CHAPTER 2
LOCALLY MOST POWERFUL RANK TESTS FOR THE
TWO-SAMPLE PROBLEM WITH MULTIPLY TYPE II
CENSORED DATA

2.1 Introduction

"Multiple censoring" means that observations within certain intervals (Type I) or observations within certain sample quantiles (Type II) are excluded from measurements. In this thesis, we assume that observations on some subsets of order statistics are not available, that is censoring is of multiple Type II. Since quite often there will not be enough time, manpower and other resources to record the failure time of each subject, this type of censoring scheme is frequently practiced. This happens in the follow up studies in epidemiology, sociology, reliability, etc.

The Type II right censoring scheme for the two-sample problem has been considered by many authors, both for parametric as well as nonparametric models (see for example, Lawless (1982) and Basu (1984)). The problem of parametric estimation based on the multiply Type II censored data has received considerable attention. Early references appear in Cohen (1975) and David (1981), and some more recent references in Balakrishnan et al. (1995).

Relatively a little has been published for the case of multiple Type II censoring in nonparametric inference. Mehrotra, et al. (1977) have obtained the LMP rank test on the lines of Hajek and Sidak (1967) in the case of triple censoring scheme where the data are left and right censored and censored once.
in the middle. They have considered the two-sample problem where the alternative involves a single parameter. Johnson and Mehrotra (1972) have considered single censoring on the right and have obtained the LMP rank tests for location and scale alternatives. They have pointed out how to obtain the asymptotic distribution of the test statistic under the alternative hypothesis using the methods of Pyke and Shorack (1968). They have remarked that the asymptotic distribution of the test statistic can also be obtained by the method of Dupac and Hajek (1969).

In this chapter we have obtained the LMP rank test in the case of multiple Type II censoring scheme and have shown using the Dupac and Hajek (1969) results for the two-sample case, that the asymptotic distribution of the test statistic is normal. These authors use the expected value of the test statistic as centering constant, which is often difficult to calculate in the case of alternative hypothesis. Applying Dupac’s (1970) and Hoeffding’s (1973) approaches a simpler centering constant is obtained under the alternative hypothesis. Exact mean and variance of the test statistic under the null hypothesis are obtained.

Section 2.2 provides the notations and basic results to derive the LMP rank tests based on the multiply Type II censored combined sample. In Section 2.3, the LMP rank test is derived for testing equality of two distribution functions against the general parameter alternative. The modification of these rank tests in the different cases of multiple Type II censoring schemes are also given here. In Section 2.4, exact mean and variance of the test statistic under
the null hypothesis are obtained. Section 2.5 is devoted to asymptotic
distribution of the test statistic under the null and alternative hypotheses. In the
last section the asymptotic mean and variance of the test statistic under the
alternative hypothesis as well as under the null hypothesis are obtained.

2.2 Notation and Basic Results

Let \( X_1, \ldots, X_m \) and \( Y_1, \ldots, Y_n \) denote two independent random samples from
two absolutely continuous distribution functions (df's) \( F(x) \) and \( G(x) \),
respectively, with the corresponding densities \( f(x) \) and \( g(x) \). Our objective is to
test the null hypothesis

\[
H_0: F(x) = G(x) \quad \text{for all } x
\]

against the general parameter alternative hypothesis

\[
H_1: G(x) = F(x; \theta) \quad \theta > 0,
\]

where \( \theta \) is some index parameter. Under \( H_0 \), we have \( \theta = 0 \) and we write
\( F(x; 0) = F(x) \). Let \( W_1 \leq \ldots \leq W_N \), with \( N = m + n \) denote the order statistics
of the combined sample and let \( (Z_1, \ldots, Z_N) \) be the random variables (rv's)
defined by

\[
Z_i = \begin{cases} 
1 & \text{if } W_i \text{ is a } Y \text{ observation} \\
0 & \text{if } W_i \text{ is an } X \text{ observation,}
\end{cases}
\]

for \( i = 1, \ldots, N \). Then \( Z_i \) are dependent Bernoulli rv's.

Multiple Type II censoring is a generalization of singly or doubly
Type II censoring. There are several versions of the multiple Type II censoring
schemes considered, particularly in parametric inference. Here we consider the following scheme for the two-sample problem.

Let \( t = 2r - 1 \) \((r \geq 2)\) be the total number of subsets which constitute the whole set of \(N\) order statistics of the combined sample. Here we assume that the combined sample is multiply Type II censored and only the following \((r-1)\) subsets of order statistics are available:

\[
\{W_{k_{2i-1}+1}, W_{k_{2i-1}+2}, \ldots, W_{k_{2i}}\} \quad i = 1, 2, \ldots, r-1. \tag{2.2.4}
\]

Further, the number of \(X\)'s and \(Y\)'s in each of the \((2r-1)\) subsets, irrespective of its being censored or observed, are assumed to be known and these numbers in the \(i\)-th subset are denoted by \(N_i = m_i + n_i = k_i - k_{i-1}, \quad i = 1, 2, \ldots, 2r - 1,\) with \(k_0 = 0\) and \(k_{2r-1} = N\). Then

\[
n_i = \sum_{j=k_{i-1}+1}^{k_i} Z_j \quad \text{and} \quad m_i = \sum_{j=k_{i-1}+1}^{k_i} (1 - Z_j).
\]

Let

\[
Z = (Z_{k_1+1}, \ldots, Z_{k_2}, Z_{k_3+1}, \ldots, Z_{k_4}, \ldots, Z_{k_{2r-3}+1}, \ldots, Z_{k_{2r-2}})
\]

be the vector of rank order statistics corresponding to the vector of observed order statistics in the combined sample,

\[
\underline{w} = (W_{k_1+1}, \ldots, W_{k_2}, W_{k_3+1}, \ldots, W_{k_4}, \ldots, W_{k_{2r-3}+1}, \ldots, W_{k_{2r-2}}).
\]

Following Rao, et al. (1960), the joint probability distribution of \(z\) and \(n = (n_1, \ldots, n_{2r-1})\) is given by

\[
P_0 (z, n) = C (m, n) \int_{\mathbb{R}^2} \left\{ \prod_{i=1}^{r} \prod_{j=k_{i-1}+1}^{k_i} f^{i-1}(w_j) g^{i}(w_j) \right\} \left\{ \prod_{i=1}^{r} \left[ F(w_{k_{i-1}+1}) - F(w_{k_{i+1}-1}) \right]^{m_i-n_i} \left[ G(w_{k_{i-1}+1}) - G(w_{k_{i+1}-1}) \right]^{n_i-m_i} \right\} dw,
\]

\[
\tag{2.2.5}
\]
where \( G(x) = F(x; \theta) \), \( g(x) = f(x; \theta) \),

\[
C(m, n) = \frac{m! \cdot n!}{\prod_{i=1}^{r} m_{2i-1}! \cdot n_{2i-1}!}
\]

and

\[
A_w = \{ w: -\infty < w_{k_1+1} < \ldots < w_{k_2} < w_{k_3+1} < \ldots < w_{k_{2r-3}+1} < \ldots < w_{k_{2r-2}} < \infty \}
\]

with \( k_0 = 0 \), \( w_0 = -\infty \), \( k_{2r-1} = N \) and \( w_{N+1} = \infty \).

Under the null hypothesis \( H_0 \),

\[
P_0 (z, n) = C(m, n) \int_{A_w} \left\{ \prod_{i=1}^{r} \prod_{j=k_{2i+1}+1}^{k_{2i}} f(w_j) \right\} \left\{ \prod_{i=1}^{r} \left[ F(w_{k_{2i+1}}) - F(w_{k_{2i-1}}) \right] \right\}^{N_{2i-1}} \, dw.
\]

(2.2.6)

Noting that

\[
\int_{a < x_1 < \ldots < x_n < b} f(x_1) \ldots f(x_n) \, dx_1 \ldots dx_n = \frac{1}{n!} [F(b) - F(a)]^n,
\]

and integrating with respect to \( w_{k_{2i+1}+1}, \ldots, w_{k_{2i}} \) over the interval

\((w_{k_{2i+1}+1}, w_{k_{2i}}), \quad i = 1, 2, \ldots, r-1\), (2.2.6) reduces to

\[
P_0 (z, n) = \frac{C(m, n)}{\prod_{i=1}^{r-1} (N_{2i} - 2)!} \int_{-\infty < w_{k_{2i+1}+1} < \ldots < w_{k_{2i+1}} < \ldots < w_{k_{2i+1}+1} < \ldots < w_{k_{2i-1}} < \ldots < w_{k_{2i-1}} < \infty} \left\{ \prod_{i=1}^{r-1} \left[ F(w_{k_{2i+1}}) - F(w_{k_{2i-1}}) \right] \right\}^{N_{2i-1}} \, dw_{k_{2i+1}} \ldots dw_{k_{2i-1}}
\]

(2.2.7)

which follows by noting that the above integrand reduces to a constant multiple of an ordered Dirichlet distribution by making the probability integral transformation

\[
F(w_{k_{2i+1}+1}) = u_{2i-1}, \quad i = 1, \ldots, r-1
\]
so that
\[0 < u_1 < u_2 < \ldots < u_{2r-2} = 1\] (see Wilks (1962) p. 182).

We suppose that the following regularity conditions are satisfied to derive LMP rank test for testing \(H_0\) against general parameter alternative hypothesis and some particular cases of general parameter alternative hypothesis.

**Condition A_1:** Let \(\Theta\) be an open interval containing 0. A family of densities \(\{f(x; \theta) : \theta \in \Theta\}\) will be said to satisfy condition A_1 if

(i) \(f(x; \theta)\) is absolutely continuous in \(\theta\) for almost every \(x\);

(ii) the limit \(f^*(x; \theta) = \frac{\partial f^*}{\partial \theta}\) exists for almost every \(x\);

(iii) \(\lim_{\theta \to 0} \int f^*(x; \theta) |dx| = \int f^*(x; 0) |dx| < \infty\) holds, with \(f^*(x; \theta)\) denoting the partial derivative of \(f(x; \theta)\) with respect to \(\theta\).

It may be noted that under these conditions and an argument similar to Hajek and Sidak ((1967) pp. 71-72), the df \(F(x; \theta)\) has partial derivative w. r. t. \(\theta\) which we denote by \(F^*(x; \theta)\). Also

\[\frac{1}{\theta} [F_0(x) - F(x)] = \int \frac{1}{\theta} (f_0(u) - f(u)) du \to \int f^*(u) du = F^*(x)\] as \(\theta \to 0\).

We use the following lemma due to Mehrotra, et al. (1977) about

\[\frac{F^*(x; \theta)}{F(x; \theta)}, \quad \frac{F^*(x; \theta)}{1 - F(x; \theta)} \quad \text{and} \quad \frac{F^*(x; \theta) - F^*(y; \theta)}{F(x; \theta) - F(y; \theta)}\]

for deriving the LMP rank tests. They have called these ratios collectively the extended hazard rates.
Lemma 2.2.1. Let the family of densities \( \{f(x; \theta) : \theta \in \Theta\} \), satisfy condition A1 given above then

(i) \((a-1) \sum_{i=1}^{l} E \left[ \frac{f^*(W_i; \theta)}{f(W_i; \theta)} \right] = \sum_{i=1}^{l} E \left[ \frac{f^*(W_i; \theta)}{f(W_i; \theta)} \right], \quad a > 1;\)

(ii) \((N-b) \sum_{i=b+1}^{N} E \left[ \frac{f^*(W_i; \theta)}{1-F(W_i; \theta)} \right] = - \sum_{i=b+1}^{N} E \left[ \frac{f^*(W_i; \theta)}{f(W_i; \theta)} \right], \quad b < N;\)

(iii) \((b-a-1) \sum_{i=a+1}^{b-1} E \left[ \frac{f^*(W_i; \theta) - f^*(W_i; \theta)}{F(W_i; \theta) - F(W_i; \theta)} \right] = \sum_{i=a+1}^{b-1} E \left[ \frac{f^*(W_i; \theta)}{f(W_i; \theta)} \right], \quad a < b-1,\)

where \(W_1 \leq W_2 \leq \ldots \leq W_N\) are the order statistics of a random sample of size \(N\) form the df \(F(x; \theta)\).

Proof. The proof of the result (iii) is given in Mehrotra, et al. (1977). The proofs of the results (i) and (ii) follow by proceeding on the lines of the proof of (iii).

2.3 The Locally Most Powerful Rank Tests

In this section, we derive the LMP rank test for the two-sample hypothesis testing problem for our general censoring scheme, proceeding on the lines of Mehrotra et al. (1977), who have considered the case \(t=5\).

Theorem 2.3.1. Let the family of densities \( \{f(x; \theta) : \theta \in \Theta\} \) satisfy condition A1. Then the test with critical region

\[
T(z, n) = \sum_{i=1}^{c} \sum_{j=1}^{k_i} a_N(j, f) z_j + \sum_{i=1}^{c} \left( \frac{n_{2i-1}}{N_{2i-1}} \right) \sum_{j=1}^{k_i} a_N(j, f) \geq \text{constant, (2.3.1)}
\]

where

\[
a_N(j, f) = E \left[ \frac{f^*(W_j)}{f(W_j)} \right], \quad (2.3.2)
\]
with $f^*(w_j) = \left[ \frac{\partial f(w_j; \theta)}{\partial \theta} \right]_{\theta=0}$, is the LMP rank test for testing $H_0$ against the alternative $H_1$ given by (2.2.2), based on multiply Type II censored data.

**Proof.** The joint probability distribution of $(z, n)$ is given by (2.2.5) with $G(x)$ written for $F(x; \theta)$ for the sake of notational simplicity. Under the null hypothesis $H_0$: $F(x; \theta) = F(x)$ or $\theta = 0$, and $P_0(z, n)$ is given by (2.2.7). The probability distributions (2.2.5) and (2.2.7) are written, respectively, as $P_{F,G}$ and $P_{F,F}$. Let

$$J_1 = \{(k_{2i-1} + 1, \ldots, k_{2i}), i = 1, \ldots, r-1\},$$

$$J_2 = \{k_1, k_2 + 1, k_3 + 1, \ldots, k_2r - 2 + 1\} \quad \text{and}$$

$$J = J_1 \cup J_2.$$

Let

$$\alpha_i = f^{k_{2i}}(w_i) g^{k_{2i}}(w_i), \quad i \in J,$$

$$\alpha_{k_{2i}+1} = F(w_{k_{2i}+1})^{m_i} (G(w_{k_{2i}+1}))^{n_i},$$

$$\alpha_{k_{2i} + 1} = \left[ F(w_{k_{2i}+1}) - F(w_{k_i}) \right]^{m_{2i}} \left[ G(w_{k_{2i}+1}) - G(w_{k_i}) \right]^{n_{2i}}, \quad i = 1, 2, \ldots, r-1.$$

Also let

$$\beta_i = f(w_i), \quad i \in J,$$

$$\beta_{k_i} = (F(w_{k_i+1}))^{N_i},$$

$$\beta_{k_{2i}+1} = \left[ F(w_{k_{2i}+1}) - F(w_{k_i}) \right]^{N_{2i}}, \quad i = 1, 2, \ldots, r-1.$$

Using the identity

$$\prod_{i \in J} \alpha_i - \prod_{i \in J} \beta_i = \sum_{i \in J} (\alpha_i - \beta_i) \prod_{j < i} \alpha_j \prod_{j > i} \beta_j,$$

the difference $(P_{F,G} - P_{F,F})$ can be expressed as
\[ P_{F,G} - P_{F,F} = C(m, n) \int_{x} \cdots \int \left( \prod_{j \in J} \alpha_j - \prod_{j \in J} \beta_j \right) \, dw \]

\[ = \theta \sum_{i \in J} Q_i^* + \theta \sum_{i \in J} (Q_i - Q_i^*) , \]

where

\[ Q_i = \frac{C(m, n)}{\theta} \int_{\mathbf{x} \in X} \cdots \int \left[ (\alpha_i - \beta_i) \prod_{j \in J, j \neq i} \alpha_j \prod_{j \in J} \beta_j \right] \, dw \]

\[ Q_i^* = \frac{C(m, n)}{\theta} \int_{\mathbf{x} \in X} \cdots \int \left[ (\alpha_i - \beta_i) \prod_{j \in J} \beta_j \right] \, dw \]  \hspace{1cm} (2.3.3)

In the following it is shown that \( Q_i^* \) converges to an appropriate term of the test statistic, whereas \( (Q_i - Q_i^*) \to 0 \) for each \( i \in J \) as \( \theta \to 0 \).

First consider \( i \in J_i \) and write

\[ g_{x_i} f^{1-x_i} - f = z_i \left( \frac{g}{f} - 1 \right) f. \]

Then after integrating out all other variables except \( w_i \) we get

\[ Q_i^* = C^* z_i \int_{\theta} \left[ \frac{g(w)}{f(w)} - 1 \right] \, dN_i(w) , \]  \hspace{1cm} (2.3.4)

where

\[ C^* = \frac{(i-1)(N-i)!(m!n!)}{N!} \prod_{i=1}^{r} \left( m_{2i-1} + n_{2i-1} \right) \]

and \( f_{N_i} \) is the density of \( w_i \) in a sample of size \( N \) from \( F \). By condition \( A_1 \) (ii), the pointwise limit of the integrand in (2.3.4) is

\[ \left( \frac{f^*(w)}{f(w)} \right) f_{N_i}(w) . \]
Further,
\[
\int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \frac{g(w) - f(w)}{f(w)} f_{N_1}(w) dw \right\}_0 \leq \frac{1}{\theta} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \frac{f^*(w;u)}{f(w;u)} du \right\}_0 dw
\]

So, by condition A₁ (iii)
\[
\limsup_{\theta \to \infty} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \frac{g(w) - f(w)}{f(w)} f_{N_1}(w) dw \right\}_0 \leq \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \frac{f^*(w)}{f(w)} f_{N_4}(w) dw \right\}_0
\]

Thus by Theorem II. 4.2 of Hajek and Sidak (1967, p.64),
\[
Q^*_i \to \mathcal{C}^* z_i E \left[ \frac{f^*(W_i)}{f(W_i)} \right].
\]

Next consider
\[
Q'_{k_{2i+1}} = \frac{C(m,n)}{\theta} \int_{\mathbb{R}^2} \left\{ \int_{\mathbb{R}} \left[ G(y) - G(x) \right]^{n_{k_{2i}+1}} - [F(y) - F(x)]^{n_{k_{2i}}} \right\} \left[ F(y) - F(x) \right]^{n_{k_{2i}+1}} f_{N_{k_{2i}+1}}(x,y) dx dy.
\]

After integrating out all other variables except \( W_{k_i} \) and \( W_{k_{2i}+1} \) we get
\[
Q'_{k_{2i+1}} = \frac{C^{**}}{\theta} \int_{\mathbb{R}^2} \left\{ \left[ G(y) - G(x) \right]^{n_{k_{2i}+1}} - [F(y) - F(x)]^{n_{k_{2i}}} \right\} \left[ F(y) - F(x) \right]^{n_{k_{2i}+1}} f_{N_{k_{2i}+1}}(x,y) dx dy, \quad (2.3.5)
\]

where
\[
C^{**} = \frac{C(m,n)}{N!} \left( (k_{2i} - 1)! (k_{2i+1} - k_{2i})! (N - 1 - k_{2i+1})! \right).
\]

Using the identity
\[
\frac{u^s - v^s}{u - v} = \sum_{j=0}^{s-1} u^{s-j-1} v^j
\]

and noting that
\[
\frac{1}{\theta} \left\{ \left[ G(y) - G(x) \right] - [F(y) - F(x)] \right\} = \frac{1}{\theta} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f^*(z;u) du dz \to \int_{-\infty}^{\infty} f^*(z) dz \quad \text{as} \quad \theta \to 0.
\]

\[
20
\]
the pointwise limit of the integrand in (2.3.5) is given by

\[ C^{* *}_{n,2i} \left[ F(y) - F(x) \right]^{n_{2i+1}} \int a(z) \, dz. \]

Also from (2.3.5)

\[ \left| Q_{k,n+1}^* \right| \leq n_{2i+1} C^{* *}_{n,2i} \left( \frac{1}{\theta} \int_0^\theta \lambda(u) \, du \right), \]

where

\[ \lambda(u) = \int \int_{xy} \left| \int_a^x f^*(z;u) \, dz \right| \left\{ \max([G(y) - G(x)], [F(y) - F(x)]) \right\} \]

\[ \left[ F(y) - F(x) \right]^{n_{2i+1}} f_{k_{n+1},k_{n+1}}(x,y) \, dx \, dy. \]

By condition A_1 (iii) it follows that

\[ \limsup_{\theta \to 0} \left| Q_{k,n+1}^* \right| \leq n_{2i+1} C \lambda(0), \]

where

\[ \lambda(0) = \lim_{\theta \to 0} \frac{1}{\theta} \int_0^\theta \lambda(u) \, du \]

\[ = \mathbb{E} \left[ \frac{F^*(W_{k,n+1}) - F^*(W_{k,n+1})}{F(W_{k,n+1}) - F(W_{k,n+1})} \right]. \]

Thus by Theorem II. 4.2 of Hajek and Sidak (1967. P.64) and by Lemma 2.2.1 we get

\[ \lim_{\theta \to 0} Q_{k,n,i}^* = \sum_{j=1}^{k} a_N(j,i) \quad i = 1, \ldots, r - 1. \]

In a similar manner it can be shown that

\[ \lim_{\theta \to 0} Q_{k,i}^* = \frac{1}{k} \sum_{j=1}^{k} a_N(j,i). \]
Next we prove below that under the assumptions of the theorem

\[ Q_i - Q_i^* \to 0 \text{ for } i \in J \text{ as } \theta \to 0. \]

After writing the expressions in a suitable form we will apply Scheffe's Theorem (see Rao (1967), p.104), which states that if \( p_n(x) \) is the pdf of a random vector \( X_n \) and \( p_n(x) \) converges to \( p(x) \), a pdf as \( n \to \infty \), then for any Borel measurable set \( A \),

\[
\int_A |p_n(x) - p(x)| \, dx \to 0 \quad \text{as } n \to \infty.
\]

We observe that \( G_n(x) \) and \( [G(y) - G(x)]^n \) can, alternatively, be written as

\[
n! \int_{B_1(x)} \prod_{i=1}^n g(x_i) \, dx \quad \text{and} \quad n! \int_{B_2(x,y)} \prod_{i=1}^n g(x_i) \, dx,
\]

where

\[
B_1(x) = (-\infty < x_1, \ldots, < x_{n_1} < x) \text{ and } B_2(x,y) = (x < x_1 < \ldots < x_{n_2} < y).
\]

Similar expressions hold for \( F_n(x) \) and \( [F(y) - F(x)]^n \). Now

\[
Q_i - Q_i^* = \frac{C(m,n)}{\theta} \int_{A^2} \left[ \left( \prod_{j=0}^{\theta} (\alpha_j - \beta_j) \right) \prod_{j=0}^{\theta} (\beta_j) \right] \, dw.
\]

For \( i = k_{2i} + 1 \), we have

\[
\alpha_{k_{2i} + 1} - \beta_{k_{2i} + 1} = \left[ F(w_{k_{2i} + 1}) - F(w_{k_{2i}}) \right]^{m_{2i+1}}
\]

\[
\left\{ \left[ G(w_{k_{2i} + 1}) - G(w_{k_{2i}}) \right]^{m_{2i+1}} - \left[ F(w_{k_{2i} + 1}) - F(w_{k_{2i}}) \right]^{m_{2i+1}} \right\}
\]

\[
\prod_{j=k_{2i} + 1}^{k_{2i+1}} \prod_{j=k_{2i} + i + 1}^{k_{2i+1}} \left[ f^{-1-2j}(w_j)g^{2j}(w_j) \right] \prod_{j=i+1}^{m_{2i+1}} \left[ f^{-1-2j}(w_j)g^{2j}(w_j) \right]
\]

\[
\prod_{j=k_{2i} + 1}^{k_{2i+1}} \prod_{j=k_{2i} + i + 1}^{k_{2i+1}} \left[ f^{-1-2j}(w_j)g^{2j}(w_j) \right] \prod_{j=i+1}^{m_{2i+1}} \left[ f^{-1-2j}(w_j)g^{2j}(w_j) \right]
\]

\[
\prod_{j=k_{2i} + 1}^{k_{2i+1}} \prod_{j=k_{2i} + i + 1}^{k_{2i+1}} \left[ f^{-1-2j}(w_j)g^{2j}(w_j) \right] \prod_{j=i+1}^{m_{2i+1}} \left[ f^{-1-2j}(w_j)g^{2j}(w_j) \right]
\]

\[
\prod_{j=k_{2i} + 1}^{k_{2i+1}} \prod_{j=k_{2i} + i + 1}^{k_{2i+1}} \left[ f^{-1-2j}(w_j)g^{2j}(w_j) \right] \prod_{j=i+1}^{m_{2i+1}} \left[ f^{-1-2j}(w_j)g^{2j}(w_j) \right]
\]

\[
\prod_{j=k_{2i} + 1}^{k_{2i+1}} \prod_{j=k_{2i} + i + 1}^{k_{2i+1}} \left[ f^{-1-2j}(w_j)g^{2j}(w_j) \right] \prod_{j=i+1}^{m_{2i+1}} \left[ f^{-1-2j}(w_j)g^{2j}(w_j) \right]
\]

\[
\prod_{j=k_{2i} + 1}^{k_{2i+1}} \prod_{j=k_{2i} + i + 1}^{k_{2i+1}} \left[ f^{-1-2j}(w_j)g^{2j}(w_j) \right] \prod_{j=i+1}^{m_{2i+1}} \left[ f^{-1-2j}(w_j)g^{2j}(w_j) \right]
\]

\[
\prod_{j=k_{2i} + 1}^{k_{2i+1}} \prod_{j=k_{2i} + i + 1}^{k_{2i+1}} \left[ f^{-1-2j}(w_j)g^{2j}(w_j) \right] \prod_{j=i+1}^{m_{2i+1}} \left[ f^{-1-2j}(w_j)g^{2j}(w_j) \right]
\]

\[
\prod_{j=k_{2i} + 1}^{k_{2i+1}} \prod_{j=k_{2i} + i + 1}^{k_{2i+1}} \left[ f^{-1-2j}(w_j)g^{2j}(w_j) \right] \prod_{j=i+1}^{m_{2i+1}} \left[ f^{-1-2j}(w_j)g^{2j}(w_j) \right]
\]

\[
\prod_{j=k_{2i} + 1}^{k_{2i+1}} \prod_{j=k_{2i} + i + 1}^{k_{2i+1}} \left[ f^{-1-2j}(w_j)g^{2j}(w_j) \right] \prod_{j=i+1}^{m_{2i+1}} \left[ f^{-1-2j}(w_j)g^{2j}(w_j) \right]
\]

\[
\prod_{j=k_{2i} + 1}^{k_{2i+1}} \prod_{j=k_{2i} + i + 1}^{k_{2i+1}} \left[ f^{-1-2j}(w_j)g^{2j}(w_j) \right] \prod_{j=i+1}^{m_{2i+1}} \left[ f^{-1-2j}(w_j)g^{2j}(w_j) \right]
\]

\[
\prod_{j=k_{2i} + 1}^{k_{2i+1}} \prod_{j=k_{2i} + i + 1}^{k_{2i+1}} \left[ f^{-1-2j}(w_j)g^{2j}(w_j) \right] \prod_{j=i+1}^{m_{2i+1}} \left[ f^{-1-2j}(w_j)g^{2j}(w_j) \right]
\]
\[
\left[ \prod_{i=1}^{n_{2i-1}} \int_{\mathcal{B}_{2i}} \prod_{j=1}^{m_{2i}} g(x_j) \, dx_j \right].
\]

where

\[
B_{2i} = \{ w_{k_{2i-1}} < x_1 < \ldots < x_{m_{2i-1}} < w_{k_{2i}} \}, \quad B_{2i+1} = \{ w_{k_{2i+1}} < x_1 < \ldots < x_{m_{2i+1}} < w_{k_{2i+2}} \}.
\]

Similarly

\[
\prod_{j=k_{2i-1}+1}^{k_{2i}} \beta_j = \left[ \prod_{i=1}^{l} \prod_{j=k_{2i-1}+1}^{k_{2i}} f(w_j) \right] \prod_{i=1}^{l} \left[ \prod_{j=1}^{m_{2i}} f(x_j) \, dx_j \right]
\]

\[
\left[ \prod_{i=1}^{n_{2i-1}} \int_{\mathcal{B}_{2i}} \prod_{j=1}^{m_{2i}} f(x_j) \, dx_j \right]
\]

Now \( \prod_{j=k_{2i-1}}^{k_{2i}} \alpha_j \) is the product of densities and hence it is a density. Also as \( \theta \to 0 \),

\[
\prod_{j=k_{2i-1}+1}^{k_{2i}} \beta_j \to \prod_{j=k_{2i-1}}^{k_{2i}} \beta_j, \text{ which is density. Hence by Scheffe's theorem}
\]

\[
\int_{B^*} \left| \prod_{j=k_{2i+1}}^{k_{2i+2}} \alpha_j - \prod_{j=k_{2i-1}}^{k_{2i}} \beta_j \right| \, dw \to 0 \text{ as } \theta \to 0 \tag{2.3.6}
\]

for any Borel measurable set \( B^* \).

Next consider the case where \( i = k_{2i-1} + s + 1 \) for \( s = 1, 2, \ldots, N_{2i} \)

\[
\prod_{j=1}^{k_{2i-1}+s} \alpha_j = \left[ \prod_{i=1}^{l} \prod_{j=k_{2i-1}+1}^{k_{2i}} f^{x_{j-1}}(w_j) g^{x_j}(w_j) \prod_{j=k_{2i+1}+1}^{k_{2i}} f^{x_{j-1}}(w_j) g^{x_j}(w_j) \right]
\]

\[
\prod_{i=1}^{l} \left\{ [F(w_{k_{2i-1}+1}) - F(w_{k_{2i+2}})]^{m_{2i-1}} [G(w_{k_{2i-1}+1}) - G(w_{k_{2i+2}})]^{n_{2i-1}} \right\}
\]

and

\[
\prod_{j=1}^{k_{2i-1}+s} \beta_j = \left[ \prod_{i=1}^{l} \prod_{j=k_{2i-1}+1}^{k_{2i}} f(w_j) \prod_{j=k_{2i+1}+1}^{k_{2i}} f(w_j) \prod_{i=1}^{l} [F(w_{k_{2i-1}+1}) - F(w_{k_{2i+1}})]^{n_{2i-1}} \right]
\]
By similar arguments as in the earlier case

\[
\int \prod_{j=1}^{k_{2}} |\alpha_j - \prod_{j=1}^{k_{2}} \beta_j| \, dw \to 0 \quad \text{as} \quad \theta \to 0
\]  

(2.3.7)

for any measurable set \( B^* \). In particular, if we let

\[ B_i^* = (-\infty < w_{k_i} < \ldots < w_j) \quad \text{and} \quad B_{j+1}^* = (w_j < \ldots < w_{k_j} < \infty), \]

then by (2.3.6) and (2.3.7), for any \( \varepsilon > 0 \), there exists a \( \theta_\varepsilon \) such that for \( 0 < \theta < \theta_\varepsilon \)

\[
|Q_i - Q_i^*| \leq C(m, n) \int B_i^* \left( \left| \prod_{j=1}^{k_{2}} |\alpha_j - \prod_{j=1}^{k_{2}} \beta_j| \right| \left| \frac{\alpha_i - \beta_i}{\theta} \prod_{j=1}^{k_{2}} \beta_j \right| \right) \, dw
\]

\[
= C(m, n) \int B_i^* \left( \left[ \int |\prod_{j=1}^{k_{2}} \alpha_j - \prod_{j=1}^{k_{2}} \beta_j| \, dw \right] \left| \frac{\alpha_i - \beta_i}{\theta} \prod_{j=1}^{k_{2}} \beta_j \right| \right) \, dw \quad \text{**} \, dw_i
\]

\[
< \varepsilon C(m, n) \int B_i^* \left| \frac{\alpha_i - \beta_i}{\theta} \right| \prod_{j=1}^{k_{2}} \beta_j \, dw_i
\]

\[
= \varepsilon K_i(m, n) \int B_i^* \left( \left| \frac{\alpha_i - \beta_i}{\theta} \right| f(w_j)[1 - F(w_j)]^{n+1} \right) \, dw_i
\]

\[ \leq \varepsilon M_i, \]

where \( K_i(m, n) \) is a constant obtained after integrating out all other variables

and \( M_i \) is finite by condition A_i (iii). Thus \( |Q_i - Q_i^*| \to 0 \).

This completes the proof.

The LMP rank test can also be obtained under the following alternative conditions.

**Condition A :**

(i) The df \( F(x; \theta) \) has a density function \( f(x; \theta) \), which along with \( f^*(x; \theta) = \frac{\partial f(x; \theta)}{\partial \theta} \), is continuous w.r.t. \( \theta \) for \(-a \leq \theta \leq a\), \( a > 0 \), for almost all \( x \). there exist functions \( M_0(x) \) and \( M_1(x) \), integrable over \((-\infty, \infty)\), such that
\[ f(x; \theta) \leq M_0(x) \text{ and } |f^*(x; \theta)| \leq M_1(x), \quad \text{for } -a \leq \theta \leq a, a > 0. \]

(ii) \( f(x) > 0 \), whenever \( f(x;0) > 0 \).

(see Puri and Sen (1971), p.108 or Capon (1961)).

**Proof of Theorem 2.3.1 (Under the Condition A).** It is well known that the LMP rank test for \( H_0 \) against \( H_i \) is given by the critical region.

\[
\frac{\partial}{\partial \theta} \log P_0(z, n) \bigg|_{\theta=0} = \frac{\partial}{\partial \theta} P_0^*(z, n) \bigg|_{\theta=0} \geq \text{constant} \tag{2.3.8}
\]

The probability \( P_0(z, n) \) is given by (2.2.7). In order to find \( \partial P_0(z, n)/\partial \theta \), we consider only the terms involving \( \theta \) in (2.2.5) and obtain

\[
\frac{\partial}{\partial \theta} \left[ \prod_{i=1}^{r-1} \prod_{j=k_{i+1}}^{k_i} f^*(w_j; \theta) \right] = \sum_{i=1}^{r-1} \sum_{j=k_{i+1}}^{k_i} f^*(w_j; \theta) \prod_{i=1}^{r-1} \prod_{j=k_{i+1}}^{k_i} f^*(w_j; \theta) \tag{2.3.9}
\]

and

\[
\frac{\partial}{\partial \theta} \left\{ \prod_{i=1}^{r-1} \left[ F(w_{k_{i+1}}; \theta) - F(w_{k_{i+2}}; \theta) \right]^{2n_{i+1}} \right\} = \sum_{j=1}^{r} \left\{ \frac{n_{2j-1}[F^*(w_{k_{2j+1}}; \theta) - F^*(w_{k_{2j+2}}; \theta)]}{[F^*(w_{k_{2j+1}}; \theta) - F^*(w_{k_{2j+2}}; \theta)]} \right\} \tag{2.3.10}
\]

From (2.3.9) and (2.3.10)

\[
\frac{\partial}{\partial \theta} \left[ \prod_{i=1}^{r-1} \prod_{j=k_{i+1}}^{k_i} f^*(w_j; \theta) \right] \prod_{i=1}^{r} \left[ F(w_{k_{2i+1}}; \theta) - F(w_{k_{2i+2}}; \theta) \right]^{2n_{i+1}} \bigg|_{\theta=0}
\]

\[
= \prod_{i=1}^{r-1} \prod_{j=k_{i+1}}^{k_i} f^*(w_j) \prod_{i=1}^{r} \left[ F(w_{k_{2i+1}}) - F(w_{k_{2i+2}}) \right]^{2n_{i+1}}
\]

\[
\left\{ \sum_{i=1}^{r-1} \sum_{j=k_{i+1}}^{k_i} \left[ \frac{f^*(w_j)}{f(w_j)} \right]^{2n_{i+1}} + \sum_{i=1}^{r} \left[ F^*(w_{k_{2i+1}}) - F^*(w_{k_{2i+2}}) \right] \right\}, \tag{2.3.11}
\]

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where
\[ F^*(w) = \frac{\partial}{\partial \theta} F(w; \theta) \bigg|_{\theta = 0}. \]

Since under the condition A we may take the derivative under the integral, we obtain by substituting from (2.3.11) into (2.3.8)

\[
\frac{\partial}{\partial \theta} p_\theta(Z, n) \bigg|_{\theta = 0} \quad \quad \frac{p_\theta(Z, n)}{p_0(Z, n)}
\]

\[
= C^*(m, n) \cdot \left( \sum_{i=1}^{n-1} \sum_{j=k_i}^{k_{i+1}} z_i \left( \frac{f^*(w_j)}{f(w_j)} \right) + \sum_{i=1}^{r} n_{2i-1} \left[ \frac{F*(w_{k_{2i-1}+1}) - F*(w_{k_{2i-1}})}{F(w_{k_{2i-1}+1}) - F(w_{k_{2i-1}})} \right] \right)
\]

\[
= \left[ \frac{N!}{\prod_{i=1}^{r} k_i} \prod_{i=1}^{N_{2i-1}} f(w_j) \right] \left[ (F(w_{k_{2i-1}+1}) - F(w_{k_{2i-1}}))^N \right] \right] \, dw
\]

(2.3.12)

where
\[
C^*(m, n) = \prod_{i=1}^{r} \binom{N_{2i-1}}{n_{2i-1}} = P_\theta(Z, n)
\]

(2.3.13)

Since the second pair of curled brackets in the integrand and of (2.3.12) is the joint pdf of the order statistics.

\[ W_{k_{1}+1}, \ldots, W_{k_{2}}, W_{k_{3}+1}, \ldots, W_{k_{4}}, \ldots, W_{k_{2r-3}+1}, \ldots, W_{k_{2r-2}} \]

under \( H_0 \), (2.3.12) reduces to

\[
\frac{\partial}{\partial \theta} p_\theta(Z, n) \bigg|_{\theta = 0} \quad \quad \frac{p_\theta(Z, n)}{p_0(Z, n)} = \sum_{i=1}^{r-1} \sum_{j=k_i}^{k_{i+1}} a_N(j, f) + \sum_{i=1}^{r} n_{2i-1} E \left[ \frac{F*(w_{k_{2i-1}+1}) - F*(w_{k_{2i-1}})}{F(w_{k_{2i-1}+1}) - F(w_{k_{2i-1}})} \right]
\]

(2.3.14)
where \( a_N(j,f) \) is as defined by (2.3.2). By Lemma 2.2.1 (iii) we have

\[
E \left[ \frac{F^*(w_{k+1}) - F^*(w_{k+2})}{F(w_{k+1}) - F(w_{k+2})} \right] = \frac{1}{N_{2i-1}} \frac{1}{N_{2i-1}} \sum_{j=k+2}^{k} a_N(j,f)
\]

In virtue of the above relation, (2.3.14) reduces to the test statistic \( T(z,n) \), given by (2.3.1), which completes the proof of the theorem.

Remarks

LMP Rank Tests for Different Patterns of multiple Type II censoring schemes

Depending on the total number \( t \) of observed and censored subsets of order statistics being odd or even and the first of these subsets being observed or censored, we will have four different patterns of multiple Type II censoring schemes. These schemes are considered one by one below which we designate as Type II, Type II (1), Type II (2) and Type II (3).

**Type II.** Let the total number \( t \) of subsets of order statistics, be odd, say, \( t = 2r-1, r \geq 2 \) and let the first subset be censored so that the last one is also censored. This is the censoring scheme discussed in the earlier sections of this chapter. In this case the LMP rank test statistic is given by (2.3.1).

**Special cases:** (i). when \( r = 2 \) the above scheme reduces to the doubly censored scheme.

(ii) when \( r = 3 \), the above scheme results in the triple censoring scheme considered by Mehrotra et al. (1977).

**Type II (1)**: \( t = 2r-1, r \geq 2 \) and the first subset of order statistics is observed. In this case \( r \) subsets which are observed are:
\[ \{W_{k_2r+1}, \ldots, W_{k_{2i}}\}, \quad i = 1, \ldots, r, \]  

(2.3.15)

with \( k_0 = 0 \) and \( k_{2r-1} = N \), and the rest of the \( r-1 \) subsets are censored.

Hence the LMP rank test statistic is given by

\[
T(z, n) = \sum_{i=1}^{r} \sum_{j=k_{2i-1}+1}^{k_{2i}} s_{N}(j, f) + \sum_{i=1}^{r} \sum_{j=k_{2i-1}+1}^{k_{2i}} a_{N}(j, f) 
\]  

(2.3.16)

**Special case:** when \( r = 2 \) the above censoring scheme reduces to the middle censoring.

**Note:** By letting \( r = 1 \) in (2.3.16) and equating its second term to zero, we get the LMP rank statistic for uncensored case with \( k_1 = N \).

**Type II (2):** \( t = 2r \), \( r \geq 1 \) and the first subset of order statistics is observed. In this case the subsets of observed order statistics are the same as in Type II (1) censoring scheme. But the \( r \) censored subsets of order statistics are

\[ \{W_{k_{2r+1}}, \ldots, W_{k_{2r}}\}, \quad i = 1, 2, \ldots, r, \]  

(2.3.17)

with \( k_{2r} = N \). The LMP rank test statistic for this scheme is given by

\[
T(z, n) = \sum_{i=1}^{r} \sum_{j=k_{2i-1}+1}^{k_{2i}} z_{j} a_{N}(j, f) + \sum_{i=1}^{r} \sum_{j=k_{2i-1}+1}^{k_{2i}} a_{N}(j, f), \]  

(2.3.18)

**Special case:** when \( r = 1 \) the above censoring scheme reduces to right censoring.

**Type II(3):** \( t = 2r \), \( r \geq 1 \), and the first subset of order statistics is censored. In this case the subsets of observed order statistics are given by (2.3.15) and the LMP rank test statistic is given by

\[
T(z, n) = \sum_{i=1}^{r} \sum_{j=k_{2i-1}+1}^{k_{2i}} z_{j} a_{N}(j, f) + \sum_{i=1}^{r} \sum_{j=k_{2i-1}+1}^{k_{2i}} a_{N}(j, f). \]  

(2.3.19)
Special case: when $r = 1$, the above censoring scheme reduces to left censoring.

2.4 Exact Mean and Variance of The Test Statistic Under $H_0$

In this section we find the exact mean and variance of the statistic $T(z, n)$ given by (2.3.1) under the null hypothesis. First we prove below certain results which are required for this purpose.

Lemma 2.4.1. (a). The distribution of the vector $n = (n_1, \ldots, n_{2r-1})$ under the null hypothesis $H_0$ is given by

$$P_0(n) = \frac{1}{N^{2r-1}} \prod_{i=1}^{2r-1} \binom{N_i}{n_i}$$

(2.4.1)

with $N_i = m_i + n_i$ and $0 \leq n_i \leq N_i$, $i = 1, 2, \ldots, 2r-1$ and $\sum_{i=1}^{2r-1} n_i = n$, $\sum_{i=1}^{2r-1} N_i = N$.

(b) $E(n_i) = \frac{n_i N_i}{N}$, \hspace{1cm} (2.4.2)

$$\text{Var}(n_i) = \frac{mn_i}{N^2(N-1)} N_i(N - N_i),$$

(2.4.3)

and

$$\text{Cov}(n_i, n_j) = -\frac{mn_i N_j}{N^2(N-1)} \text{ for } i \neq j.$$ \hspace{1cm} (2.4.4)

Proof of (a). Summing $P_0(z, n)$ given by (2.2.7) over all possible values of the vector $z$ defined in (2.2.5), we get

$$P_0(n) = \frac{1}{N^{2r-1}} \prod_{i=1}^{2r-1} \binom{N_{2i-1}}{n_{2i-1}} \prod_{i=1}^{2r-1} \binom{N_{2i}}{n_{2i}}.$$ \hspace{1cm} (2.4.5)
This follows by noting that in the 2i-th subset of order statistics
\{W_{k_{2i-1}+1}, \ldots, W_{k_{2i}}\} (i = 1, 2, \ldots, r-1), \(m_{2i}\) order statistics are X's and \(n_{2i}\) are Y's. Hence among the corresponding \(z_i\)'s, \(m_{2i}\) are 0's and \(n_{2i}\) are 1's, and these can be arranged in \(\binom{N_{2i}}{n_{2i}}\) ways. This is true for all the \((r - 1)\) observed subsets of order statistics.

**Proof of (b).** These results follow by noting that \(P_0(\mathbf{n})\) is the probability mass function (pmf) of a \((2r-2)\) variate hypergeometric distribution (See Wilks (1962), p. 136).

**Lemma 2.4.2.** Consider the vector \(\mathbf{Z} = \{Z_{k_{2i-1}+1}, \ldots, Z_{k_{2i}}\}, i = 1, 2, \ldots, r-1\), with \(k_{2i} - k_{2i-1} = m_{2i} + n_{2i} = N_{2i}\). Then for \(j = 1, 2, \ldots, N_{2i}\),

\[
E(Z_{k_{2i-1}+j} | n_{2i}) = \frac{n_{2i}}{N_{2i}},
\]

\[
\text{Var}(Z_{k_{2i-1}+j} | n_{2i}) = \frac{m_{2i}n_{2i}}{N_{2i}^2},
\]

\[
\text{Cov}(Z_{k_{2i-1}+j}, Z_{k_{2i-1}+j+1} | n_{2i}) = \frac{m_{2i}n_{2i}}{N_{2i}^2(N_{2i} - 1)} \quad \text{for } j \neq 1
\]

and

\[
\text{Cov}(Z_{k_{2i-1}+j}, Z_{k_{2i-1}+1} | n_{2i}, n_{2i}) = 0 \quad \text{for } i \neq i'.
\]

**Proof** \(E(Z_{k_{2i-1}+j} | n_{2i}) = P(Z_{k_{2i-1}+1} = 1 | n_{2i}) = \frac{n_{2i} - 1}{N_{2i}} = \frac{n_{2i}}{N_{2i}}\)

\[
\text{Var}(Z_{k_{2i-1}+j} | n_{2i}) = E(Z_{k_{2i-1}+j}^2 | n_{2i}) - (E(Z_{k_{2i-1}+j} | n_{2i}))^2
\]
\[ \text{Cov}(Z_{k_{2i+1}}, Z_{k_{2i+1}+1} | n_{2i}) = E(Z_{k_{2i+1}}, Z_{k_{2i+1}+1} | n_{2i}) - E(Z_{k_{2i+1}} | n_{2i}) E(Z_{k_{2i+1}+1} | n_{2i}) \]
\[ = \frac{n_{2i}(n_{2i-1})}{N_{2i}(N_{2i}-1)} \frac{n_{2i}^2}{N_{2i}} \]
\[ = \frac{-m_{2i}n_{2i}}{N_{2i}(N_{2i}-1)}. \]

Since \( n_{2i} \) and \( n_{2i} \) are fixed, \( Z_{k_{2i+1}} \) and \( Z_{k_{2i+1}+1} \) will be uncorrelated. So (2.4.9) is obvious.

**Theorem 2.4.3** Under the null hypothesis \( H_0 \),

\[ E(T(z, n)) = \frac{n}{N} \sum_{i=1}^{N} a(j), \quad (2.4.10) \]

\[ \text{Var}(T(z, n)) = \frac{mn}{N(N-1)} \sum_{i=1}^{k_{2i+1}} \sum_{j=k_{2i+1}}^{k_{2i}} (a(j) - \bar{a})^2 - \frac{mn}{N(N-1)} \sum_{i=1}^{k_{2i+1}} \sum_{j=k_{2i+1}+1}^{k_{2i+1}} (a(j) - \bar{a}_{2i+1})^2, \quad (2.4.11) \]

where

\[ \bar{A} = \frac{1}{N} \sum_{i=1}^{k_{2i+1}} \sum_{j=k_{2i+1}}^{k_{2i}} a(j), \quad \bar{a}_{2i+1} = \frac{1}{N_{2i-1}} \sum_{j=k_{2i-1}+1}^{k_{2i+1}} a(j) \]

and for brevity we have written \( a(j) \) for \( a_N(j, f) \).

**Proof** The test statistic \( T(z, n) \) can be written as

\[ T(z, n) = \sum_{i=1}^{k_{2i+1}} \sum_{j=k_{2i+1}}^{k_{2i}} z_i a(j) + \sum_{i=1}^{N_{2i-1}} n_{2i-1} \bar{a}_{2i-1}, \quad (2.4.12) \]
where 
\[
\tilde{a}_i = \frac{1}{N_k} \sum_{j=k_{i-1} + 1}^{k_i} a(j).
\]

Then 
\[
E[E\{T(z, n) \mid n\}] = E\left[\sum_{i=1}^{r-l} \sum_{j=\tilde{a}_{2i-1}}^{\tilde{a}_{2i}} E(Z_i \mid n_j) a(j) + \sum_{i=l}^{N} n_{2i-1} \tilde{a}_{2i-1}\right]
\]
\[
= E\left(\sum_{i=1}^{r-l} n_{2i-1} \tilde{a}_{2i} + \sum_{i=l}^{N} n_{2i-1} \tilde{a}_{2i-1}\right), \quad \text{from (2.4.6)}
\]
\[
= E\left(\sum_{i=1}^{r-l} n_i \tilde{a}_i\right)
\]
\[
= \frac{n}{N} \sum_{i=1}^{N} n_i \tilde{a}_i, \quad \text{from (2.4.2)}
\]
\[
= \frac{n}{N} \sum_{i=1}^{N} a(i).
\]

For the variance, we have 
\[
\text{Var}[T(z, n)] = \text{Var}[E\{T(z, n) \mid n\}] + E[\text{Var}\{T(z, n) \mid n\}] \quad (2.4.13)
\]

Now 
\[
\text{Var}[E\{T(z, n) \mid n\}] = \text{Var}\left[\sum_{i=1}^{r-l} n_i \tilde{a}_i\right], \quad \text{as in (2.4.13)}
\]
\[
= \sum_{i=1}^{2r-l} \tilde{a}_i^2 \text{var}(n_i) + \sum_{i<j}^{2r-l} \tilde{a}_i \tilde{a}_j \text{cov}(n_i n_j)
\]
\[
= \frac{mn}{N^2(N-1)} \left[\sum_{i=1}^{2r-l} N_i \tilde{a}^2_i - \frac{1}{N} \left(\sum_{i=1}^{2r-l} N_i \tilde{a}_i\right)^2\right], \quad (2.4.14)
\]

from (2.4.3) and (2.4.4).

Next consider
Var\{T(\text{Z, n}) \mid n\} = \text{var}\left[ \sum_{i=1}^{k_{n}} \sum_{j=k_{n-1}+1}^{k_{n}} z_{j}a(j) + \sum_{i=1}^{i} n_{2i-1}\bar{a}_{2i-1} \right] | n

= \text{var}\left[ \sum_{i=1}^{i} \sum_{j=k_{n-1}+1}^{k_{n}} z_{j}a(j) \right] | n, \text{Since the other two terms will be}

zero for fixed n

\begin{align*}
&= \sum_{i=1}^{i} \left[ \sum_{j=k_{n-1}+1}^{k_{n}} a^2(j) \text{var}(z_j \mid n_{2i}) + \sum_{j=k_{n-1}+1}^{k_{n}} a(j)a(j') \text{cov}(z_j, z_{j'}) \mid n_{2i} \right] \\
&\quad + \sum_{i=1}^{i} \text{cov} \left[ \left( \sum_{j=k_{n-1}+1}^{k_{n}} z_{j}a(j), \sum_{j=k_{n-1}+1}^{k_{n}} z_{j}a(j') \mid n_{2i}, n_{2i} \right) \right] \\
&= \sum_{i=1}^{i} \left[ \sum_{j=k_{n-1}+1}^{k_{n}} a^2(j) \left( \frac{m_{2in_{2i}}}{{N_{2i}^2}} - \sum_{j=k_{n-1}+1}^{k_{n}} a(j) a(j') \left( \frac{m_{2in_{2i}}}{{N_{2i}^2(N_{2i}-1)}} \right) \right) \right],
\end{align*}

from (2.4.7) – (2.4.9),

\begin{equation}
= \sum_{i=1}^{i} \frac{m_{2in_{2i}}}{{N_{2i}(N_{2i}-1)}} \left[ \sum_{j=k_{n-1}+1}^{k_{n}} a^2(j) - N_{2i}\bar{a}_{2i}^2 \right]. \tag{2.4.15}
\end{equation}

Now

\begin{align*}
E(m_{2in_{2i}}) &= E[n_{2i}(N_{2i}-n_{2i})] \\
&= N_{2i}E(n_{2i}) - \left[ \text{var}(n_{2i}) + (E(n_{2i}))^2 \right] \\
&= \frac{mnN_{2i}(N_{2i}-1)}{N(N-1)}, \tag{2.4.16}
\end{align*}

which follows from (2.4.2) and (2.4.3) after a little simplification. Therefore

\begin{equation}
E[\text{Var}\{T(\text{Z, n}) \mid n\}] = \frac{mn}{N(N-1)} \sum_{i=1}^{i} \left[ \sum_{j=k_{n-1}+1}^{k_{n}} a^2(j) - N_{2i}\bar{a}_{2i}^2 \right]. \tag{2.4.17}
\end{equation}
Adding (2.4.14) and (2.4.17) we obtain

$$\text{Var}[T(z,n)] = \frac{mn}{N(N-1)} \left[ \sum_{i=1}^{2r-1} N_i \bar{a}_i^2 - \frac{1}{N} \left( \sum_{i=1}^{2r-1} N_i \bar{a}_i \right)^2 \right] + \left[ \sum_{j=1}^{k_2} \left( \sum_{i=j+1}^{k_2} a^2(j) - N_2 \bar{a}_2^2 \right) \right]$$

with \( \bar{a}_i = \frac{1}{N_i} \sum_{j=k_{i-1}+1}^{k_i} a(j) \).

Writing

$$\sum_{i=1}^{2r-1} N_i \bar{a}_i^2 = \sum_{i=1}^{2r-1} N_i \bar{a}_i^2 + \sum_{i=1}^{N_2} \bar{a}_{2i-1}^2$$

and

$$\sum_{i=1}^{r} \sum_{j=k_{2i-1}+1}^{k_{2i}} a^2(j) = \sum_{i=1}^{r} \sum_{j=k_{2i-1}+1}^{k_{2i}} a^2(j)$$

in the above variance, it reduces to the form given by (2.4.11). This completes the proof of the theorem.

**Remark:** The first term of (2.4.11) is the variance of the statistic based on complete sample and the second term is due to censoring.

### 2.5 Asymptotic Distribution of the Test Statistic

In this section we find the asymptotic distribution of the test statistic \( T(z,n) \) for testing \( H_0 \) against the alternative \( H_1 \) given by (2.2.2). The statistic \( T(z,n) \) which can be expressed as in (2.5.18) is a linear rank order statistic. Here we use the scores generating function (sgf) \( \phi_p(u) \), given by (2.5.22) to generate the scores \( a_{n,k} (i, f) \) defined by (2.5.19). The function \( \phi_p(u) \) has jumps at a finite number of points. The Chernoff-Savage (1958) theorem and its generalization by Govindarajalu, Lecam and Raghavachari (1967) require the sgf to be absolutely
continuous in the interval \([0, 1]\). So their theorems are not applicable for finding the asymptotic distribution of \(T(z, n)\).

Johnson and Mehrotra (1972) have pointed out how to obtain the asymptotic distribution of the test statistic based on right censored data using the results of Pyke and Shorack (1968). They have also remarked that the asymptotic distribution can also be obtained by applying the relevant theorems of Dupac and Hajek (1969). These authors have not provided a simpler expression for the asymptotic expectation of the test statistic. Dupac (1970) and Hoeffding (1973) have established that the expectation of the test statistic can be replaced by a simpler centering constant by imposing a slightly stronger condition than the square integrability on the sgf \(\phi(u)\). We give below Dupac and Hajek's (1969) theorems for the two-sample problem with slight modifications suggested by Dupac (1970) and Hoeffding (1973).

As in Section 2.2 let \(X_1, \ldots, X_m\) and \(Y_1, \ldots, Y_n\) be two independent random samples from absolutely continuous df's \(F(x)\) and \(G(x)\) respectively. Let \(Z_{n,i} = 1\) or 0 according as the \(i\)-th smallest observation in the combined sample is a \(Y\) observation or an \(X\) observation. Now consider the statistics of the form

\[
S_N = \sum_{i=1}^{N} a_N(i)z_{N,i}, \tag{2.5.1}
\]

where \(a_N(1), \ldots, a_N(N)\) are known constants which are called scores. We give below the conditions to be satisfied by the sgf \(\phi(u)\) and the scores \(a_N(i)\), for the asymptotic normality of the statistic \(T(z, n)\).
Condition B. The scores $a_N(i)$ are generated by a function $\phi(u)$, $0 < u < 1$, which is representable as

$$\phi(u) = \phi_1(u) - \phi_2(u),$$

(2.5.2)

where $\phi_i(u)$ ($i = 1, 2$) is nondecreasing and satisfies the Hoeffding-Dupac condition

$$J(\phi_i) = \int_0^1 |\phi_i(u)| [u(1-u)]^{-1/2} du < \infty.$$ (2.5.3)

Condition C. The scores $a_N(i)$ are obtained either by taking

$$a_N(i) = \phi \left( \frac{i}{N+1} \right), 1 \leq i \leq N,$$

(2.5.4)

or by a procedure satisfying

$$\sum_{i=1}^{N} |a_N(i) - \phi \left( \frac{i}{N+1} \right)| = O(1),$$

(2.5.5)

or

$$\sum_{i=1}^{N} |a_N(i) - \phi \left( \frac{i}{N+1} \right)| = o(N^{1/2})$$

(2.5.6)

Definition (Hajek (1968), p.328). A function $\phi(u)$, $0 < u < 1$, is said to be absolutely continuous inside $(0, 1)$, if for every $\varepsilon > 0$ and $0 < \alpha < \frac{1}{2}$, there exists a $\delta = \delta(\varepsilon, \alpha)$ such that for each finite set of disjoint intervals $(a_k, b_k)$ the relation

$$\sum_{k=1}^{n} |b_k - a_k| < \delta \quad \text{and} \quad \alpha < a_k, b_k < 1 - \alpha$$

implies

$$\sum_{k=1}^{n} |\phi(b_k) - \phi(a_k)| < \varepsilon$$

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Thus $\phi$ is absolutely continuous inside $(0,1)$, if it is absolutely continuous on $(\alpha, 1-\alpha)$ for every $\alpha \in (0, \frac{1}{2})$.

It is a well-known theorem of calculus that $\phi(u)$ is absolutely continuous inside $(0,1)$ if and only if there exists a function $\phi'(x)$ integrable on every interval $(\alpha, 1-\alpha)$, $0 < \alpha < 1/2$, and such that

$$\phi(b) - \phi(a) = \int_a^b \phi'(u) \, du, \quad 0 < a < b < 1. \tag{2.5.7}$$

We further know that $\phi'(u)$ represents the derivative of $\phi(u)$ almost everywhere. (See Hajek (1968), pp.328-329 for further details).

**Proposition 2.5.1. (Hoeffding (1973)).** There is a numerical constant $c$ such that if $\phi$ is nondecreasing then

$$\sum_{i=1}^{n} |E\phi(U_{(i)}) - \phi\left(\frac{i}{N+1}\right)| \leq c\sqrt{N}J(\phi), \tag{2.5.8}$$

where $J(\phi)$ is as given in (2.5.3).

**Proof.** See Hoeffding (1973).

**Proposition 2.5.2.** A sgf $\phi(u)$ satisfies the Hoeffding-Dupac condition (2.5.3) if

$$\int_0^1 |\phi(u)|^{2+\delta} < \infty \quad \text{for some } \delta > 0. \tag{2.5.9}$$

**Proof :** Consider Holder's inequality

$$E|XY| \leq (E|X|^p)^{\frac{1}{p}} (E|Y|^q)^{\frac{1}{q}} \quad \text{with } \frac{1}{p} + \frac{1}{q} = 1, \quad p > 1.$$ 

Let $q = 2 - \varepsilon$, where $0 < \varepsilon < 1$, so that

$$p = \frac{q}{q-1} = \frac{2-\varepsilon}{1-\varepsilon} = 2 + \frac{\varepsilon}{1-\varepsilon}. \quad \text{37}$$
With these values of $p$ and $q$, Holder's inequality becomes

$$\int_0^1 |\phi(u)|^p u^{1-\frac{1}{p}} du \leq \left( \int_0^1 |\phi(u)|^{p+1-\frac{1}{p}} du \right)^{1-\frac{1}{p}} \left( \int_0^1 u^{1-\frac{1}{p}} du \right)^{\frac{1}{p}}. \quad (2.5.10)$$

Now the second integral on the right hand side of (2.5.10) is finite and for the first integral to be finite, $|\phi(u)|$ has to be a little more than square integrable.

To be precise $|\phi(u)|^{2+\epsilon(l-\epsilon)^{-1}}$ for some $0 < \epsilon < 1$ should be integrable. It may be noted that $\epsilon (l-\epsilon)^{-1} \leq 1$ iff $\epsilon \leq 1/2$.

Thus a sufficient condition for the Hoeffding-Dupac condition to hold is that $|\phi(u)|^{2+\delta}$ should be integrable for some $\delta > 0$.

**Lemma 2.5.3.** The conditions (2.5.5) and (2.5.6) will be satisfied for $a_N(i) = \mathbb{E} \Phi(U(i))$, if

(i) $\Phi$ has a bounded second derivative, or

(ii) $\Phi(u) = \Phi_1(u) - \Phi_2(u)$, $0 < u < 1$, where $\Phi_i(u), \ i = 1, 2$ both are non decreasing, absolutely continuous inside $(0,1)$ and $J(\Phi_i) < \infty$.

**Proof (i).** Expanding $\Phi(U(i))$ around $EU(i)$ using Taylor’s formula and then taking expectation,

$$\mathbb{E}\Phi(U(i)) \leq \Phi \left( \frac{i}{N+1} \right) + \frac{i(N-i+1)}{(N+1)^2(N+2)} K,$$

where $K$ is a constant such that $|\Phi''(u)| \leq K$. Since

$$\frac{i(N-i+1)}{(N+1)^2(N+2)} \leq \frac{1}{N} \text{ for all } i,$$

we have

$$\sum_{i=1}^{N} \left| \mathbb{E}\Phi(U(i)) - \Phi \left( \frac{i}{N+1} \right) \right| \leq \frac{K}{2}.$$

Thus (2.5.5) is satisfied.
(ii) By Lemma 5.1 of Hajek (1968) and Lemma 1 of Hoeffding (1973), for
every $\alpha > 0$ there exists a decomposition

$$\phi(u) = \Psi(u) + \phi^{(1)}(u) - \phi^{(2)}(u), \quad 0 < u < 1,$$

such that $\Psi$ is a polynomial, $\phi^{(1)}$ and $\phi^{(2)}$ are nondecreasing and

$$J(\phi^{(1)}) + J(\phi^{(2)}) < \alpha.$$

Now

$$\sum_{i=1}^{N} \left| \mathbb{E}\phi(U_{(i)}) - \phi\left(\frac{i}{N+1}\right) \right| = \sum_{i=1}^{N} \left| \mathbb{E}\left\{\psi(U_{(i)}) + \phi^{(1)}(U_{(i)}) - \phi^{(2)}(U_{(i)})\right\} - \phi\left(\frac{i}{N+1}\right) \right|$$

$$\leq \sum_{i=1}^{N} \left\{ \left| \mathbb{E}\psi(U_{(i)}) - \psi\left(\frac{i}{N+1}\right) + \phi^{(1)}(U_{(i)}) - \phi^{(2)}\left(\frac{i}{N+1}\right) \right| + \left| \mathbb{E}\phi^{(2)}(U_{(i)}) - \phi^{(2)}\left(\frac{i}{N+1}\right) \right| \right\}$$

$$\leq K(\Psi) + c_{1}\sqrt{N} J(\phi^{(1)}) + c_{2}\sqrt{N} J(\phi^{(2)}),$$

by part (i) of this lemma and by Proposition 2.5.1. Thus

$$\sum_{i=1}^{N} \left| \mathbb{E}\phi(U_{(i)}) - \phi\left(\frac{i}{N+1}\right) \right| \leq K(\Psi) + c\sqrt{N} (J(\phi^{(1)}) + J(\phi^{(2)}))$$

$$\leq K(\Psi) + c\alpha\sqrt{N},$$

where $c = \max(c_{1}, c_{2})$ and $\alpha$ is arbitrarily small positive number. Hence

$$\sum_{i=1}^{N} \left| \mathbb{E}\phi(U_{(i)}) - \phi\left(\frac{i}{N+1}\right) \right| = o\left(\sqrt{N}\right)$$

and (2.5.6) is satisfied.
Theorem 2.5.4 (Dupac-Hajek's Theorem 4 with Hoeffding-Dupac's Modification). Consider the statistic $S_N$ for the two-sample problem. Assume that the sgf $\phi$ satisfies the condition $B$, the scores $a_N(i)$ satisfy the condition $C$, and that for some fixed $\lambda_0$,

$$\frac{m}{N} = \lambda_0 + O\left(\frac{1}{N}\right). \quad (0 < \lambda_0 < 1)$$

(2.5.12)

Put $H_0 = \lambda_0 F + (1-\lambda_0)G$, $L_0 = FH_0^{-1}$ and $M_0 = GH_0^{-1}$, where $FH_0^{-1}$ denotes the composition of functions $F$ and $H_0^{-1}$ and $GH_0^{-1}$ has a similar meaning. Put

$$\tau_0^2 = 2\lambda_0(1-\lambda_0) \int_{\phi_0} (1-\lambda_0)M'_0(u)M'_0(v)L_0(u)[1-L_0(v)] - \lambda_0 L'_0(u)L'_0(v)M_0(u)$$

$$\cdot [1-M_0(v)] \, d\phi(u) \, d\phi(v).$$

(2.5.13)

and postulate that $\tau_0^2 > 0$. Then $S_N$ is asymptotically normal with natural parameters $(ES_N, \text{Var}(S_N))$ and also with $(\mu_0, N\tau_0^2)$, where

$$\mu_0 = N \int_0^1 \phi(u) \, dM_0(u)$$

(2.5.14)

Theorem 2.5.5 (Dupac-Hajek's Theorem 5 with Hoeffding-Dupac's Modification). Consider the statistic $S_N$ for the two sample problem. Assume that the sgf $\phi$ satisfies the condition $B$ and the scores $a_N(i)$ satisfy the condition $C$. Further assume that

$$0 < \liminf \frac{m}{N} \leq \limsup \frac{m}{N} < 1$$

(2.5.15)

and that the densities $f = F'$ and $g = G'$ exist and $f(x) + g(x) > 0$ for $x \in (\alpha, \beta)$, whereas $f(x) + g(x) = 0$ for $x \in [\alpha, \beta]$. Put $H_N = \lambda_N F + (1-\lambda_N)G$, $L_N = FH_N^{-1}$, and $M_N = GH_N^{-1}$ and

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\[ \tau^2_N = 2\lambda_N(1-\lambda_N) \int_0^1 \left( (1-\lambda_N)M'_N(u)M'_N(v)L_N(u)[1-L_N(v)] \right. \\
+ \lambda_N L'_N(u)L'_N(v)M_N(u)[1-M_N(v)] \left. \right) d\phi(u)d\phi(v), \]  
(2.5.16)

where \( \lambda_N = m_N/N \). Assume that

\[ \liminf \tau^2_N > 0. \]

Then \( S_N \) is asymptotically normal with natural parameters \((E_{SN}, \text{Var}(S_N))\) and also with \((\mu_N, N\tau^2_N)\), where

\[ \mu_N = N(1-\lambda_N) \int_0^1 \phi(u)dM_N(u). \]  
(2.5.17)

**Asymptotic Normality of \( T(z, n) \)**

To show that the asymptotic distribution of \( T(z, n) \) is normal, all that we need to show is that the conditions of Theorems 2.5.4 or 2.5.5 are satisfied and that \( T(z, n) \) is a statistic of the form \( S_N \) given by (2.5.1). The LMP rank test statistic \( T(z, n) \) may be alternatively expressed as

\[ T(z, n) = \sum_{i=1}^N a_{N,i}(j,f)z_{N,i}, \]  
(2.5.18)

where

\[ a_{N,k}(j,f) = a_N(j,f), \quad j \in \{k_{2i-2}+1, \ldots, k_{2i}\} \quad i = 1, 2, \ldots, r-1 \]

\[ = \bar{a}_{2i-1} \quad j \in \{k_{2i-2}+1, \ldots, k_{2i-1}\} \quad i = 1, 2, \ldots, r, \]  
(2.5.19)

where

\[ \bar{a}_{2i-1} = \frac{1}{N_{2i-1}} \sum_{j=k_{2i-2}+1}^{k_{2i-1}} a_N(j,f) \]

with \( a_N(j,f) \) is as defined by (2.3.2).
Let
\[ p_i = \lim_{N \to \infty} \frac{k_i}{N}, \quad i = 1, 2, \ldots, 2r-1 \text{ and } 0 = p_0 < p_1 < p_2 < \ldots < p_{2r-2} < p_{2r-1} = 1 \]
(2.5.20)

However, for convenience we will assume that
\[ kj = \lfloor Np_i \rfloor \text{ if } \lfloor Np_i \rfloor \text{ is an integer} \]
\[ = \lfloor Np_i \rfloor + 1 \text{ otherwise}, \] (2.5.21)

where \([x]\) denotes the largest integer not exceeding \(x\).

We define the sgf \( \phi_p(u) \), corresponding to an asymptotically most powerful rank test (AMPRT) for the multiply Type II censored data as follows:
\[ \phi_p(u) = \phi(u), \quad p_{2i-1} < u \leq p_{2i}, \quad i = 1, 2, \ldots, r-1 \]
\[ = \frac{1}{p_{2i-1} - p_{2i-2}} \int_{p_{2i-2}}^{p_{2i-1}} \phi(u) \, du \]
(2.5.22)

where
\[ \phi_{2i-1} = \frac{1}{p_{2i-1} - p_{2i-2}} \int_{p_{2i-2}}^{p_{2i-1}} \phi(u) \, du \]

This is the extended form of the corresponding sgf given by Gastwirth (1965) for Type II right censored data and used by Johnson and Mehrotra (1972) for obtaining the asymptotic distribution of their statistic.

Now we examine below the conditions B in respect of the sgf \( \phi_p(u) \).

Suppose the original sgf \( \phi(u) \) satisfies this condition. Then it can be easily seen that the sgf \( \phi_p(u) \) also satisfies this condition, with \( \phi_{2p}(u) \) \((j = 1, 2)\) defined similarly as \( \phi_p(u) \) with \( \phi(u) \) in the latter replaced by \( \phi_j(u) \).
Considering the Hoeffding-Dupac condition (2.5.3), for \( \phi_{1,p}(u) \), we have

\[
\int_0^1 |\phi_{1,p}(u)| u(1-u)^{\alpha_j} \, du = \sum_{i=1}^{r-1} \int_{\theta_{p,i-1}}^{\theta_{p,i}} |\phi_{1,p}(u)| u(1-u)^{\alpha_j} \, du + \sum_{i=1}^{r-1} \int_{\theta_{p,i-1}}^{\theta_{p,i}} |u(1-u)|^{\alpha_j} \, du,
\]

(2.5.23)

where \( \theta_{2i-1} \) is as given in (2.5.22).

Since it is assumed that the sgf \( \phi(u) \) satisfies (2.5.3), it follows that both the integrals of (2.5.23) are finite. Hence \( \phi_{1,p} \) satisfies (2.5.3). Similarly \( \phi_{2,p} \) also satisfies this condition. Also if \( \phi_j(u) \) is nondecreasing so if \( \phi_{p,j}(u), \, j = 1, 2 \). Thus the sgf \( \phi_p(u) \) satisfies the condition B if \( \phi(u) \) satisfies the same.

As for the condition C for \( a_{N,k}(i, f) \), we have

\[
\sum_{i=1}^{N} a_{N,k}(i, f) = \sum_{i=1}^{r-1} \sum_{j=k_{i-1}}^{k_i} a(j) - \phi \left( \frac{j}{N+1} \right)
\]

(2.5.24)

where \( \theta_{2i-1} \) and \( \theta_{2i-1} \) are as defined in (2.5.19) and (2.5.22), respectively. When \( p_i = k_i/N \), the second term on the left-hand side of (2.5.24) equals to

\[
\sum_{i=1}^{r-1} \sum_{j=k_{i-1}+1}^{k_i} a(j) - \phi \left( \frac{j}{N+1} \right)
\]

(2.5.25)

Now

\[
\int_{\theta_{p,i-1}}^{\theta_{p,i}} \phi(u) \, du = \int_{\theta_{p,i-1}}^{\theta_{p,i}} \phi(u) \, du = \sum_{j=k_{i-1}}^{k_i} \int_{\frac{j}{N+1}}^{\frac{j+1}{N+1}} \phi(u) \, du = \frac{1}{N} \sum_{j=k_{i-1}+1}^{k_i} \phi \left( \frac{j}{N+1} \right)
\]

(2.5.26)

for large \( N \).

Therefore for large \( N \), (2.5.25) reduces to
\[
\sum_{i=1}^{\sum_{j=1}^{k_{N-i}} N + 1} \left| a(j) - \phi \left( \frac{j}{N+1} \right) \right|, \text{ which is}
\]
\[
\leq \sum_{i=1}^{\sum_{j=1}^{k_{N-i}} N + 1} \left| a(j) - \phi \left( \frac{j}{N+1} \right) \right|. \tag{2.5.27}
\]

Thus from (2.5.24) and (2.5.27) we have
\[
\sum_{i=1}^{N} a_{N,k}(i) - \phi \left( \frac{i}{N+1} \right) \leq \sum_{i=1}^{\sum_{j=1}^{k_{N-i}} N + 1} a(j) - \phi \left( \frac{j}{N+1} \right) + \sum_{i=1}^{\sum_{j=1}^{k_{N-i}} N + 1} a(j) - \phi \left( \frac{j}{N+1} \right)
\]
\[
= \sum_{i=1}^{N} a(i) - \phi \left( \frac{i}{N+1} \right).
\]

From this it follows that, if the scores for the uncensored case satisfy the condition (2.5.5), (or (2.5.6)) then the corresponding scores for the censored case will also satisfy this condition.

We give below a corollary which is a modified version of Theorem 2.5.5, suitable for our multiple Type II censored situation and we use it to establish the asymptotic normality of \( T(z, n) \).

**Corollary 2.5.6.** Consider the LMP rank statistic \( T(z, n) \) given by (2.3.1). Let \( p_i \) and \( \phi_p(u) \) be defined by (2.5.20) and (2.5.22) respectively. Assume that the \( \text{sgf } \phi \) and the scores \( a_N(i) \) corresponding to the uncensored case satisfy the conditions B and C respectively. Further assume that the ratio \( \lambda_N = m/N \) and the densities \( f \) and \( g \) satisfy the conditions given in Theorem 2.5.5. Let \( t_N^2 \) be given by (2.5.16) with \( \phi \) replaced by \( \phi_p \) and assume that
\[
\lim \inf t_N^2 > 0.
\]
Then the statistic $T(z, n)$ is asymptotically normal with natural parameters $(\mu, \text{Var}T)$ and also with $(\mu_N, N\tau^2_N)$, where

$$
\mu_N = N(1-\lambda_N) \int_0^1 \phi_p(u)dM_N(u) \quad (2.5.28)
$$

### 2.6 Asymptotic Mean and Variance of the Statistic $T(z, n)$

In this section we obtain asymptotic mean and variance of the statistic $T(z, n)$ under $H_1$ as well as under $H_0$.

#### Asymptotic Mean Under $H_1$

From (2.5.28)

$$
\frac{\mu_N}{n} = \int_0^1 \phi_p(u)dM_N(u) = \sum_{i=1}^{N} \int_{p_{2i-1}}^{p_{2i}} \phi(u)M'(u)du + \sum_{i=1}^{N} \phi_{2i-1}[M(p_{2i-1}) - M(p_{2i-2})] \quad (2.6.1)
$$

#### Asymptotic Variance Under $H_1$

The asymptotic variance of the statistic $T(z, n)$ under the alternative $H_1$ is given by

$$
V(T(z, n)) = N\tau^2_N, \quad (2.6.2)
$$

where $\tau^2_N$ is given by (2.5.16). Suppressing the suffix $N$ on all the quantities in $\tau^2_N$ and readjusting the terms in the integrand, we get

$$
\frac{\tau^2}{2\lambda(1-\lambda)} = \int_0^{\lambda} \int_{0<u<v<1} [(1-\lambda)M'(u)M'(v)L(u) + \lambda L'(u)L'(v)M(u)] d\phi_p(u)d\phi_p(v) - \int_0^{\lambda} \int_{0<u<v<1} [(1-\lambda)M'(u)M'(v)L(u)L(v) + \lambda L'(u)L'(v)M(u)M(v)] d\phi_p(u)d\phi_p(v)
$$

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\begin{equation}
= [(1-\lambda)I_1(M, L) + \lambda I_1(L, M)] - [(1-\lambda)I_2(M, L) + \lambda I_2(L, M)], \text{ say.} \tag{2.6.3}
\end{equation}

Considering the integral \(I_1(M, L)\) we have

\[
I_1(M, L) = \int_0^1 \int_0^1 M'(u) M'(v) L(u) \phi_p(u) \phi_p(v) \left( - \int_0^u L'(t) dt \right) \phi_p(u) \phi_p(v)
\]

\[
= \int_0^1 L'(t) \left[ \int_0^1 M'(u) \phi_p(u) \phi_p(v) \right] dt
\]

\[
= \frac{1}{2} \int_0^1 L'(t) \left[ \int_0^1 M'(u) \phi_p(u) \right]^2 dt
\]

So,

\[
2I_1(M, L) = \sum_{i=1}^{m-1} \int_{p_{2i-1}}^{p_{2i}} L'(t) \left( \int_0^1 M'(u) \phi_p(u) \right)^2 dt + \sum_{i=1}^{m-1} \int_{p_{2i-1}}^{p_{2i}} L'(t) \left( \int_0^1 M'(u) \phi_p(u) \right)^2 dt
\]

\[
= \sum_{i=1}^{r} \left[ L(p_{2i-1}) - L(p_{2i-2}) \right] \left[ \sum_{j=1}^{r} \int_{p_{2j-1}}^{p_{2j}} M'(u) \phi_p(u) + \sum_{j=2i-1}^{2r-2} M'(p_j) \phi_j \right]^2
\]

\[
+ \sum_{i=1}^{m-1} \int_{p_{2i-1}}^{p_{2i}} L'(t) \left[ \int_0^1 M'(u) \phi_p(u) + \sum_{j=1}^{r} \int_{p_{2j-1}}^{p_{2j}} M'(u) \phi_p(u) + \sum_{j=2i}^{2r-2} M'(p_j) \phi_j \right]^2 dt, \tag{2.6.4}
\]

where \(\phi_j\) is the magnitude of the jump of the function \(\phi_p(u)\) at the point \(p_j\) and is given by

\[
\phi_j = \phi(p_{2j-1}) - \phi(p_{2j-1}), j = 1, \ldots, r-1
\]

and

\[
\phi_j = \phi(p_{2j}) - \phi(p_{2j}), j = 1, 2, \ldots, r. \tag{2.6.5}
\]

The integral \(I_1(L, M)\) is obtained by interchanging \(L'(\cdot)\) and \(M'(\cdot)\) in the expression (2.6.4).
The integral $I_2(M, L)$ is given by

$$I_2(M, L) = \int_0^1 \int_0^1 M'(u)M'(v)L(u)L(v)d\phi_p(u)d\phi_p(v)$$

$$= \frac{1}{2} \left[ \int_0^1 M'(u)L(u)d\phi_p(u) \right]^2$$

$$= \frac{1}{2} \left[ \sum_{i=1}^{r-1} \int_{p_{2i-1}}^{p_{2i}} M'(u)L(u)d\phi(u) + \sum_{i=1}^{r-2} M'(p_i)L(p_i)\phi_i \right]^2.$$

(2.6.6)

The integral $I_2(L, M)$ is obtained by writing $L'(.)$ and $M(.)$ in place of $M'(.)$ and $L(.)$, respectively, in (2.6.6). Finally substituting the expressions for $I_1(M, L)$, $I_1(L,M)$, $I_2(M, L)$ and $I_2(L, M)$ in (2.6.3), we get $\tau^2/[2\lambda(1-\lambda)]$ and hence we get $V[T(z, n)]$ from (2.6.2).

**Asymptotic mean under $H_0$**

Under $H_0$, $F(x) = G(x)$, and consequently we have

$$H(x) = F(x), \quad L(u) = M(u) = F(F^{-1}(u)) = u \quad \text{and} \quad L'(u) = M'(u) = 1. \quad (2.6.7)$$

So, under $H_0$, it follows from (2.6.1) and (2.6.7) that

$$\frac{\mu_N}{n} = \sum_{i=1}^{r-1} \int_{p_{2i-1}}^{p_{2i}} \phi(u)du + \sum_{i=1}^{r-2} \phi_{2i-1}(p_{2i-1} - p_{2i-2})$$

$$= \sum_{i=1}^{r-1} \int_{p_{2i-1}}^{p_{2i}} \phi(u)du + \sum_{i=1}^{r-2} \int_{p_{2i-2}}^{p_{2i}} \phi(u)du, \quad \text{by the definition of } \phi_{2i-1}$$

$$= \int_0^1 \phi(u)du \quad (2.6.8)$$

**Asymptotic variance under $H_0$**

Under $H_0$, it follows from (2.6.1) and (2.6.7) that $I_1(M, L) = I_1(L, M)$, $I_2(M, L) = I_2(L, M)$, and hence

$$\frac{\tau^2}{2\lambda(1-\lambda)} = I_1(M, L) - I_2(M, L) \quad (2.6.9)$$
Using (2.6.7) in (2.6.4), we get

\[
2I_1(M, L) = \sum_{i=1}^{r-1} \int_{P_{2i-1}}^{P_{2i}} \left[ \int_{P_{2i-1}}^{P_{2i}} d\phi(u) + \sum_{j=2i-1}^{2r-2} \phi_j \right] dt + \sum_{i=1}^{r-1} \int_{P_{2i-1}}^{P_{2i}} d\phi(u) + \sum_{j=2i-1}^{2r-2} \phi_j \right] dt
\]

\[
= \sum_{i=1}^{r-1} (P_{2i-1} - P_{2i-2}) \left[ \sum_{j=2i-1}^{2r-2} \left( \phi(P_{2j}) - \phi(P_{2j-1}) \right) + \sum_{j=2i-1}^{2r-2} \phi_j \right] \left( \phi(P_{2j}) - \phi(P_{2j-1}) \right) + \sum_{j=2i-1}^{2r-2} \phi_j \right] \left( \phi(P_{2j}) - \phi(P_{2j}) \right) dt,
\]

(2.6.10)

From (2.6.5) we have

\[
\sum_{j=2i-1}^{2r-2} \phi_j = \sum_{j=2i-1}^{2r-2} (\phi_{2j-1} + \phi_{2j}) = \sum_{j=2i-1}^{2r-2} (\Phi_{2j-1} - \Phi_{2j}) - \sum_{j=2i-1}^{2r-2} (\phi(P_{2j}) - \phi(P_{2j-1})),
\]

(2.6.11)

Similarly

\[
\sum_{j=2i-1}^{2r-2} \phi_j = \Phi_{2j-1} - \sum_{j=2i-1}^{2r-2} (\phi(P_{2j}) - \phi(P_{2j-1})),
\]

(2.6.12)

Substituting from (2.6.11) and (2.6.12) in (2.6.10), we get

\[
2I_1(M, L) = \sum_{i=1}^{r-1} (P_{2i-1} - P_{2i-2}) \left( \Phi_{2i-1} - \Phi_{2i-2} \right)^2 + \sum_{i=1}^{r-1} \int_{P_{2i-1}}^{P_{2i}} (\Phi_{2i-1} - \phi(t))^2 dt
\]

\[
= \sum_{i=1}^{r-1} (P_{2i-1} - P_{2i-2}) \left( \Phi_{2i-1}^2 - 2\Phi_{2i-1} \Phi_{2i-2} + \Phi_{2i-2}^2 \right)
\]

\[
+ \sum_{i=1}^{r-1} [(P_{2i} - P_{2i-1}) (\Phi_{2i-1} - 2\Phi_{2i-1} \Phi_{2i-2})] + \int_{P_{2i-1}}^{P_{2i}} \phi^2(t) dt
\]

(2.6.13)

(2.6.14)
Now
\[
\sum_{i=1}^{r} (p_{2i-1} - p_{2i-2}) + \sum_{i=1}^{r-1} (p_{2i} - p_{2i-1}) = p_{2r-1} = 1
\]  \hspace{1cm} (2.6.15)

and
\[
\sum_{i=1}^{r} (p_{2i-1} - p_{2i-2}) \bar{\phi}_{2i-1} + \sum_{i=1}^{r-1} (p_{2i} - p_{2i-1}) \bar{\phi}_{2i} = \int_{0}^{1} \phi(u) du = \bar{\phi}
\]  \hspace{1cm} (2.6.16)

Substituting from (2.6.15) and (2.6.16) in (2.6.14) we get
\[
2I_1(M, L) = \bar{\phi}_{2r-1} - 2\bar{\phi}_{2r-2} \bar{\phi} + \sum_{i=1}^{r-1} (p_{2i-1} - p_{2i-2}) \bar{\phi}_{2i-1}^2 + \sum_{i=1}^{r} \int_{p_{2i-1}}^{p_{2i}} \phi^2(u) du
\]  \hspace{1cm} (2.6.17)

When \(H_0\) is true (2.6.6) reduces to
\[
2I_2(M, L) = \left( \sum_{i=1}^{r-1} \int_{p_{2i-1}}^{p_{2i}} ud\phi(u) + \sum_{i=1}^{r} p_i \phi_i \right)^2
\]
\[
= \left( \sum_{i=1}^{r-1} \{p_{2i-1} \bar{\phi}_{2i} + p_{2i} \bar{\phi}_{2i+1} - p_{2i-1} \bar{\phi}_{2i} - p_{2i} \bar{\phi}_{2i+1}\} \right)^2
\]
\[
= \left( \sum_{i=1}^{r-1} \{(p_{2i} - p_{2i+1}) \bar{\phi}_{2i} - (p_{2i+1} - p_{2i}) \bar{\phi}_{2i+1} + p_{2i+1} \bar{\phi}_{2i+1} - p_{2i} \bar{\phi}_{2i}\} \right)^2
\]
\[
= \left( -\bar{\phi} - p_i \phi_i \right) + \sum_{j=2}^{r-1} p_{2j-1} \bar{\phi}_{2j-1} - \sum_{i=1}^{r-1} p_{2i} \bar{\phi}_{2i-1}
\]
\[
= (\bar{\phi}_{2r-1} - \bar{\phi})^2, \text{ since } p_{2r-1} = 1.
\]  \hspace{1cm} (2.6.18)

Substituting from (2.6.17) and (2.6.18) in (2.6.9), we obtain
\[
\frac{T^2}{\lambda(1-\lambda)} = \left[ \int_{0}^{1} \phi^2(u) du - \bar{\phi}^2 \right] - \left[ \sum_{i=1}^{r} \{\int_{p_{2i-2}}^{p_{2i-1}} \phi^2(u) du - (p_{2i-1} - p_{2i-2}) \bar{\phi}_{2i-1}^2 \} \right]
\]
\[
= \sum_{i=1}^{r-1} \int_{p_{2i-1}}^{p_{2i}} \phi^2(u) du + \sum_{i=1}^{r} (p_{2i-1} - p_{2i-2}) \bar{\phi}_{2i-1}^2 - \bar{\phi}^2
\]  \hspace{1cm} (2.6.19)

Using (2.6.19) we get \(V(T(z, n))\) under \(H_0\) from (2.6.2).