1, 2), \text{ then } p_i^{-1}(V) (i = 1, 2) \text{ is open in } X_1 \times X_2 \text{ as the projection mapping } p_i \text{ is } f\text{-continuous [60]. Since } f \text{ is } fg^*\text{-continuous, } f^{-1}(p_i^{-1}(V)) = (p_i \circ f)^{-1}(V) (i = 1, 2) \text{ is } g^*\text{-open in } X. \text{ Hence } p_i \circ f \text{ is } fg^*\text{-continuous.}

3.3.22 Definition: A function } f : X \to Y \text{ is called } fg^*\text{-Irresolute function if the inverse image of every } g^*\text{-closed fuzzy set in } Y \text{ is } g^*\text{-closed in } X.

3.3.23 Theorem: A function } f : X \to Y \text{ is } fg^*\text{-Irresolute iff the inverse image of every } g^*\text{-open fuzzy set in } Y \text{ is } g^*\text{-open in } X.

Proof: The routine proof is omitted.

3.3.24 Theorem: Every } fg^*\text{-Irresolute function is } fg^*\text{-continuous.}

Proof: Let } f : X \to Y \text{ be } fg^*\text{-Irresolute function. Let } F \text{ be closed fuzzy set in } Y. \text{ Then } F \text{ is } g^*\text{-closed in } Y. \text{ Since } f \text{ is } fg^*\text{-Irresolute, } f^{-1}(F) \text{ is } g^*\text{-closed in } X. \text{ Hence } f \text{ is } fg^*\text{-continuous function.}

The converse of the above theorem need not be true as seen from the following example.

3.3.25 Example: Let } X = Y = \{a, b, c\}. \text{ Fuzzy sets } A, B, C \text{ and } D \text{ be defined as follows: } A = \{(a, 1), (b, 0), (c, 0)\}, \ B = \{(a, 0), (b, 1), (c, 0)\}, \ C = \{(a, 1), (b, 1), (c, 0)\} \text{ and } D = \{(a, 1), (b, 0), (c, 1)\}. \text{ Consider } T = \{0, 1, A, B, C, D\} \text{ and } \sigma = \{0, 1, C\} \text{ then } (X, T) \text{ and } (Y, \sigma) \text{ are fts.}
Define \( f : X \to Y \) by \( f(a) = b \), \( f(b) = c \), and \( f(c) = a \). Then \( f \) is \( \text{fg}* \)-continuous but not \( \text{fg}* \)-irresolute as the fuzzy set \( E = \{(a, 0), (b, 1), (c, 1)\} \) is \( g* \)-closed in \( Y \) but \( f^{-1}(E) = C \) is not \( g* \)-closed in \( X \).

3.3.26 **Theorem:** Let \( X \) be a fuzzy-\( T_{1/2} \) fts. If \( f : X \to Y \) is fuzzy gc-irresolute then \( f \) is \( \text{fg}* \)-irresolute.

**Proof:** Let \( F \) be \( g* \)-closed fuzzy set in \( Y \). Then \( F \) is \( g \)-closed in \( Y \). Since \( f \) is gc - irresolute, \( f^{-1}(F) \) is \( g \)-closed in \( X \). And then \( f^{-1}(F) \) is \( g* \)-closed in \( X \), since \( X \) is fuzzy \( T_{1/2} \). Hence \( f \) is \( \text{fg}* \)-irresolute function.

3.3.27 **Theorem:** Let \( Y \) be fuzzy-\( T_{1/2} \) fts. If \( f : X \to Y \) is \( \text{fg}* \)-irresolute then \( f \) is fuzzy gc- irresolute.

**Proof:** Let \( F \) be \( g \)-closed fuzzy set in \( Y \). Then \( F \) is \( g* \)-closed in \( Y \) since \( Y \) is fuzzy-\( T_{1/2} \). Since \( f \) is \( f \text{g*} \)-irresolute, \( f^{-1}(F) \) is \( g* \)-closed in \( X \) and so \( f^{-1}(F) \) is \( g \)-closed in \( X \). Hence \( f \) is fuzzy gc - irresolute.

3.3.28 **Theorem:** Let \( X \) be \( \text{fb} \)-space. If \( f : X \to Y \) is fuzzy \( \text{b} \)-irresolute then \( f \) is \( \text{fg}* \)-irresolute.

**Proof:** Let \( F \) be \( g* \)-closed \( \text{fb} \)-space in \( Y \). Then \( F \) is \( g* \)-closed in \( Y \). Since \( f \) is \( \text{fb} \)-irresolute, \( f^{-1}(F) \) is \( \text{b} \)-closed in \( X \). And therefore \( f^{-1}(F) \) is \( g* \)-closed in \( X \). Hence \( f \) is \( \text{fg}* \)-irresolute function.

3.3.29 **Theorem:** Let \( Y \) be \( \text{fb} \)-space. If \( f : X \to Y \) is \( \text{fg}* \)-irresolute then \( f \) is \( \text{fb} \)-irresolute function.

**Proof:** Let \( F \) be \( \text{b} \)-closed fuzzy set in \( Y \). Then \( F \) is \( \text{b} \)-closed \( \text{fb} \)-space in \( Y \) since \( Y \) is \( \text{fb} \)-space. And so \( F \) is \( g*- \)closed in \( Y \). Since \( f \) is \( \text{fg}* \)-irresolute, \( f^{-1}(F) \) is \( g*- \)-closed in \( X \) and \( \text{b} \)-closed in \( X \). Hence \( f \) is \( \text{fb} \)-irresolute function.

3.3.30 **Theorem:** Let \( Y \) be fuzzy - \( T_{1/2}^* \) space. If \( f : X \to Y \) is \( \text{fg}* \)-continuous then \( f \) is \( \text{fg}* \)-irresolute function.
**Proof:** Let $F$ be $g^*$-closed in $Y$. Then $F$ is closed in $Y$ since $Y$ is fuzzy - $T^{*,1/2}$. Since $f$ is $fg^*$-continuous, $f^{-1}(F)$ is $g^*$-closed in $X$. And hence $f$ is $fg^*$-irresolute function.

3.3.31 **Theorem:** Let $f : X \rightarrow Y$, $g : Y \rightarrow Z$ be two functions. If $f$ and $g$ are $fg^*$-irresolute functions then $gof : X \rightarrow Z$ is $fg^*$-irresolute function.

**Proof:** Let $F$ be $g^*$-closed in $Z$, then $g^{-1}(F)$ is $g^*$-closed in $Y$ since $g$ is $fg^*$-irresolute. Since $f$ is $fg^*$-irresolute, $f^{-1}(g^{-1}(F))$ is $g^*$-closed in $X$. That is $(g \circ f)^{-1}(F) = f^{-1}(g^{-1}(F))$ is $g^*$-closed in $X$. Hence $gof$ is $fg^*$-irresolute function.

3.3.32 **Theorem:** Let $f : X \rightarrow Y$, $g : Y \rightarrow Z$ be two functions. If $f$ is $fg^*$-continuous and $g$ is $fg^*$-irresolute and $Y$ is fuzzy - $T^{*,1/2}$ then $gof : X \rightarrow Z$ is $fg^*$-irresolute function.

**Proof:** Let $F$ be $g^*$-closed in $Z$, then $g^{-1}(F)$ is $g^*$-closed in $Y$ since $g$ is $fg^*$-irresolute. Since $Y$ is fuzzy - $T^{*,1/2}$, $g^{-1}(F)$ is closed in $Y$ and so $f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F)$ is $g^*$-closed in $X$, since $f$ is $fg^*$-continuous. Hence $gof$ is $fg^*$-irresolute function.

3.3.33 **Theorem:** Let $f : X \rightarrow Y$, $g : Y \rightarrow Z$ be two functions. If $f$ is $fg^*$-irresolute and $g$ is $fg^*$-continuous then $gof : X \rightarrow Z$ is $fg^*$-continuous.

**Proof:** Let $F$ be closed fuzzy set in $Z$. Then $g^{-1}(F)$ is $g^*$-closed in $Y$ since $g$ is $fg^*$-continuous. Since $f$ is $fg^*$-irresolute, $f^{-1}(g^{-1}(F)) = (gof)^{-1}(F)$ is $g^*$-closed in $X$. Hence $gof$ is $fg^*$-continuous.

3.3.34 **Theorem:** Let $f : X \rightarrow Y$ be an onto, $fg^*$-irresolute and a closed map. If $X$ is a fuzzy - $T^{*,1/2}$, then $Y$ is also fuzzy - $T^{*,1/2}$.

**Proof:** Let $F$ be $g^*$-closed fuzzy set in $Y$. Then $f^{-1}(F)$ is $g^*$-closed in $X$ since $f$ is $fg^*$-irresolute. Since $X$ is fuzzy - $T^{*,1/2}$, $f^{-1}(F)$ is closed in $X$. And so $f(f^{-1}(F)) = F$ is closed in $Y$ as $f$ is a closed and onto function. Hence $Y$ is fuzzy - $T^{*,1/2}$ space.
3.3.35 **Theorem**: Let $f : X \to Y$ be a fuzzy $g_c^*$ - irresolute and a closed map. Then $f (A)$ is a $g^*$ - closed fuzzy set of $Y$ for every $g^*$ - closed fuzzy set $A$ of $X$.

**Proof**: Let $A$ be a $g^*$ - closed fuzzy set of $X$. Let $U$ be $g^*$ - open fuzzy set of $Y$, such that $f(A) \leq U$. Since $f$ is fuzzy $g_c^*$ - irresolute, $f^{-1} (U)$ is a $g^*$ - open fuzzy set of $X$. Since $A \leq f^{-1} (U)$ and $A$ is a $g^*$ - closed set of $X$, $\text{cl}(A) \leq f^{-1} (U)$. Then $f (\text{cl} (A)) \leq f(f^{-1}(U)) \leq U$. Since $f$ is a closed map, $f(\text{cl}(A)) = \text{cl}(f (\text{cl} A))$. This implies that $\text{cl} (f (A)) \leq \text{cl}(f (\text{cl} (A))) \leq f(\text{cl} A) \leq U$. That is $\text{cl}(f(A)) \leq U$. Therefore $f (A)$ is a $g^*$ - closed fuzzy set of $Y$.

We introduce the following.

3.3.36 **Definition**: A map $f : X \to Y$ is said to be fuzzy $g^*$ - open if the image of every open fuzzy set in $X$ is $g^*$ - open fuzzy set in $Y$.

3.3.37 **Definition**: A map $f : X \to Y$ is said to be fuzzy $g^*$ - closed if the image of every closed fuzzy set in $X$ is $g^*$ - closed in $Y$.

3.3.38 **Theorem**: Every fuzzy open map is fuzzy $g^*$ - open.

**Proof**: Let $f : X \to Y$ be a fuzzy open map. Let $V$ be an open fuzzy set in $X$. Then $f (V)$ is open in $Y$ since $f$ is fuzzy open map. And therefore $f (V)$ is $g^*$ - open in $Y$. Hence $f$ is fuzzy $g^*$ - open map.

The converse of the above theorem need not be true as shown from the following example.

3.3.39 **Example**: Let $X = Y = \{a, b, c\}$. Fuzzy sets $A$, $B$ and $C$ be defined as follows: $A = \{(a, 0), (b, .1), (c, .2)\}$, $B = \{(a, .4), (b, .5), (c, .7)\}$ and $C = \{(a, 1), (b, .9), (c, .8)\}$. Consider $T = \{0, 1, A\}$, $\sigma = \{0, 1, B\}$. Then $(X, T)$, $(Y, \sigma)$ are fts. Define $f : X \to Y$ by $f (a) = a$, $f (b) = b$ and $f (c) = c$. Then $f$ is $fg^*$ - open map but not open map as the fuzzy set $A$ is open in $X$, its image $f (A) = A$ is not open in $Y$ which is $g^*$ - open fuzzy set in $Y$.

3.3.40 **Theorem**: If $f : X \to Y$ is $fg^*$-open map and $Y$ is a fuzzy $\text{T}^*_1$ - open map, then $f$ is a open map.
Proof: Let \( f: X \to Y \) be a fuzzy \( g^* \)-open map. Let \( V \) be an open fuzzy set in \( X \). Then \( f(V) \) is \( g^* \)-open in \( Y \). Since \( Y \) is fuzzy-\( T^*_{1/2} \) space, \( f(V) \) is open fuzzy set in \( Y \). Hence \( f \) is open map.

3.3.41 Theorem: Every \( fg^* \)-open map is fuzzy \( g \)-open.

Proof: Let \( f: X \to Y \) be a \( fg^* \)-open map. Let \( V \) be an open fuzzy set in \( X \). Then \( f(V) \) is \( g^* \)-open in \( Y \) since \( f \) is \( fg^* \)-open map. And therefore \( f(V) \) is \( g \)-open fuzzy set in \( Y \). Hence \( f \) is fuzzy \( g \)-open map.

The converse of the above theorem need not be true as shown from the following example.

3.3.42 Example: Let \( X = Y = \{a, b, c\} \). Fuzzy sets \( A, B \) and \( C \) be defined as follows: \( A = \{(a, .2), (b, .5), (c, .3)\} \), \( B = \{(a, .8), (b, .5), (c, .7)\} \), \( C = \{(a, .5), (b, .2), (c, .3)\} \). Consider \( T = \{0, 1, A\} \), \( \sigma = \{0, 1, A, B\} \). (\( X, T \)) and (\( Y, \sigma \)) are fts. Define \( f: X \to Y \) by \( f(a) = b \), \( f(b) = a \), \( f(c) = c \).

Then the function \( f \) is fuzzy \( g \)-open map but not \( fg^* \)-open as the image of open fuzzy set \( A \) in \( X \) is \( f(A) = C \) is \( b \)-open but not \( g^* \)-open in \( Y \).

3.3.43 Theorem: If \( f: X \to Y \) is fuzzy \( g \)-open and \( Y \) is fuzzy-\( T^*_{1/2} \) space, then \( f \) is \( fg^* \)-open map.

Proof: Let \( V \) be an open fuzzy set in \( X \). Then \( f(V) \) is \( g \)-open in \( Y \). Since \( Y \) is fuzzy-\( T^*_{1/2} \), \( f(V) \) is \( g^* \)-open in \( Y \). And hence \( f \) is \( fg^* \)-open map.

3.3.44 Theorem: Every fuzzy closed map is \( fg^* \)-closed map.

Proof: Let \( f: X \to Y \) be fuzzy closed map. Let \( F \) be closed fuzzy set in \( X \). Then \( f(F) \) is closed in \( Y \). And therefore \( f(F) \) is \( g^* \)-closed in \( Y \). And hence \( f \) is \( fg^* \)-closed map.

The converse of the above theorem need not be true as shown from the following example.

3.3.45 Example: In the example 3.3.39 the function \( f \) is \( g^* \)-closed but not closed map as the fuzzy set \( C \) is closed in \( X \) and its image \( f(C) = C \) is \( g^* \)-closed in \( Y \) but not closed in \( Y \).
3.3.46 Theorem: If \( f: X \rightarrow Y \) is \( fg^* \) - closed and \( Y \) is fuzzy - \( T_{1/2}^* \). Then \( f \) is fuzzy closed map.

Proof: Let \( f: X \rightarrow Y \) be \( fg^* \) - closed map. Let \( F \) be closed fuzzy set in \( X \). Then \( f(F) \) is \( g^* \) - closed in \( Y \). Since \( Y \) is fuzzy - \( T_{1/2}^* \), \( f(F) \) is closed in \( Y \). Hence \( f \) is fuzzy closed map.

3.3.47 Theorem: A map \( f: X \rightarrow Y \) is \( fg^* \) - closed iff for each fuzzy set \( S \) of \( Y \) and for each open fuzzy set \( U \) such that \( f^{-1}(S) \leq U \) there is a \( g^* \) - open fuzzy set \( V \) of \( Y \) such that \( S \leq V \) and \( f^{-1}(V) \leq U \).

Proof: Suppose \( f \) is \( fg^* \) - closed map. Let \( S \) be a fuzzy set of \( Y \) and \( U \) be an open fuzzy set of \( X \) such that \( f^{-1}(U) \leq U \). Then \( V = Y - f(X - U) \) is a \( g^* \) - open fuzzy set in \( Y \) such that \( S \leq V \) and \( f^{-1}(V) \leq U \).

Conversely, suppose that \( F \) is a closed fuzzy set of \( X \). Then \( f^{-1}(Y - f(F)) \leq X - F \) and \( X - F \) is open. By hypothesis, there is a \( g^* \) - open fuzzy set \( V \) of \( Y \) such that \( Y - f(F) \leq V \) and \( f^{-1}(V) \leq X - F \). Therefore \( F \leq X - f^{-1}(V) \). Hence \( Y - V \leq f(F) \leq f(X - f^{-1}(V)) \leq Y - V \) which implies \( f(F) = Y - V \). Since \( Y - V \) is \( g^* \) - closed, \( f(F) \) is \( g^* \) - closed and thus \( f \) is a \( fg^* \) - closed map.

3.3.48 Theorem: If a map \( f: X \rightarrow Y \) is fuzzy gc-irresolute and \( fg^* \) - closed and \( A \) is \( g^* \) - closed fuzzy set of \( X \), then \( f(A) \) is \( g^* \) - closed in \( Y \).

Proof: Let \( f(A) \leq O \), where \( O \) is \( g \) - open in \( Y \). Since \( f \) is fuzzy gc- irresolute, \( f^{-1}(O) \) is a \( g \) - open fuzzy set such that \( A \leq f^{-1}(O) \). Hence \( \text{cl}(A) \leq f^{-1}(O) \) since \( A \) is \( g^* \) - closed fuzzy set . Since \( f \) is \( g^* \) - closed map \[ \text{cl}(A) \text{ is closed in } X \], \( f(\text{cl}(A)) \) is \( g^* \) - closed and \( f(\text{cl}(A)) \leq O \), which implies \( \text{cl}(f(\text{cl} A)) \leq O \) since \( f(\text{cl} A) \) is \( g^* \) - closed set, that is \( \text{cl}(f(A)) \leq \text{cl}(f(\text{cl} A)) \leq O \), and so \( \text{cl}(f(A)) \leq O \). Hence \( f(A) \) is \( g^* \) - closed in \( Y \).
3.3.49 **Theorem:** Let $f : X \to Y$ be $f$-continuous and fuzzy $g^*$-closed. If $A$ is $g^*$-closed fuzzy set in $X$ and $Y$ is fuzzy $T_{1/2}$ then $f(A)$ is $g^*$-closed in $Y$.

**Proof:** Let $f(A) \subseteq O$, where $O$ is $g$-open in $Y$. Then $O$ is open in $Y$ since $Y$ is fuzzy $T_{1/2}$. Since $f$ is $f$-continuous, $f^{-1}(O)$ is an open and so $g$-open fuzzy set such that $A \subseteq f^{-1}(O)$. Hence $\text{cl}(A) \subseteq f^{-1}(O)$ as $A$ is $g^*$-closed. Also since $f$ is $g^*$-closed map [$\text{cl}(A)$ is closed in $X$], $f(\text{cl} A)$ is a $g^*$-closed and $f(f(\text{cl} A)) \subseteq O$. Which implies that $\text{cl}(f(\text{cl} A)) \subseteq O$ as $f(\text{cl} (A))$ is $g^*$-closed fuzzy set. Hence $\text{cl}(f(\text{cl} A)) \subseteq O$, that is $\text{cl}(f(A)) \subseteq O$. So $f(A)$ is $g^*$-closed in $Y$.

3.3.50 **Theorem:** If $f : X \to Y$ is fuzzy closed and $h : Y \to Z$ is $fg^*$-closed then $hof : X \to Z$ is $fg^*$-closed map.

**Proof:** Let $F$ be closed fuzzy set in $X$. Then $f(F)$ is closed in $Y$. Since $h$ is $fg^*$-closed, $h(f(F))$ is $g^*$-closed in $Z$. That is $(hof)(F) = h(f(F))$ is $g^*$-closed in $Z$. Hence $hof$ is $fg^*$-closed map.

3.3.51 **Theorem:** If $f : X \to Y$ is $fg^*$-closed and $h : Y \to Z$ is $fg^*$-closed maps and $Y$ is a fuzzy $T^*_{1/2}$ then $hof : X \to Z$ is $fg^*$-closed map.

**Proof:** Let $F$ be closed fuzzy set in $X$. Then $f(F)$ is $g^*$-closed in $Y$. Since $Y$ is a fuzzy $T^*_{1/2}$ space, $f(F)$ is closed in $Y$. And so $h(f(F))$ is $g^*$-closed in $Z$ since $h$ is $fg^*$-closed. That is $(hof)(F) = h(f(F))$ is $g^*$-closed in $Z$. Hence $hof$ is $fg^*$-closed map.

3.3.52 **Theorem:** If $f$ is a $f$-continuous, $fg^*$-closed map from a fuzzy normal space $X$ onto a fts $Y$, then $Y$ is fuzzy normal.

**Proof:** Let $a, b$ be two closed fuzzy sets of $Y$ such that $a \subseteq 1 - b$. Then $f^{-1}(a)$ and $f^{-1}(b)$ are closed fuzzy sets of $X$ such that $f^{-1}(a) \subseteq 1 - f^{-1}(b)$. Since $X$ is fuzzy normal, there exists open fuzzy sets $u, v$ in $X$ such that $f^{-1}(a) \subseteq u, f^{-1}(b) \subseteq v$ and $u \leq 1 - v$. Since $f$ is $fg^*$-closed map, by theorem 3.3.47, there exists $g^*$-open fuzzy sets $g, h$ in $Y$ such that $a \leq g,
b \leq h, f^{-1}(g) \leq u and f^{-1}(h) \leq v. Since u \leq 1 - v, we have int(g), int(h) are open fuzzy sets such that g^* \leq 1 - h^*. Since g is g*-open, a is closed and so is g-closed and a \leq g implies a \leq int(g). Similarly b \leq int(h) and g^* \leq 1 - h^*. Hence Y is fuzzy normal.

The following is due to S.R. Malghan and S.S. Benchalli [31].

3.3.53 Theorem[31]: A fts (X, T) is regular iff for each x \in X and a g \in T with g(x) = 1 there exists h \in T with h(x) = 1, such that h \leq \bar{h} \leq g.

3.3.54 Theorem: Let f: X \rightarrow Y be an f-continuous, open and fg*-closed surjection. If X is regular fts then Y is regular.

Proof: Let q \in Y and p \in X such that f(p) = q. Let g be an open fuzzy set in Y such that g(q) = 1. Then (f^{-1}(g))(p) = g(f(p)) = 1 and f^{-1}(g) is open fuzzy set in X. Since X is regular, there is an open fuzzy set h in X such that h(p) = 1 and h \leq \bar{h} \leq f^{-1}(g). Since f is f-open, f(h) is an open fuzzy set such that (f(h))(q) = 1 and f(h) \leq f(\bar{h}) \leq g. Since f is fg*-closed, f(cl(h)) is g*-closed such that f(cl(h)) \leq g, g is open fuzzy set and so is g-open. It follows that cl(f(cl(h))) \leq g. And hence f(h) \leq cl(f(cl(h))) \leq cl(f(clh)) \leq g. That is f(h) \leq cl(f(h)) \leq g. Hence Y is regular fts.

3.3.55 Theorem: Let f : X \rightarrow Y, h : Y \rightarrow Z be two maps such that hof : X \rightarrow Z is fg*-closed map.

i) If f is f-continuous and surjective, then h is fg*-closed.

ii) If h is fg*-irresolute and injective, then f is fg*-closed.

Proof: (i) Let H be a closed fuzzy set of Y. Then f^{-1}(H) is closed in X and so (hof)(f^{-1}(H)) = h(H) is g*-closed in Z. Thus h is a g*-closed map.
Let F be a closed fuzzy set of X. Then \((\text{hof})(F)\) is \(g^*\) - closed in \(Z\) and so \(h^{-1}(\text{hof})(F)\) is \(g^*\) - closed in \(Y\). Since \(h\) is injective, \(f(F) = h^{-1}(\text{hof})(F)\) is \(g^*\) - closed in \(Y\). Therefore \(f\) is a \(fg^*\) - closed map.

**3.3.56 Theorem:** The function \(f : X \rightarrow Y\) is \(fg^*\) - closed and \(Y\) is fuzzy - \(T^*_{1/2}\), then for each open fuzzy set \(A \subseteq X\), \(\text{cl}(f(A)) \subseteq f(\text{cl}(A))\).

**Proof:** Suppose \(f\) is \(fg^*\) - closed map. If \(A\) is open, \(\text{cl}(A)\) is closed. Then \(f(\text{cl}(A))\) is \(g^*\) - closed in \(Y\) and hence \(f(\text{cl}(A))\) is closed as \(Y\) is fuzzy - \(T^*_{1/2}\). Since \(f(A) \subseteq f(\text{cl}(A))\), \(\text{cl}(f(A)) \subseteq \text{cl}(f(\text{cl}(A))) = f(\text{cl}(A))\).

That is \(\text{cl}(f(A)) \subseteq f(\text{cl}(A))\).

We introduce the following.

**3.3.57 Definition:** Let \(X\) and \(Y\) be fuzzy topological spaces. A bijection map \(f : X \rightarrow Y\) is called fuzzy \(g^*\) - homeomorphism (\(fg^*\) - homeomorphism) if \(f\) and \(f^{-1}\) are fuzzy \(g^*\) - continuous.

**3.3.58 Theorem:** Every fuzzy homeomorphism is fuzzy \(g^*\) - homeomorphism.

**Proof:** Follows from the two definitions involved.

The converse of the above theorem need not be true as seen from the following example.

**3.3.59 Example:** Let \(X = Y = \{a, b, c\}\). Fuzzy sets \(A\), \(B\) and \(C\) be defined as follows: \(A = \{(a, 1), (b, 0), (c, 0)\}\), \(B = \{(a, 1), (b, 1), (c, 0)\}\) and \(C = \{(a, 1), (b, 0), (c, 1)\}\). Consider \(T = \{0, 1, A, C\}\), \(\sigma = \{0, 1, B\}\). Then \((X, T)\) and \((Y, \sigma)\) are fts. Define \(f : X \rightarrow Y\) by \(f(a) = a, f(b) = c, f(c) = b\), then \(f\) is fuzzy \(g^*\) - homeomorphism but not fuzzy homeomorphism as \(A\) is open fuzzy set in \(X\) and its image \(f(A) = A\) is not open in \(Y\), \(f^{-1} : Y \rightarrow X\) is not \(f\) - continuous.

**3.3.60 Theorem:** Let \(f : X \rightarrow Y\) be a bijective function. Then the following are equivalent.

a) \(f\) is \(fg^*\) - homeomorphism
b) $f$ is $fg^*$ - continuous and $fg^*$ - open maps.
c) $f$ is $fg^*$ - continuous and $fg^*$ - closed maps.

**Proof:** (a) $\Rightarrow$ (b): Let $f$ be $fg^*$ - homeomorphism. Then $f$ is $fg^*$ - continuous and $f^{-1}$ is $fg^*$ - continuous. To prove that $f$ is $fg^*$ - open map. Let $U$ be an open fuzzy set in $X$. Then, since $f^{-1} : Y \to X$ is $fg^*$ - continuous, $f(U) = (f^{-1})^{-1}(U)$ is $fg^*$ - open in $Y$. Therefore $f(U)$ is $fg^*$ - open in $Y$. Hence $f$ is $fg^*$ - open map.

(b) $\Rightarrow$ (a): Let $f$ be $fg^*$ - open and $fg^*$ - continuous map. To prove that $f^{-1} : Y \to X$ is $fg^*$ - continuous. Let $U$ be an open fuzzy set in $X$. Then $f(U)$ is $fg^*$ - open in $Y$ since $f$ is $fg^*$ - open map. Now $(f^{-1})^{-1}(U) = f(U)$ is $fg^*$ - open in $Y$. Therefore $f^{-1} : Y \to X$ is $fg^*$ - continuous. And hence $f$ is $fg^*$ - homeomorphism.

(b) $\Rightarrow$ (c): Let $f$ be $fg^*$ - continuous and $fg^*$ - open map. To prove that $f$ is $fg^*$ - closed map. Let $F$ be a closed fuzzy set in $X$. Since $f$ is $fg^*$ - open map, $f(1 - F)$ is $fg^*$ - open in $Y$. Now $f(1 - F) = 1 - f(F)$. Therefore $f(F)$ is $fg^*$ - closed in $Y$. Hence $f$ is a $fg^*$ - closed map.

(c) $\Rightarrow$ (b): Let $f$ be $fg^*$ - continuous and $fg^*$ - closed map. To prove that $f$ is $fg^*$ - open map. Let $U$ be an open fuzzy set in $X$. Then $1 - U$ is closed fuzzy set in $X$. Since $f$ is a $fg^*$ - closed map, $f(1 - U)$ is $fg^*$ - closed in $Y$. Now $f(1 - U) = 1 - f(U)$. Therefore $f(U)$ is $fg^*$ - open in $Y$. Hence $f$ is $fg^*$ - open map.

3.3.61 Theorem: If $f : X \to Y$ is $fg^*$ - homeomorphism and $g : Y \to Z$ is $fg^*$ - homeomorphism and $Y$ is a fuzzy - $T^*_{1/2}$ then $gof : X \to Z$ is $fg^*$ - homeomorphism.

**Proof:** To show that $gof$ and $(gof)^{-1}$ are $fg^*$ - continuous. Let $U$ be an open fuzzy set in $Z$. Since $g : Y \to Z$ is $fg^*$ - continuous, $g^{-1}(U)$ is $fg^*$ - open in $Y$. Then $g^{-1}(U)$ is open fuzzy set in $Y$ as $Y$ is fuzzy - $T^*_{1/2}$.
Also since \( f : X \rightarrow Y \) is \( fg^* \)-continuous, \( f^{-1}(g^{-1}(U)) = (gof)^{-1}(U) \) is \( g^* \)-open in \( X \). Therefore \( gof \) is \( fg^* \)-continuous.

Again, let \( U \) be an open fuzzy set in \( X \). Since \( f^{-1} : Y \rightarrow X \) is \( fg^* \)-continuous, \( (f^{-1})^{-1}(U) = f(U) \) is \( g^* \)-open in \( Y \). And so \( f(U) \) is open fuzzy set in \( Y \) as \( Y \) is fuzzy - \( T^*_{1/2} \). Also since \( g^{-1} : Z \rightarrow Y \) is \( fg^* \)-continuous, \( (g^{-1})^{-1}(f(U)) = g(f(U)) \) is \( g^* \)-open in \( Z \). Therefore \( ((gof)^{-1})^{-1}(U) = (gof)(U) \) is \( g^* \)-open in \( Z \). Hence \((gof)^{-1}\) is \( fg^* \)-continuous. Thus \( gof \) is \( fg^* \)-homeomorphism.

### 3.3.62 Definition:
A bijection map \( f : X \rightarrow Y \) is called \( fg^*c \)-homeomorphism if \( f \) and \( f^{-1} \) are \( fg^* \)-irresolute.

### 3.3.63 Remark:
The family of all \( fg^*c \)-homeomorphism (resp. \( fg^* \)-homeomorphism and fuzzy homeomorphism) from \((X, T)\) onto itself is denoted by \( fg^*c \ h(X, T) \) (resp. \( fg^* h \ (X, T) \) and \( f h (X,T) \)).

### 3.3.64 Theorem:
Let \( X, Y, Z \) be fuzzy topological spaces and \( f : X \rightarrow Y \), \( g : Y \rightarrow Z \) be \( fg^*c \)-homeomorphisms then their composition \( gof : X \rightarrow Z \) is \( fg^*c \)-homeomorphism.

**Proof:** Let \( U \) be \( g^* \)-open fuzzy set in \( Z \). Then since \( g : Y \rightarrow Z \) is \( fg^* \)-irresolute, \( g^{-1}(U) \) is \( g^* \)-open in \( Y \). Also since \( f : X \rightarrow Y \) is \( fg^* \)-irresolute, \( f^{-1}(g^{-1}(U)) = (gof)^{-1}(U) \) is \( g^* \)-open in \( X \). Therefore \( gof : X \rightarrow Z \) is \( fg^* \)-irresolute. Again, let \( U \) be a \( g^* \)-open fuzzy set in \( X \). Then, since \( f^{-1} : Y \rightarrow X \) is \( fg^* \)-irresolute, \( (f^{-1})^{-1}(U) = f(U) \) is \( g^* \)-open in \( Y \). Also \( g^{-1} : Z \rightarrow Y \) is \( fg^* \)-irresolute, \( (g^{-1})^{-1}(f(U)) = g(f(U)) = (gof)(U) \) is \( g^* \)-open in \( Z \). Therefore \((gof)^{-1} : Z \rightarrow X \) is \( fg^* \)-irresolute. And hence \( gof \) is \( fg^*c \)-homeomorphism.

### 3.3.65 Theorem:
Every \( fg^*c \)-homeomorphism is \( fg^* \)-homeomorphism.

**Proof:** Let \( f : X \rightarrow Y \) be \( fg^*c \)-homeomorphism. Then \( f \) and \( f^{-1} \) are \( fg^* \)-irresolute functions. And so \( f \) and \( f^{-1} \) are \( fg^* \)-continuous functions. And hence \( f \) is \( fg^* \)-homeomorphism.
Now, the stronger forms of fuzzy $g^*$ - continuous functions namely strongly $fg^*$ - continuous, perfectly $fg^*$ - continuous and completely $fg^*$ - continuous functions have been introduced and studied.

3.3.66 Definition: A function $f : X \rightarrow Y$ is called strongly $fg^*$ - continuous if the inverse image of every $g^*$ - open fuzzy set in $Y$ is open in $X$.

3.3.67 Theorem: A function $f : X \rightarrow Y$ is strongly $fg^*$ - continuous iff the inverse image of every $g^*$ - closed fuzzy set in $Y$ is closed in $X$.

Proof: Assume that $f$ is strongly $fg^*$ - continuous. Let $F$ be $g^*$ - closed in $Y$. Then $1 - F$ is $g^*$ - open. Since $f$ is strongly $fg^*$ - continuous, $f^{-1}(1 - F)$ is open in $X$. But $f^{-1}(1 - F) = 1 - f^{-1}(F)$ and so $f^{-1}(F)$ is closed in $X$.

Conversely, suppose that the inverse image of every $g^*$ - closed fuzzy set in $Y$ is closed in $X$. Let $V$ be $g^*$ - open in $Y$, then $1 - V$ is $g^*$ - closed in $Y$. By hypothesis, $f^{-1}(1 - V)$ is closed in $X$. Now $f^{-1}(1 - V) = 1 - f^{-1}(V)$ and so $f^{-1}(V)$ is open fuzzy set in $X$. Hence $f$ is strongly $fg^*$ - continuous.

3.3.68 Theorem: Every strongly $fg^*$ - continuous function is a $f$ - continuous function.

Proof: Let $f : X \rightarrow Y$ be strongly $fg^*$ - continuous function. Let $V$ be open fuzzy set in $Y$ and so $V$ is $g^*$ - open in $Y$. Then $f^{-1}(V)$ is open in $X$. Hence $f$ is $f$ - continuous.

The converse of the above theorem need not be true as seen from the following example.

3.3.69 Example: Let $X = Y = \{a, b, c\}$. Fuzzy sets $A$, $B$ and $C$ be defined as follows: $A = \{(a, .6), (b, .5), (c, .7)\}$, $B = \{(a, .4), (b, .5), (c, .3)\}$ and $C = \{(a, .7), (b, .5), (c, .8)\}$. Consider $T = \{0, 1, A, B\}$ and $\sigma = \{0, 1, B\}$. Then $(X, T)$ and $(Y, \sigma)$ are fts. Define $f : X \rightarrow Y$ by $f(a) = a$, $f(b) = b$, $f(c) = c$. Now, $f$ is strongly $fg^*$ - continuous function.
\( f(c) = c \). Then \( f \) is \( f \)-continuous as \( B \) is open in \( Y \) and \( f^{-1}(B) = B \) is open in \( X \). But \( f \) is not strongly \( fg^* \)-continuous as the fuzzy set \( C \) is \( g^* \)-closed in \( Y \) and \( f^{-1}(C) = C \) is not closed in \( X \).

### 3.3.70 Theorem

Every strongly \( gf \)-continuous function is strongly \( fg^* \)-continuous.

**Proof:** Let \( f : X \to Y \) be strongly \( gf \)-continuous. Let \( V \) be \( g^* \)-open fuzzy set in \( Y \). Then \( V \) is \( g \)-open in \( Y \). And then \( f^{-1}(V) \) is open in \( X \) since \( f \) is strongly \( fg \)-continuous function. Hence \( f \) is strongly \( fg^* \)-continuous function.

### 3.3.71 Theorem

Every strongly \( f \)-continuous function is strongly \( fg^* \)-continuous.

**Proof:** Let \( f : X \to Y \) be strongly \( f \)-continuous function. Let \( v \) be \( g^* \)-open fuzzy set in \( Y \). Then \( f^{-1}(v) \) is both open and closed in \( X \) as \( f \) is strongly \( f \)-continuous. Hence \( f \) is strongly \( fg^* \)-continuous.

The converse of the above theorem need not be true as shown from the following example.

### 3.3.72 Example

Let \( X = Y = \{a, b, c\} \). Fuzzy sets \( A_1, A_2, A_3, A_4, A_5, A_6 \) be defined as follows: \( A_1 = \{(a, 1), (b, 0), (c, 0)\} \), \( A_2 = \{(a, 0), (b, 1), (c, 0)\} \), \( A_3 = \{(a, 0), (b, 0), (c, 1)\} \), \( A_4 = \{(a, 1), (b, 1), (c, 0)\} \), \( A_5 = \{(a, 1), (b, 0), (c, 1)\} \), \( A_6 = \{(a, 0), (b, 1), (c, 1)\} \). Consider \( T = \{0, 1, A_1, A_2, A_4\} \), \( \sigma = \{0, 1, A_4\} \). Then \((X, T) \) and \((Y, \sigma) \) are fts. Define \( f : X \to Y \) by \( f(a) = b, f(b) = a, f(c) = c \). Then \( f \) is strongly \( fg^* \)-continuous but not strongly \( f \)-continuous as the fuzzy set \( A_1 \) in \( Y \) is such that \( f^{-1}(A_1) = A_2 \) is open but not closed in \( X \).

### 3.3.73 Theorem

Let \( f : X \to Y \) be \( f \)-continuous and \( Y \) be fuzzy - \( T^*_1/2 \). Then \( f \) is strongly \( fg^* \)-continuous.
Proof: Let $V$ be $g^*$-open fuzzy set in $Y$. Then $V$ is open in $Y$ since $Y$ is fuzzy -$T^{*}_{1/2}$. And then $f^{-1}(V)$ is open in $X$ as $f$ is $f$-continuous. Hence $f$ is strongly $fg^*$-continuous.

3.3.74 Theorem: If $f : X \rightarrow Y$ is strongly $fg^*$-continuous and $Y$ is fuzzy -$T^{*}_{1/2}$, then $f$ is strongly $gf$-continuous.

Proof: Let $V$ be $g$-open fuzzy set in $Y$. Then $V$ is $g^*$-open in $Y$, since $Y$ is fuzzy -$T^{*}_{1/2}$. Since $f$ is strongly $fg^*$-continuous, $f^{-1}(V)$ is open in $X$. Hence $f$ is strongly $gf$-continuous.

3.3.75 Theorem: Every strongly $fb$-continuous function is strongly $fg^*$-continuous.

Proof: Let $f : X \rightarrow Y$ be strongly $fb$-continuous function. Let $V$ be $g^*$-open fuzzy set in $Y$, then $V$ is $b$-open in $Y$. And then $f^{-1}(V)$ is open in $X$ since $f$ is strongly $fb$-continuous function. Hence $f$ is strongly $fg^*$-continuous function.

3.3.76 Theorem: If $f : X \rightarrow Y$ is strongly $fg^*$-continuous and $Y$ is $fb$-space. Then $f$ is strongly $fb$-continuous function.

Proof: Let $V$ be $b$-open fuzzy set in $Y$. Then $V$ is open in $Y$, since $Y$ is $fb$-space. And so $V$ is $g^*$-open in $Y$. Since $f$ is strongly $fg^*$-continuous, $f^{-1}(V)$ is open in $X$. And hence $f$ is strongly $fb$-continuous function.

3.3.77 Theorem: If $f : X \rightarrow Y$ is strongly $fg^*$-continuous and $g : Y \rightarrow Z$ is strongly $fg^*$-continuous. Then the composition map $gof : X \rightarrow Z$ is strongly $fg^*$-continuous function.

Proof: Let $V$ be $g^*$-open fuzzy set in $Z$. Then $g^{-1}(V)$ is open in $Y$ since $g$ is strongly $fg^*$-continuous. And therefore $g^{-1}(V)$ is $g^*$-open in $Y$. Also since $f$ is strongly $fg^*$-continuous, $f^{-1}(g^{-1}(V)) = (gof)^{-1}(V)$ is open in $X$. Hence gof is strongly $fg^*$-continuous function.
3.3.78 Theorem: Let \( f : X \rightarrow Y, g : Y \rightarrow Z \) be maps such that \( f \) is strongly \( fg^* \)-continuous and \( g \) is \( fg^* \)-continuous then \( gof : X \rightarrow Z \) is \( f \)-continuous.

Proof: Let \( F \) be closed fuzzy set in \( Z \). Then \( g^{-1}(F) \) is \( g^* \)-closed in \( Y \) since \( g \) is \( fg^* \)-continuous. Also since \( f \) is strongly \( fg^* \)-continuous, \( f^{-1}(g^{-1}(F)) = (gof)^{-1}(F) \) is closed in \( X \). Hence \( gof \) is \( f \)-continuous.

3.3.79 Theorem: If \( f : X \rightarrow Y \) is strongly \( fg^* \)-continuous, \( g : Y \rightarrow Z \) is \( fg^* \)-irresolute then \( gof : X \rightarrow Z \) is strongly \( fg^* \)-continuous.

Proof: Let \( V \) be \( g^* \)-open fuzzy set in \( Z \). Then \( g^{-1}(V) \) is \( g^* \)-open in \( Y \) since \( g \) is \( fg^* \)-irresolute. And then \( f^{-1}(g^{-1}(V)) = (gof)^{-1}(V) \) is open in \( X \) since \( f \) is strongly \( fg^* \)-continuous. Hence \( gof \) is strongly \( fg^* \)-continuous.

3.3.80 Theorem: Every strongly \( fg^* \)-continuous image of a fuzzy compact space is fuzzy compact.

Proof: Let \( f : X \rightarrow Y \) be a strongly \( fg^* \)-continuous function from a fuzzy compact space \( X \) onto a fts \( Y \). Let \( \{U_\lambda : \lambda \in \Lambda \} \) be any fuzzy open cover of \( Y \) and so fuzzy \( g^* \)-open cover of \( Y \). Since \( f \) is strongly \( fg^* \)-continuous, \( \{f^{-1}(U_\lambda) : \lambda \in \Lambda \} \) is an open cover of \( X \). Since \( X \) is fuzzy compact, there is a finite sub cover \( \{f^{-1}(U_\lambda) : i = 1,2,...,n \} \) of \( \{f^{-1}(U_\lambda) \} \). If implies that \( \{U_{\lambda i} : i = 1,2,...,n \} \) is a finite sub cover of \( \{U_\lambda : \lambda \in \Lambda \} \). Hence \( f(X) = Y \) is fuzzy compact.

3.3.81 Definition: A function \( f : X \rightarrow Y \) is called perfectly \( fg^* \)-continuous if the inverse image of every \( g^* \)-open fuzzy set in \( Y \) is both open and closed fuzzy set in \( X \).

3.3.82 Theorem: A map \( f : X \rightarrow Y \) is perfectly \( fg^* \)-continuous iff the inverse image of every \( g^* \)-closed fuzzy set in \( Y \) is both open and closed fuzzy set in \( X \).

Proof: Assume that \( f \) is perfectly \( fg^* \)-continuous. Let \( F \) be \( g^* \)-closed fuzzy set in \( Y \). Then \( 1 - F \) is \( g^* \)-open in \( Y \). And therefore \( f^{-1}(1 - F) \) is
both open and closed fuzzy set in \( X \). But \( f^{-1}(1 - F) = 1 - f^{-1}(F) \) and so \( f^{-1}(F) \) is both open and closed fuzzy set in \( X \).

Conversely, the inverse image of every \( g^*\)-closed fuzzy set in \( Y \) is both open and closed in \( X \). Let \( V \) be \( g^*\)-open fuzzy set in \( Y \). Then \( 1 - V \) is \( g^*\)-closed in \( Y \). Then by hypothesis, \( f^{-1}(1 - V) \) is both open and closed fuzzy set in \( X \). But \( f^{-1}(1 - V) = 1 - f^{-1}(V) \). Therefore \( f^{-1}(V) \) is both open and closed fuzzy set in \( X \). Hence \( f \) is perfectly \( fg^* \)-continuous.

**3.3.83 Theorem:** Every perfectly \( fg^* \)-continuous function is \( f \)-continuous.

**Proof:** Let \( f : X \rightarrow Y \) be perfectly \( fg^* \)-continuous. Let \( v \) be open fuzzy set in \( Y \) and so \( v \) is \( g^*\)-open fuzzy set in \( Y \). Since \( f \) is perfectly \( fg^* \)-continuous, \( f^{-1}(v) \) is both open and closed fuzzy set in \( X \). That is \( f^{-1}(v) \) is open fuzzy set in \( X \). Hence \( f \) is \( f \)-continuous.

The converse of the above theorem need not be true as shown in the following example.

**3.3.84 Example:** In the example 3.3.69, the function \( f \) is \( f \)-continuous but not perfectly \( fg^* \)-continuous as the fuzzy set \( 1 - C = \{(a, .3), (b, .5), (c, .2)\} \) is \( g^* \)-open in \( Y \) and \( f^{-1}(1 - C) = 1 - C \) which is not both open and closed in \( X \).

**3.3.85 Theorem:** Every perfectly \( fg^* \)-continuous function is perfectly \( f \)-continuous.

**Proof:** Let \( f : X \rightarrow Y \) be perfectly \( fg^* \)-continuous. Let \( v \) be open fuzzy set in \( Y \), then \( v \) is \( g^* \)-open in \( Y \). Since \( f \) is perfectly \( fg^* \)-continuous, \( f^{-1}(v) \) is both open and closed in \( X \) and hence \( f \) is perfectly \( f \)-continuous.

The converse of the above theorem need not be true as seen from the following example.

**3.3.86 Example:** In the example 3.3.69, the function \( f \) is perfectly \( f \)-continuous as the fuzzy set \( B \) is open in \( Y \) and its inverse image...
f^{-1}(B) = B is both open and closed in X. But f is not perfectly fg*-continuous as the fuzzy set \(1 - C = \{(a, .3), (b, .5), (c, .2)\}\) is g*-open in Y and \(f^{-1}(1 - C) = 1 - C\) is not both open and closed in X.

3.3.87 Theorem: Every perfectly gf-continuous function is perfectly fg*-continuous.

Proof: Let \(f: X \to Y\) be perfectly gf-continuous. Let \(v\) be g*-open fuzzy set in Y. Then \(v\) is g-open in Y. And then \(f^{-1}(v)\) is both open and closed in X. Hence \(f\) is perfectly fg*-continuous function.

3.3.88 Theorem: If \(f: X \to Y\) is perfectly f-continuous and Y is fuzzy - \(^*\)T_{1/2}. Then \(f\) is perfectly fg*-continuous.

Proof: Let \(v\) be g*-open in Y. Then \(v\) is open in Y as Y is fuzzy - \(^*\)T_{1/2}. Since \(f\) is perfectly f-continuous, \(f^{-1}(v)\) is both open and closed in X. And therefore \(f\) is perfectly fg*-continuous function.

3.3.89 Theorem: If \(f: X \to Y\) is perfectly fg*-continuous and Y is fuzzy - \(^*\)T_{1/2}. Then \(f\) is perfectly fg-continuous.

Proof: Let \(v\) be g-open fuzzy set in Y. Then \(v\) is g*-open in Y since Y is fuzzy - \(^*\)T_{1/2}. Since \(f\) is perfectly fg*-continuous, \(f^{-1}(v)\) is both open and closed in X. And hence \(f\) is perfectly fg-continuous function.

3.3.90 Theorem: Every perfectly fg*-continuous function is strongly fg*-continuous.

Proof: Let \(f: X \to Y\) be perfectly fg*-continuous. Let \(v\) be g*-open fuzzy set in Y. Then \(f^{-1}(v)\) is both open and closed in X. Therefore \(f^{-1}(v)\) is open in X. Hence \(f\) is strongly fg*-continuous.

The converse of the above theorem need not be true as seen from the following example.
3.3.91 Example: In the example 3.3.72, the function \( f \) is strongly \( \text{fg}^* \)-continuous but not perfectly \( \text{fg}^* \)-continuous as the fuzzy set \( A_3 \) is \( \text{g}^* \)-closed in \( Y \) and \( f^{-1}(A_3) = A_3 \) is not both open and closed in \( X \).

3.3.92 Theorem: Every perfectly \( \text{fb} \)-continuous function is perfectly \( \text{fg}^* \)-continuous.

Proof: Let \( f: X \rightarrow Y \) be perfectly \( \text{fb} \)-continuous. Let \( v \) be \( \text{g}^* \)-open in \( Y \). Then \( v \) is \( \text{b} \)-open in \( Y \). And then \( f^{-1}(v) \) is both open and closed in \( X \) since \( f \) is perfectly \( \text{fb} \)-continuous. Hence \( f \) is perfectly \( \text{fg}^* \)-continuous.

3.3.93 Theorem: If \( f : X \rightarrow Y \) is perfectly \( \text{fg}^* \)-continuous and \( Y \) is \( \text{fb} \)-space. Then \( f \) is perfectly \( \text{fb} \)-continuous.

Proof: Let \( v \) be \( \text{b} \)-open fuzzy set in \( Y \). Then \( v \) is open and so \( \text{g}^* \)-open in \( Y \) since \( Y \) is \( \text{fb} \)-space. And then \( f^{-1}(v) \) is both open and closed in \( X \), since \( f \) is perfectly \( \text{fg}^* \)-continuous. Hence \( f \) is perfectly \( \text{fb} \)-continuous function.

3.3.94 Theorem: If \( f : X \rightarrow Y, \ g : Y \rightarrow Z \) be two perfectly \( \text{fg}^* \)-continuous functions then \( g \circ f : X \rightarrow Z \) is perfectly \( \text{fg}^* \)-continuous function.

Proof: Let \( v \) be \( \text{g}^* \)-open fuzzy set in \( Z \). Then \( g^{-1}(v) \) is both open and closed in \( Y \), since \( g \) is perfectly \( \text{fg}^* \)-continuous. And therefore \( g^{-1}(v) \) is \( \text{g}^* \)-open in \( Y \). Also since \( f \) is perfectly \( \text{fg}^* \)-continuous, \( f^{-1}(g^{-1}(v)) = (g \circ f)^{-1}(v) \) is both open and closed in \( X \). Hence \( g \circ f \) is perfectly \( \text{fg}^* \)-continuous.

3.3.95 Theorem: If \( f : X \rightarrow Y \) is perfectly \( \text{fg}^* \)-continuous and \( g : Y \rightarrow Z \) is \( \text{fg}^*- \) irresolute function then \( g \circ f : X \rightarrow Z \) is perfectly \( \text{fg}^* \)-continuous function.

Proof: Let \( v \) be \( \text{g}^* \)-open fuzzy set in \( Z \). Then \( g^{-1}(v) \) is \( \text{g}^* \)-open in \( Y \) since \( g \) is \( \text{fg}^*- \) irresolute function. Also since \( f \) is perfectly
fg*-continuous, $f^{-1}(g^{-1}(v)) = (gof)^{-1}(v)$ is both open and closed in X. Hence gof is perfectly fg*-continuous.

3.3.96 Definition: A map $f : X \rightarrow Y$ is said to be completely fg*-continuous if the inverse image of every g*-open fuzzy set in Y is regular open fuzzy set in X.

3.3.97 Theorem: A map $f : X \rightarrow Y$ is completely fg*-continuous iff the inverse image of every g*-closed fuzzy set in Y is regular-closed fuzzy set in X.

Proof: Suppose $f$ is completely fg*-continuous. Let $F$ be g*-closed in Y. Then $1 - F$ is g*-open in Y. And then $f^{-1}(1 - F)$ is regular open in X. Now $f^{-1}(1 - F) = 1 - f^{-1}(F)$ and therefore $f^{-1}(F)$ is regular closed in X.

Conversely, assume that the inverse image of every g*-closed fuzzy set in Y is regular-closed in X. Let $V$ be g*-open in Y. Then $1 - V$ is g*-closed in Y. By hypothesis, $f^{-1}(1 - V)$ is regular closed in X. Now $f^{-1}(1 - V) = 1 - f^{-1}(V)$. And therefore $f^{-1}(V)$ is regular open fuzzy set in X. Hence $f$ is completely fg*-continuous function.

3.3.98 Theorem: Every completely fg*-continuous function is f-continuous.

Proof: Let $f : X \rightarrow Y$ be completely fg*-continuous function. Let $V$ be open fuzzy set in Y. Then $V$ is g*-open in Y. And then $f^{-1}(V)$ is regular open and therefore $f^{-1}(V)$ is open in X. Hence $f$ is f-continuous.

The converse of the above theorem need not be true as seen from the following example.

3.3.99 Example: In the example 3.3.69, the function $f$ is f-continuous but not completely fg*-continuous as the fuzzy set $1 - C$ is g*-open in Y and $f^{-1}(1 - C) = 1 - C$ is not regular-open in X.

3.3.100 Theorem: Every completely fg*-continuous function is completely f-continuous.
Proof: Let $f : X \rightarrow Y$ be completely $f_{g^*}$-continuous. Let $V$ be open fuzzy set in $Y$. Then $V$ is $g^*$-open in $Y$. And then $f^{-1}(V)$ is regular open in $X$. Hence $f$ is completely $f$-continuous.

The converse of the above theorem need not be true as shown from the following example.

3.3.101 Example: In the example 3.3.69, the function $f$ is completely $f$-continuous as the fuzzy set $B$ is open in $Y$ and $f^{-1}(B) = B$ is regular open in $X$. But $f$ is not completely $f_{g^*}$-continuous as the fuzzy set $1 - C$ is $g^*$-open in $Y$ and $f^{-1}(1 - C) = 1 - C$ is not regular open in $X$.

3.3.102 Theorem: Every completely $f_{g^*}$-continuous function is strongly $f_{g^*}$-continuous function.

Proof: Let $f : X \rightarrow Y$ be completely $f_{g^*}$-continuous. Let $V$ be $g^*$-open in $Y$. Then $f^{-1}(V)$ is regular open in $X$. And therefore $f^{-1}(V)$ is open in $X$. Hence $f$ is strongly $f_{g^*}$-continuous.

The converse of the above theorem need not be true as shown from the following example.

3.3.103 Example: In the example 3.3.72, the function $f$ is strongly $f_{g^*}$-continuous but not completely $f_{g^*}$-continuous as the fuzzy set $A_3$ is $g^*$-closed in $Y$ and its inverse image $f^{-1}(A_3) = A_3$ is not regular closed in $X$.

3.3.104 Theorem: If $f : X \rightarrow Y$ is completely $f$-continuous and $Y$ is fuzzy $- T^*_{1/2}$. Then $f$ is completely $f_{g^*}$-continuous function.

Proof: Let $v$ be $g^*$-open in $Y$. Then $v$ is open in $Y$ since $Y$ is fuzzy $- T^*_{1/2}$. Also since $f$ is completely $f$-continuous, $f^{-1}(v)$ is regular open in $X$. And hence $f$ is completely $f_{g^*}$-continuous function.

3.3.105 Theorem: Every completely $f^b$-continuous function is completely $f_{g^*}$-continuous function.
**Proof**: Let \( f: X \to Y \) be completely \( fb \) - continuous. Let \( v \) be \( g^* \) - open fuzzy set in \( Y \). Then \( v \) is \( b \) - open in \( Y \). And then \( f^{-1}(v) \) is regular open in \( X \). Hence \( f \) is completely \( fg^* \) - continuous function.

**3.3.106 Theorem**: If \( f: X \to Y \) is completely \( fg^* \) - continuous and \( Y \) is \( fb \) - space. Then \( f \) is completely \( fb \) - continuous function.

**Proof**: Let \( v \) be \( b \) - open fuzzy set in \( Y \). Then \( v \) is open in \( Y \) since \( Y \) is \( fb \) - space. And so \( v \) is \( g^* \) - open in \( Y \). Then \( f^{-1}(v) \) is regular open in \( X \) since \( f \) is completely \( fg^* \) - continuous. Hence \( f \) is completely \( fb \)-continuous function.

**3.3.107 Theorem**: If \( f: X \to Y \) is completely \( fg^*\)-continuous and \( g: Y \to Z \) is \( fg^* \) - irresolute function then \( gof: X \to Z \) is completely \( fg^* \) - continuous function.

**Proof**: Let \( v \) be \( g^* \) - open fuzzy set in \( Z \). Then \( g^{-1}(v) \) is \( g^* \) - open in \( Y \) since \( g \) is \( fg^* \) - irresolute. And then \( f^{-1}(g^{-1}(v)) = (gof)^{-1}(v) \) is regular open in \( X \). Hence \( gof \) is completely \( fg^* \) - continuous function.

**3.3.108 Theorem**: If \( f: X \to Y \) and \( g: Y \to Z \) be two completely \( fg^*\)-continuous functions then \( gof: X \to Z \) is completely \( fg^*\)-continuous function.

**Proof**: Let \( v \) be \( g^* \) - open fuzzy set in \( Z \). Then \( g^{-1}(v) \) is regular open in \( Y \) since \( g \) is completely \( fg^* \) - continuous. And so \( g^{-1}(v) \) is open and then \( g^* \) - open in \( Y \). Also since, \( f \) is completely \( fg^* \) - continuous function, \( f^{-1}(g^{-1}(v)) = (gof)^{-1}(v) \) is regular open in \( X \). And hence \( gof \) is completely \( fg^* \) - continuous function.

**3.3.109 Theorem**: Every completely \( fg^* \) - continuous image of a fuzzy nearly compact space is fuzzy compact.

**Proof**: Let \( f: X \to Y \) be a completely \( fg^* \) - continuous mapping from a fuzzy nearly compact space \( X \) onto a \( Y \). Let \( \{U_{\lambda}: \lambda \in \Lambda\} \) be any fuzzy open cover and so fuzzy \( g^*\)-open cover of \( Y \). Since \( f \) is completely
fg* - continuous, \{f^{-1}(U_\lambda) : \lambda \in \Lambda\} is a fuzzy regular open cover of X. Since X is fuzzy nearly compact space, there exist a finite sub cover \{f^{-1}(U_{\lambda_i}) : i = 1, \ldots, n\} of \{f^{-1}(U_\lambda)\}. It implies that \{U_{\lambda_i} : i = 1, 2, \ldots, n\} is a finite sub cover of \{U_\lambda : \lambda \in \Lambda\}. Hence \(f(X) = Y\) is fuzzy compact.

3.4 g*- COMPACTNESS AND OTHER RELATED CONCEPTS IN FTS

In this section the concepts of fuzzy g*- compactness, countable g*- compactness and g* - Lindelöf property have been introduced and studied.

It is also introduced, the concepts of g*- regular and g*- normal in fts and studied their properties.

3.4.1 Definition: A collection \{A_\lambda : \lambda \in \Lambda\} of g* - open fuzzy sets in a fts X is called g* - open cover of a fuzzy set B in X if \(B \leq \bigvee_{\lambda \in \Lambda} A_\lambda\).

3.4.2 Definition: A fts X is called g* - compact if every g* - open cover of X has a finite sub cover.

3.4.3 Definition: A fuzzy set A in fts X is said to be g* - compact relative to X if for every collection \{A_\lambda : \lambda \in \Lambda\} of g* - open fuzzy sets of X such that \(A \leq \bigvee_{\lambda \in \Lambda} A_\lambda\), there exists a finite subset \(\Lambda_0\) of \(\Lambda\) such that \(A \leq \bigvee_{\lambda \in \Lambda_0} A_\lambda\).

3.4.4 Definition: A fuzzy set A of X is said to be g* - compact if A is g* - compact relative to X.

3.4.5 Theorem: A g* - closed crisp subset of a g* - compact fts X is g* - compact.

Proof: Let \(Y\) be a g* - closed crisp sub space of g* - compact fts X. To prove that \(Y\) is g*-compact. Let \(\{U_\lambda : \lambda \in \Lambda\}\) be any g* - open cover of Y. Then \(u = \{U_\lambda : \lambda \in \Lambda\} \cup \{1 - Y\}\) is a g* - open cover of X. Let \(x \in X\) then \(x \in Y\) or \(x \notin Y\). If \(x \notin Y\), then \(x \in 1 - Y\). That is \((1 - Y)(x) = 1\).
where \(1 - Y \in \{ U_\lambda : \lambda \in \Lambda \} \cup \{ 1 - Y \} \). Suppose \( x \in Y \). Since \( \{ U_\lambda : \lambda \in \Lambda \} \) is \( g^* \)-open cover of \( Y \), there exists \( U_{\lambda_0} \) such that \( U_{\lambda_0}(x) = 1 \), thus \( u = \{ U_\lambda : \lambda \in \Lambda \} \cup \{ 1 - Y \} \) is \( g^* \)-open cover of \( X \). Since \( X \) is \( g^* \)-compact, \( u \) has a finite sub cover say \( u = \{ U_{\lambda_1}, \ldots, U_{\lambda_k} \} \cup \{ 1 - Y \} \) for \( X \). Then the family \( v = \{ U_{\lambda_1}, \ldots, U_{\lambda_k} \} \) is finite sub cover of \( \{ U_\lambda : \lambda \in \Lambda \} \) for \( Y \). Let \( x \in Y \). Then \( x \in X \) and \( x \notin 1 - Y \). Since \( u \) is a finite sub cover of \( X \), there exists \( U_{\lambda_i} \) (\( i = 1, 2, \ldots, k \)) such that \( U_{\lambda_i}(x) = 1 \). It follows that \( v \) is a \( g^* \)-open cover of \( Y \). Thus every \( g^* \)-open cover of \( Y \) has a finite sub cover. Hence \( Y \) is \( g^* \)-compact fts.

3.4.6 Theorem: The image of a \( fg^* \)-compact fts under a \( fg^* \)-continuous map is fuzzy compact.

**Proof:** Let \( f : X \to Y \) be a \( fg^* \)-continuous map from a \( g^* \)-compact fts \( X \) onto a fts \( Y \). Let \( u = \{ A_\lambda : \lambda \in \Lambda \} \) be a fuzzy open cover of \( Y \). Then the collection \( v = \{ f^{-1}(A_\lambda) : \lambda \in \Lambda \} \) is a \( g^* \)-open cover of \( X \), since \( f \) is \( fg^* \)-continuous. Since \( X \) is \( g^* \)-compact, \( v \) has a finite sub cover say \( \{ f^{-1}(A_{\lambda_1}), \ldots, f^{-1}(A_{\lambda_k}) \} \). Then \( \{ A_{\lambda_1}, \ldots, A_{\lambda_k} \} \) is a finite sub cover of \( u \) for \( Y \) and hence \( Y \) is fuzzy compact.

3.4.7 Theorem: If a map \( f : X \to Y \) is \( fg^* \)-irresolute and a subset \( B \) is \( fg^* \)-compact relative to \( X \), then the image \( f(B) \) is \( fg^* \)-compact relative to \( Y \).

**Proof:** Let \( \{ A_\lambda : \lambda \in \Lambda \} \) be any collection of \( g^* \)-open fuzzy sets of \( X \) such that \( f(B) \leq \bigvee_{\lambda \in \Lambda} A_\lambda \). Then \( B \leq \bigvee_{\lambda \in \Lambda} f^{-1}(A_\lambda) \). By hypothesis, there exists a finite subset \( \Lambda_0 \) of \( \Lambda \) such that \( B \leq \bigvee_{\lambda \in \Lambda_0} f^{-1}(A_\lambda) \). Therefore we have

\[ f(B) \leq \bigvee_{\lambda \in \Lambda_0} A_\lambda. \]

Which shows that \( f(B) \) is \( fg^* \)-compact relative to \( Y \).
3.4.8 Theorem: The image of fg*- compact fts under a fg*- irresolute function is fg* - compact.

Proof: Let f : X → Y be fg*- irresolute function from a fg*-compact fts X onto a fts Y. Let u = {A_\lambda : \lambda \in \Lambda} be a g* - open cover of Y. Then the collection u' = {f^{-1}(A_\lambda) : \lambda \in \Lambda} is a g* - open cover of X, since f is fg* - irresolute function. Since X is fg* - compact, u' has a finite sub cover say v = {f^{-1}(A_{\lambda_1}), ..., f^{-1}(A_{\lambda_n})}. Then \{A_{\lambda_1}, ..., A_{\lambda_n}\} is a finite sub cover of Y. And hence Y is fg* - compact.

3.4.9 Theorem: Let f : X → Y be fg* - continuous map from a fg*- compact fts X onto fts Y. If Y is fuzzy - T*_{1/2} , then Y is fg* - compact.

Proof: Let X be a fg* - compact fts. Let u = {A_\lambda : \lambda \in \Lambda} be a g* - open cover of Y. Since Y is fuzzy - T*_{1/2} , the cover u is a open cover of X. Then the collection v = {f^{-1}(A_\lambda) : \lambda \in \Lambda} is a g* - open cover of X, since f is fg* - continuous. Also since X is fg* - compact, v has a finite sub cover say v' = {f^{-1}(A_{\lambda_1}), ..., f^{-1}(A_{\lambda_n})}. Then \{A_{\lambda_1}, ..., A_{\lambda_n}\} is a finite sub cover of Y. And hence Y is fg* - compact.

3.4.10 Theorem: Every fg* - compact space is fuzzy compact.

Proof: Let X be a fg* - compact fts. Let u = {A_\lambda : \lambda \in \Lambda} be an open cover of X. Therefore u = {A_\lambda : \lambda \in \Lambda} is a g*- open cover of X. Since X is g* - compact, u has a finite sub cover for X. Hence X is fuzzy compact.

3.4.11 Theorem: If X is fuzzy compact and a fuzzy-T*_{1/2} , then X is fg* - compact.

Proof: Let u = {A_\lambda : \lambda \in \Lambda} be a g*-open cover of X. Since X is fuzzy - T*_{1/2} , then the cover u is an open cover of X. Also, since X is fuzzy compact, u has a finite sub cover. Hence X is fg* - compact.

3.4.12 Theorem: Every fg - compact space is fg* - compact.
Proof: Let $X$ be a $fg$-compact fts. Let $\{A_\lambda : \lambda \in \Lambda\}$ be a $g^*$-open cover of $X$. Since every $g^*$-open fuzzy set is $g$-open. The cover $u$ is a $g$-open cover of $X$. Since $X$ is $fg$-compact, the cover $u$ has a finite sub cover. Hence $X$ is $fg^*$-compact.

3.4.13 Theorem: If $X$ is $fg^*$-compact and a fuzzy - $T_{1/2}$, then $X$ is $fg$-compact.

Proof: Let $u = \{A_\lambda : \lambda \in \Lambda\}$ be a $g$-open cover of $X$. Since $X$ is fuzzy-$T_{1/2}$, the cover $u$ is a $g^*$-open cover of $X$. Since $X$ is $fg^*$-compact, the cover $u$ has a finite sub cover. Hence $X$ is $fg^*$-compact.

3.4.14 Theorem: If $f: X \to Y$ is a strongly $fg^*$-continuous map from a fuzzy compact space $X$ onto fts $Y$. Then $Y$ is $fg^*$-compact.

Proof: Let $u = \{A_\lambda : \lambda \in \Lambda\}$ is a $g^*$-open cover of $Y$. Then $v = \{f^{-1}(A_\lambda) : \lambda \in \Lambda\}$ is an open cover of $X$, since $f$ is strongly $fg^*$-continuous. Since $X$ is $f$-compact, the cover $v$ has a finite sub cover say $v = \{f^{-1}(A_{\lambda_1}),...,f^{-1}(A_{\lambda_n})\}$. Since $f$ is onto then $\{A_{\lambda_1},...,A_{\lambda_n}\}$ is a finite sub cover of $u$ for $Y$. Therefore $Y$ is $fg^*$-compact.

3.4.15 Theorem: If $f: X \to Y$ is completely $fg^*$-continuous map from a nearly fuzzy compact fts $X$ onto fts $Y$. Then $Y$ is $fg^*$-compact.

Proof: Let $u = \{A_\lambda : \lambda \in \Lambda\}$ be any $g^*$-open cover of $Y$. Since $f$ is completely $fg^*$-continuous, then $v = \{f^{-1}(A_\lambda) : \lambda \in \Lambda\}$ is a regular open cover of $X$. Also since $X$ is nearly fuzzy compact, $v$ has a finite sub cover say $\{f^{-1}(A_{\lambda_i}) : i=1,2,...,n\}$. Then $\{A_{\lambda_1},...,A_{\lambda_n}\}$is a finite sub cover of $u$ for $Y$. Hence $Y$ is $fg^*$-compact.

3.4.16 Theorem: A fts $X$ is $fg^*$-compact iff every family of $g^*$-closed fuzzy sets of $X$ having f. i.p. has a non-empty intersection.
Proof: Suppose $X$ is a $fg^*$-compact fts. Let $G = \{G_\lambda : \lambda \in \Lambda\}$ be a family of $g^*$-closed fuzzy sets of $X$ having f.i.p. To show that $\bigwedge_{\lambda \in \Lambda} G_\lambda \neq O$.

Assume to the contrary that $\bigwedge_{\lambda \in \Lambda} G_\lambda = O$. Then $\bigvee_{\lambda \in \Lambda} (1 - G_\lambda) = 1$. The cover $\{1 - G_\lambda : \lambda \in \Lambda\}$ is a $g^*$-open cover of $X$. Since $X$ is $fg^*$-compact, the cover has a finite sub cover for $X$, there exists a finite sub set $\Lambda_0 \subseteq \Lambda$ such that $\bigvee_{\lambda \in \Lambda_0} (1 - G_\lambda) = 1$. And then $\bigwedge_{\lambda \in \Lambda_0} G_\lambda = 0$. Which contradicts the fact that $G$ has f.i.p. Hence $\bigwedge_{\lambda \in \Lambda} G_\lambda \neq O$.

Conversely, assume that every family of $g^*$-closed fuzzy sets of $X$ having f.i.p. has non-empty intersection. Let $G = \{G_\lambda : \lambda \in \Lambda\}$ be a $g^*$-open cover of $X$. Suppose to the contrary that for any finite $\Lambda_0 \subseteq \Lambda$, $\bigvee_{\lambda \in \Lambda_0} G_\lambda = 1$. And so $\bigwedge_{\lambda \in \Lambda_0} (1 - G_\lambda) = 0$. The family $\{1 - G_\lambda : \lambda \in \Lambda\}$ of $g^*$-closed fuzzy sets has f.i.p. And so by assumption $\bigwedge_{\lambda \in \Lambda} (1 - G_\lambda) \neq O$.

Which implies that $\bigvee_{\lambda \in \Lambda} G_\lambda = 1$. This contradicts that $G$ is a cover for $X$. And hence $G$ has a finite sub cover. Hence $X$ is $fg^*$-compact.

3.4.17 Theorem: The image of a $fg^*$-compact fts under a strongly $fg^*$-continuous function is $fg^*$-compact fts.

Proof: The routine proof is omitted.

3.4.18 Definition: A fts $X$ is said to be countably $g^*$-compact if every countable $g^*$-open cover of $X$ has a finite sub cover.

3.4.19 Theorem: Every countably $g^*$-compact fts is countably compact fts.

Proof: Follows from the two concepts.
3.4.20 Theorem: If $X$ is countably compact fts and a fuzzy - $T_{1/2}^*$ space, then $X$ is countably $g^*$-compact.

Proof: The easy verification is omitted.

3.4.21 Theorem: Every $g^*$-compact fts is countably $g^*$-compact fts.

Proof: Follows from the two definitions.

3.4.22 Theorem: A $g^*$-closed crisp subset of a countably $g^*$-compact fts is countably $g^*$-compact.

Proof: Let $Y$ be a $g^*$-closed crisp subspace of a countably $g^*$-compact fts $X$. To prove that $Y$ is countably $g^*$-compact. Let $u = \{U_\lambda : \lambda \in \Lambda\}$ be any countable $g^*$-open cover of $Y$. Then $v = \{U_\lambda : \lambda \in \Lambda\} \cup \{1 - Y\}$ is a countable $g^*$-open cover of $X$. Let $x \in X$ then $x \in Y$ or $x \notin Y$. If $x \notin Y$, then $x \in 1 - Y$. That is $(1 - Y)(x) = 1$. Where $1 - Y \in \{U_\lambda : \lambda \in \Lambda\} \cup \{1 - Y\}$.

Suppose $x \in Y$. Since $\{U_\lambda : \lambda \in \Lambda\}$ is countable $g^*$-open cover of $Y$, there exists $U_{\lambda_0}$ such that $U_{\lambda_0}(x) = 1$. Thus $v = \{U_\lambda : \lambda \in \Lambda\} \cup \{1 - Y\}$ is a countable $g^*$-open cover of $X$. Since $X$ is countably $g^*$-compact fts, $v$ has a finite sub cover say $v' = \{U_{\lambda_1}, U_{\lambda_2}, \ldots, U_{\lambda_k}\} \cup \{1 - Y\}$ for $X$. Then the family $u' = \{U_{\lambda_1}, U_{\lambda_2}, \ldots, U_{\lambda_k}\}$ is a finite sub cover of $u$ for $Y$. Let $x \in Y$, then $x \in X$ and $x \notin 1 - Y$, since $v'$ is a finite sub cover of $X$, there exists $U_{\lambda_i}$ ( $i = 1, \ldots, k$) such that $U_{\lambda_i}(x) = 1$. It follows that $u'$ is a finite $g^*$-open cover of $Y$. Thus every countable $g^*$-open cover $u$ of $Y$ has a finite sub cover $u$. Hence $Y$ is a countably $g^*$-compact fts.

3.4.23 Theorem: The image of a countably $g^*$-compact fts under a $fg^*$-continuous map is countably fuzzy compact.

Proof: Let $f : X \rightarrow Y$ be $fg^*$-continuous map from a countably $g^*$-compact fts $X$ onto fts $Y$. Let $u = \{U_\lambda : \lambda \in \Lambda\}$ be a countable open cover of $Y$ by open fuzzy sets in $Y$. Then the collection $v = \{f^{-1}(U_\lambda) : \lambda \in \Lambda\}$ is countable $g^*$-open cover of $X$, since $f$ is $fg^*$-continuous function. Since $X$ is countably $g^*$-compact, $v$ has a finite
sub cover say \{f^{-1}(U_{\lambda_1}), f^{-1}(U_{\lambda_2}), \ldots, f^{-1}(U_{\lambda_n})\}. Then \{U_{\lambda_1}, U_{\lambda_2}, \ldots, U_{\lambda_n}\} is a finite sub cover of Y. And hence Y is countably fuzzy compact.

3.4.24 Theorem: The image of a countably g*-compact fts under fg*- irresolute map is countably g*- compact.

Proof: Let \( f : X \rightarrow Y \) be fg*- irresolute map from a countably g*- compact fts X onto fts Y. Let \( u = \{ U_{\lambda} : \lambda \in \Lambda \} \) be a countable g*- open cover for Y. Then \( v = \{ f^{-1}(U_{\lambda}) : \lambda \in \Lambda \} \) is countable g* - open cover for X, since \( f \) is fg*- irresolute function. Again since X is countably g*-compact, \( v \) has a finite sub cover say \{f^{-1}(U_{\lambda_i}) : i = 1, \ldots, k\}.

Therefore \( X = \bigvee_{i=1}^{k} f^{-1}(U_{\lambda_i}) \), implies

\[
Y = \bigvee_{i=1}^{k} U_{\lambda_i}.
\]

Therefore \( \{U_{\lambda_i} : i = 1, 2, \ldots, k\} \) is finite sub cover of \( u \) for Y. Hence Y is countably g*-compact.

3.4.25 Theorem: Let \( f : X \rightarrow Y \) be fg*- continuous map from a countably g* - compact fts X onto Y. If Y is fuzzy-T*_{1/2} space, then Y is countably g* - compact.

Proof: Follows easily.

3.4.26 Theorem: If \( f : X \rightarrow Y \) is strongly fg*-continuous map from a countably compact fts X onto fts Y. Then Y is countably g* - compact.

Proof: The routine proof is omitted.

3.4.27 Theorem: A fts X is countable g* - compact iff every countable family of g* - closed fuzzy sets of X having finite intersection property has a non-empty intersection.

Proof: The routine proof is not included.
3.4.28 **Definition:** A fts \((X, T)\) is said to be \(g^*\)-Lindelöf if every \(g^*\)-open cover of \(X\) has a countable sub cover.

This concept can be investigated and several results analogous to \(g\)-compact and compact fts can be obtained.

3.4.29 **Definition:** A fts \((X, T)\) is said to be \(g^*\)-regular fts if for each \(x \in X\) and a \(g^*\)-closed fuzzy set \(A\) with \(A(x) = 0\), there exist open fuzzy sets \(G, H\) such that \(G(x) = 1\), \(A < H\) and \(G < 1 - H\).

3.4.30 **Theorem:** Every \(g^*\)-regular fts is regular fts.

**Proof:** Follows from the two concepts.

3.4.31 **Example:** Let \(X = \{a, b, c\}\). Fuzzy sets \(A, B, C\) and \(D\) be defined as follows: 
\[
A = \{(a, 1), (b, 0), (c, 0)\}, \quad B = \{(a, 0), (b, 1), (c, 1)\}, \\
C = \{(a, 0), (b, 1), (c, 0)\} \quad \text{and} \quad D = \{(a, 0), (b, 0), (c, 1)\}.
\]

\((X, T)\) is fts with topology \(T = \{0, 1, A, B\}\). Then \((X, T)\) is \(g^*\)-regular fts, \(a \in X\) and \(B\) is \(g^*\)-closed with \(B(a) = 0\), then \(A\) and \(B\) are open fuzzy sets such that \(A(a) = 1\), \(B \leq B\) and \(A \leq 1 - B\).

3.4.32 **Theorem:** Every \(b\)-regular fts is \(g^*\)-regular fts.

**Proof:** Let \(X\) be a \(b\)-regular fts. Let \(x \in X\) and \(A\) be \(fg^*\)-closed in \(X\) with \(A(x) = 0\). Then \(A\) is \(b\)-closed fuzzy set in \(X\). Since \(X\) is \(b\)-regular fts, there exists open fuzzy sets \(G, H\) such that \(G(x) = 1\), \(A \leq H\) and \(G \leq 1 - H\). Hence \(X\) is \(g^*\)-regular fts.

The converse of the above theorem need not be true as seen from the following example.

3.4.33 **Example:** In the example 3.4.31, the fts \((X, T)\) is \(g^*\)-regular fts but not \(b\)-regular fts as there does not exist open fuzzy sets say \(G, H\) with \(G \leq 1 - H\) which contain \(c \in X\) and \(a\)-closed fuzzy set \(C\).
3.4.34 **Theorem:** If a fts $X$ is $g^*$-regular and a $fb$-T space, then $X$ is $b$-regular fuzzy topological space.

**Proof:** Let $X$ be $g^*$-regular fts. Let $x \in X$ and $A$ be $b$-closed fuzzy set in $X$ with $A(x) = 0$. Since $X$ is fuzzy $b$-T space, $b$-closed fuzzy set $A$ is $g^*$-closed in $X$. So we have $x \in X$ and $A$ be $g^*$-closed fuzzy set in $X$ with $A(x) = 0$. Then, since $X$ is $g^*$-regular, there exists open fuzzy sets $G$, $H$ in $X$ such that $G(x) = 1$, $A \leq H$ and $G \leq 1 - H$. Hence $X$ is $b$-regular fts.

3.4.35 **Theorem:** If a fts $X$ is regular and a fuzzy-$T^*_{1/2}$, then $X$ is $g^*$-regular fts.

**Proof:** Let $X$ be regular fts. Let $x \in X$ and $A$ be $g^*$-closed fuzzy set in $X$ with $A(x) = 0$. Then $A$ is closed fuzzy set in $X$ since $X$ is fuzzy-$T^*_{1/2}$. Again since $X$ is regular fts, there exists open fuzzy sets $G$, $H$ such that $G(x) = 1$, $A \leq H$. Hence $X$ is $g^*$-regular fts.

3.4.36 **Theorem:** The following three properties are equivalent.

1) $X$ is $g^*$-regular fts.

2) For each $x \in X$ and a $g^*$-open fuzzy set $U$ with $U(x) = 1$, there exists an open fuzzy set $V$ with $V(x) = 1$, such that $V \leq \overline{V} \leq U$.

3) For each $x \in X$ and a $g^*$-closed fuzzy set $A$ with $A(x) = 0$. There is an open fuzzy set $V$ with $V(x) = 1$, such that $A \leq 1 - V$ or $\overline{V} \leq 1 - A$.

**Proof:** (1) $\Rightarrow$ (2): Let $X$ be $g^*$-regular fts. Let $x \in X$ and a $g^*$-open fuzzy set $U$ with $U(x) = 1$. Then we have $x \in X$ and $g^*$-closed fuzzy set $1 - U$ with $(1 - U)(x) = 0$. By hypothesis, there exists open fuzzy sets $W$ and $V$ such that $V(x) = 1$ and $1 - U \leq W$ and $V \leq 1 - W$. Now $V \leq 1 - W$ implies that $V \leq \overline{1 - W} = 1 - W \leq U$. Therefore $V \leq \overline{V} \leq U$. 
Let $x \in X$ and a $g^*$-closed fuzzy set $A$ with $A(x) = 0$. We have $x \in X$ and a $g^*$-open fuzzy set $1 - A$ with $(1 - A)(x) = 1$. By hypothesis, there exists an open fuzzy set $V$ with $V(x) = 1$, such that $V \leq \overline{V} \leq 1 - A$.

(2) $\Rightarrow$ (3): Let $x \in X$ and a $g^*$-closed fuzzy set $A$ with $A(x) = 0$. By hypothesis, there exists an open fuzzy set $V$ with $V(x) = 1$ and $V \leq 1 - A$. Let $B = 1 - \overline{V}$. Then $B$ is open fuzzy set and $\overline{V} \leq 1 - A$ implies $A \leq 1 - \overline{V} = B$. Also $V \leq \overline{V} = 1 - B$. Hence $A \leq B$ and $V \leq 1 - B$. Hence $X$ is $g^*$-regular fts.

(3) $\Rightarrow$ (1): Let $x \in X$ and $A$ be $g^*$-closed fuzzy set with $A(x) = 0$. By hypothesis, there exists an open fuzzy set $V$ with $V(x) = 1$ and $V \leq 1 - A$. Let $B = 1 - \overline{V}$. Then $B$ is open fuzzy set and $\overline{V} \leq 1 - A$ implies $A \leq 1 - \overline{V} = B$. Also $V \leq \overline{V} = 1 - B$. Hence $A \leq B$ and $V \leq 1 - B$. Hence $X$ is $g^*$-regular fts.

### 3.4.37 Theorem:
A fuzzy subspace of a $g^*$-regular fts is $g^*$-regular.

**Proof:** Let $Y$ be a fuzzy subspace of $g^*$-regular fts $X$. Let $x \in Y$ and $A$ be $g^*$-closed fuzzy set in $Y$ with $A(x) = 0$. Then there is a closed fuzzy set and so $g^*$-closed fuzzy set $B$ of $X$ such that $A = B \wedge Y$ and $B(x) = 0$. Since $X$ is $g^*$-regular fts, there exist open fuzzy sets $G, H$ such that $G(x) = 1$, $B \leq H$ and $G \leq 1 - H$. Then $G \wedge Y$ and $H \wedge Y$ are open fuzzy sets such that $(G \wedge Y)(x) = 1$, $A \leq H \wedge Y$ and $G \wedge Y \leq 1 - (H \wedge Y)$. Hence $Y$ is $g^*$-regular.

### 3.4.38 Theorem:
If $f : X \rightarrow Y$ is an open, $fg^*$-irresolute bijection and $X$ is $g^*$-regular fts then $Y$ is $g^*$-regular fts.

**Proof:** Let $y \in Y$ and $A$ be $g^*$-closed fuzzy set of $Y$ with $A(y) = 0$. Since $f$ is $fg^*$-irresolute, $f^{-1}(A)$ is $g^*$-closed fuzzy set in $X$. Put $f(x) = y$, then $(1 - f^{-1}(A))(x) = 1$. Since $X$ is $g^*$-regular fts, there exists open fuzzy sets $G, H$ such that $G(x) = 1$, $f^{-1}(A) \leq H$ and $G \leq 1 - H$. Since $f$ is open and bijective, we have $(f(G))(y) = 1$, $A \leq f(H)$ and $f(G) \leq 1 - f(H)$. This shows that $Y$ is $g^*$-regular fts.
3.4.39 **Definition:** A fts $X$ is said to be $g^*$-normal if for every $g^*$-closed fuzzy set $K$ and $g^*$-open fuzzy set $B$ such that $K \leq B$, there exist a fuzzy set $A$ such that $K \leq A^e \leq \bar{A} \leq B$.

3.4.40 **Theorem:** For a fts $X$ the following statements are equivalent.

1) $X$ is a $g^*$-normal fts.

2) For any two $g^*$-closed fuzzy sets $A$ and $B$ in $X$ such that $A \leq 1 - B$, there exist open fuzzy sets $C$, $D$ such that $A \leq C$, $B \leq D$ and $C \leq 1 - D$.

3) For any two $g^*$-closed fuzzy sets $A$ and $B$ in $X$ such that $A \leq 1 - B$, there is an open fuzzy set $C$ such that $A \leq C$ and $\bar{C} \leq 1 - B$.

4) For any two $g^*$-closed fuzzy sets $A$ and $B$ in $X$ such that $A \leq 1 - B$, there are open fuzzy sets $C$, $D$ such that $A \leq C$, $B \leq D$ and $\bar{C} \leq 1 - D$.

**Proof:**

1) $\Rightarrow$ 2): Let $A$ and $B$ be any two $g^*$-closed fuzzy sets with $A \leq 1 - B$. Then since $1 - B$ is a $g^*$-open fuzzy set, from (1), there exists a fuzzy set $E$ such that $A \leq E^e \leq \bar{E} \leq 1 - B$. Put $C = E^e$ and $D = 1 - \bar{E}$. Then $C$ and $D$ are open fuzzy sets. And then $A \leq C$ and $B \leq 1 - \bar{E} = D$. Also $C = E^e \leq E \leq \bar{E} \leq 1 - D$. Thus $A \leq C$, $B \leq D$ and $C \leq 1 - D$.

2) $\Rightarrow$ 3): Let $A$ and $B$ be $g^*$-closed fuzzy sets with $A \leq 1 - B$. Then from (2), there exist open fuzzy sets $C$, $D$ such that $A \leq C$, $B \leq D$ and $C \leq 1 - D$. Now $\bar{C} \leq 1 - D = 1 - D \leq 1 - B$. Therefore $\bar{C} \leq 1 - B$.

3) $\Rightarrow$ 4): Let $A$ and $B$ be $g^*$-closed fuzzy sets such that $A \leq 1 - B$. Then from (3), there is an open fuzzy set $C$ such that $A \leq C$ and
\( \overline{C} \leq 1 - B \). Now \( \overline{C} \) is closed fuzzy set and so \( g^* \)-closed fuzzy set. \( \overline{C} \) and \( B \) are \( g^* \)-closed fuzzy sets with \( \overline{C} \leq 1 - B \). So \( B \leq 1 - \overline{C} \). Again from (3), there exists a open fuzzy set \( D \) such that \( B \leq D \) and \( \overline{D} \leq 1 - \overline{C} \). So \( \overline{C} \leq 1 - \overline{D} \). Hence \( A \leq C, B \leq D \) and \( \overline{C} \leq 1 - \overline{D} \).

(4) \( \Rightarrow \) (1): Let \( K \) be a \( g^* \)-closed fuzzy set and \( B \) be \( g^* \)-open fuzzy set such that \( K \leq B \), then \( B = 1 - C \), where \( C \) is \( g^* \)-closed fuzzy set. Now \( K \leq 1 - C \), where \( K, C \) are \( g^* \)-closed fuzzy sets. From (4), there exist open fuzzy sets \( D, E \) such that \( K \leq D, C \leq E \) and \( \overline{D} \leq 1 - \overline{E} \). Take \( A = D \), then \( A \) is open fuzzy set and \( K \leq A^* \leq \overline{A} \leq 1 - \overline{E} \leq 1 - C = B \). Therefore \( K \leq A^* \leq \overline{A} \leq B \). Hence \( X \) is \( g^* \)-normal fts. This completes the proof.

3.4.41 Example: In the example 3.4.31, \( X \) is \( g^* \)-normal fts, as the fuzzy sets \( A \) and \( B \) are \( g^* \)-closed fuzzy sets with \( A \leq B \). The fuzzy sets \( A \) and \( B \) are open fuzzy sets such that \( \land A \leq 1 - B \). Hence \( (X, T) \) is \( g^* \)-normal fts.

3.4.42 Theorem: Every \( g^* \)-normal fts is normal fts.
Proof: The proof follows from the two definitions.

3.4.43 Theorem: If \( X \) is normal fts and a fuzzy - \( T^*_{1/2} \), then \( X \) is \( g^* \)-normal fts.
Proof: Let \( K \) be a \( g^* \)-closed fuzzy set and \( B \) is \( g^* \)-open fuzzy set in \( X \) such that \( K \leq B \), since \( X \) is fuzzy - \( T^*_{1/2} \), \( K \) is closed fuzzy set and \( B \) is open fuzzy set such that \( K \leq B \). Again since \( X \) is a normal fts, there exist a fuzzy set \( A \) such that \( K \leq A^* \leq \overline{A} \leq B \). And hence \( X \) is \( g^* \)-normal fts.

3.4.44 Theorem: Every b-normal fts is \( g^* \)-normal fts.
Proof: Let $X$ be a $b$-normal fts. Let $K$ be a $g^*$-closed and $B$ be $g^*$-open fuzzy sets in $X$ such that $K \leq B$, then $K$ is $b$-closed and $B$ is $b$-open fuzzy sets and $K \leq B$. Since $X$ is $b$-normal fts, there exist a fuzzy set $A$ such that $K \leq A^* \leq \overline{A} \leq B$. Hence $X$ is $g^*$-normal fts.

The converse of the above theorem need not be true as seen from the following example.

3.4.45 Example: In the example 3.4.31, $X$ is $g^*$-normal fts. But $X$ is not $b$-normal fts, as the fuzzy sets $C, D$ are $b$-closed fuzzy sets with $C \leq 1 - D$, but there do not exist open fuzzy sets $A$ and $B$ such that $C \leq A, D \leq B$ and $A \leq 1 - B$.

3.4.46 Theorem: If $X$ is $g^*$-normal fts and a fuzzy $b$-space then $X$ is $b$-normal fts.

Proof: Let $K$ be a $b$-closed fuzzy set and $B$ be $b$-open fuzzy set in $X$ such that $K \leq B$, since $X$ is fuzzy $b$-space, $K$ is closed and $B$ is open fuzzy sets in $X$ and so $K$ is $g^*$-closed, $B$ is $g^*$-open fuzzy sets with $K \leq B$. Again since $X$ is $g^*$-normal fts, there exist a fuzzy set $A$ such that $K \leq A^* \leq \overline{A} \leq B$. Hence $X$ is $b$-normal fts.

3.4.47 Theorem: If $f : X \to Y$ is an open, $fg^*$-irresolute bijection and $X$ is $g^*$-normal fts then $Y$ is $g^*$-normal fts.

Proof: Let $A$ and $B$ be $g^*$-closed fuzzy sets of $Y$ with $A \leq 1 - B$. Since $f$ is $fg^*$-irresolute function, $f^{-1}(A)$ and $f^{-1}(B)$ are $g^*$-closed fuzzy sets of $X$ such that $f^{-1}(A) \leq 1 - f^{-1}(B)$. Since $X$ is $g^*$-normal fts, there exists open fuzzy sets $C$ and $D$ such that $f^{-1}(A) \leq C, f^{-1}(B) \leq D$. 

and $C \leq 1 - D$. Since $f$ is open and bijective, we have $A \leq f(C)$, $B \leq f(D)$ and $f(C) \leq 1 - f(D)$. Here $f(C)$ and $f(D)$ are open fuzzy sets in $Y$. This proves that $Y$ is $g^*$-normalfts.

The concepts of fuzzy $g^*-T_0$, fuzzy $g^*-T_1$ and fuzzy $g^*-T_2$ fts can be introduced and studied using the concept of $g^*$-open and $g^*$-closed fuzzy sets.