Chapter 2

Quotient Ternary Semimodules

2.1 Introduction

The concepts of partitioning ideal (= $Q$-ideal) of a semiring $R$, maximal homomorphism of semirings are initiated by P. J. Allen [1] and studied by many authors (see [1, 7, 19, 21, 24, 28, 37]). In this chapter, we introduce the notion of partitioning ternary subsemimodule of a ternary semimodule, which is useful to develop the quotient structure of ternary semimodule. Also we introduce the notion of maximal ternary semimodule homomorphism and steady ternary semimodule homomorphism respectively. The main part in this chapter is devoted to stating and proving analogous to several well known results in the theory of semimodules over semirings. We also study relation between maximal and steady ternary $R$-semimodule homomorphisms.

2.2 Partitioning ternary subsemimodules

In this section, we extend some definitions and results of P. J. Allen [1], P. J. Allen, J. Neggers and H. S. Kim [2] and S. E. Atani [7] to ternary semimodules over ternary
Definition 2.1. A ternary subsemimodule $N$ of a ternary $R$-semimodule $M$ will be called partitioning ternary subsemimodule (= $Q$-ternary subsemimodule), if there exists a subset $Q$ of $M$ such that

1) $M = \bigcup\{q + N : q \in Q\}$;

2) if $q_1, q_2 \in Q$, then $(q_1 + N) \cap (q_2 + N) \neq \emptyset \iff q_1 = q_2$.

Since every ternary semiring is a ternary semimodule over itself, every partitioning ideal of a ternary semiring $R$ is a partitioning ternary subsemimodule of a ternary $R$-semimodule $R$.

Lemma 2.2. Let $N$ be a partitioning ternary subsemimodule of a ternary $R$-semimodule $M$. If $x \in M$, then there exists a unique $q \in Q$ such that $x + N \subseteq q + N$. Hence $x = q + a$ for some $a \in N$.

Proof. Easy.


Lemma 2.3. If $N$ is a partitioning ternary subsemimodule of a ternary $R$-semimodule $M$, then there exists a unique $q_0 \in Q$ such that $N = q_0 + N$.

Proof. Since $N$ is a partitioning ternary subsemimodule, by Lemma 2.2, there exists a unique $q_0 \in Q$ such that $0 = q_0 + a_0$ for some $a_0 \in N$. If $b \in N$, then by Lemma 2.2, there exists a unique $q \in Q$ such that $b = q + a$ for some $a \in N$. Therefore, $q + a = b = b + 0 = b + q_0 + a_0 \in q_0 + N$. Hence $N \subseteq q_0 + N$. Again by Lemma 2.2, there
exists a unique $q' \in Q$ such that $q_0 + q_0 = q' + c$ for some $c \in N$. Now $q_0 = q_0 + 0 = q_0 + q_0 + a_0 = q' + c + a_0 \in q' + N$. Also $q_0 \in q_0 + N$. Hence $(q' + N) \cap (q_0 + N) \neq \phi$ and so $q_0 = q'$. Thus $q_0 + N = q' + c + a_0 + N = q_0 + c + a_0 + N = c + q_0 + a_0 + N = c + N \subseteq N$.

Now $N = q_0 + N$ where $q_0 \in Q$ is a unique element. Let $N$ be a partitioning ternary subsemimodule of a ternary $R$-semimodule $M$. Then $M/N(Q) = \{q + N : q \in Q\}$ forms a ternary $R$-semimodule under the following addition “$\ominus$” and ternary scalar multiplication “$\odot$”, $(q_1 + N) \ominus (q_2 + N) = q_3 + N$ where $q_3 \in Q$ is a unique element such that $q_1 + q_2 + N \subseteq q_3 + N$, and $r_1 \odot r_2 \odot (q_1 + N) = q_4 + N$ where $q_4 \in Q$ is a unique element such that $r_1 r_2 q_1 + N \subseteq q_4 + N$. This ternary $R$-semimodule $M/N(Q)$ will be called a quotient ternary semimodule of $M$ by $N$ and denoted by $(M/N(Q), \ominus, \odot)$ or just $M/N(Q)$. By Lemma 2.3, there exists a unique $q_0 \in Q$ such that $q_0 + N = N$. This $q_0 + N$ is the zero element of $M/N(Q)$.

If $N$ is a ternary subsemimodule of a ternary $R$-semimodule $M$, then it is possible that $N$ can be considered to be a partitioning ternary subsemimodule with respect to many different subsets $Q$ of $M$. However, the next theorem proves that the structure $(M/N(Q), \ominus, \odot)$ is essentially independent of $Q$.

**Theorem 2.4.** If $N$ is a partitioning ternary subsemimodule with respect to two subsets $Q_1$ and $Q_2$ of a ternary $R$-semimodule $M$, then $M/N(Q_1) \cong M/N(Q_2)$.

**Proof.** Define $f : M/N(Q_1) \to M/N(Q_2)$ by $f(q_1 + N) = q_2 + N$ where $q_2 \in Q_2$ is a unique such that $q_1 + N \subseteq q_2 + N$. Clearly, $f$ is well defined.

1) Let $q_1 + N, q_1' + N \in M/N(Q_1)$ and $r_1, r_2 \in R$. Therefore,

$$f((q_1 + N) \ominus (q_1' + N)) = f(q_1' + N) = q_2 + N \quad (2.2.1)$$
where \( q''_1 \in Q_1 \) is a unique such that \( q_1 + q'_1 + N \subseteq q''_1 + N \) and \( q_2 \in Q_2 \) is a unique such that \( q''_1 + N \subseteq q_2 + N \). Also

\[
    f(q_1 + N) \oplus f(q'_1 + N) = (q'_2 + N) \oplus (q''_2 + N) = q'''_2 + N \tag{2.2.2}
\]

where \( q'_2, q''_2 \in Q_2 \) are unique such that \( q_1 + N \subseteq q'_2 + N \) and \( q'_1 + N \subseteq q''_2 + N \) and \( q'''_2 \in Q_2 \) is a unique such that \( q'_2 + q''_2 + N \subseteq q'''_2 + N \). Now

\[
    q_1 + q'_1 \in q_1 + q'_1 + N \subseteq q''_1 + N \subseteq q_2 + N \tag{2.2.3}
\]

Also by Lemma 1.17,

\[
    q_1 + N \subseteq q'_2 + N \text{ and } q'_1 + N \subseteq q''_2 + N \Rightarrow q_1 + q'_1 + N \subseteq q'_2 + q'_1 + N \subseteq q'_2 + q''_2 + N \subseteq q'''_2 + N. \text{ Therefore }
\]

\[
    q_1 + q'_1 \in q_1 + q'_1 + N \subseteq q'''_2 + N \tag{2.2.4}
\]

From (2.2.3) and (2.2.4), \( q_2 = q'''_2 \). Hence by (2.2.1) and (2.2.2), \( f((q_1 + N) \oplus (q'_1 + N)) = f(q_1 + N) \oplus f(q'_1 + N) \).

Let

\[
    f(r_1 \odot r_2 \odot (q_1 + N)) = f(q'_1 + N) = q_2 + N \tag{2.2.5}
\]

where \( q'_1 \in Q_1 \) is a unique such that \( r_1 r_2 q_1 + N \subseteq q'_1 + N \) and \( q_2 \in Q_2 \) is a unique such that \( q'_1 + N \subseteq q_2 + N \). Also

\[
    r_1 \odot r_2 \odot f(q_1 + N) = r_1 \odot r_2 \odot (q'_2 + N) = q''_2 + N \tag{2.2.6}
\]

where \( q'_2, q''_2 \in Q_2 \) are unique such that \( q_1 + N \subseteq q'_2 + N \) and \( r_1 r_2 q'_2 + N \subseteq q''_2 + N \). Now

\[
    r_1 r_2 q_1 \in r_1 r_2 q_1 + N \subseteq q'_1 + N \subseteq q_2 + N \tag{2.2.7}
\]
Also by Lemma 1.17,
\[ q_1 + N \subseteq q'_2 + N \Rightarrow r_1r_2q_1 + N \subseteq r_1r_2q'_2 + N \subseteq q''_2 + N. \]
Therefore
\[ r_1r_2q_1 \in r_1r_2q_1 + N \subseteq q''_2 + N \] (2.2.8)

From (2.2.7) and (2.2.8), \( q_2 = q''_2 \). Hence by (2.2.5) and (2.2.6), \( f(r_1 \odot r_2 \odot (q_1 + N)) = r_1 \odot r_2 \odot f(q_1 + N). \)

2) Let \( q_2 + N \in M/N(q_2) \). Since \( q_2 \in M \), there exists a unique \( q_1 \in Q_1 \) such that \( q_2 + N \subseteq q_1 + N \). But then there exists a unique \( q'_2 \in Q_2 \) such that \( q_1 + N \subseteq q'_2 + N \). Now \( q_2 = q'_2 \) implies \( q_2 + N = q'_2 + N \) and hence \( f(q_1 + N) = q_2 + N \).

So \( f \) is onto.

3) Suppose that \( f(q_1 + N) = f(q'_1 + N) = q_2 + N \) say, where \( q_2 \in Q_2 \) is a unique element such that \( q_1 + N \subseteq q_2 + N \) and \( q'_1 + N \subseteq q_2 + N \). Choose \( t_1 \in Q_1 \) such that \( q_2 + N \subseteq t_1 + N \). But then \( q_1 = t_1 = q'_1 \). So \( q_1 + N = q'_1 + N \). So \( f \) is one-one.

Thus \( f : M/N(Q_1) \to M/N(Q_2) \) is an isomorphism.

**Theorem 2.5.** If \( N \) is a partitioning ternary subsemimodule with respect to two subsets \( Q_1 \) and \( Q_2 \) of a ternary \( R \)-semimodule \( M \), then \( M/N(Q_1) \) and \( M/N(Q_2) \) are equal as sets.

**Proof.** Let \( q_1 + N \in M/N(Q_1) \). Then \( q_1 \in Q_1 \subseteq M \) and hence by Lemma 2.2, there exists a unique \( q_2 \in Q_2 \) such that \( q_1 + N \subseteq q_2 + N \). Again there exists a unique \( q_3 \in Q_1 \) such that \( q_2 + N \subseteq q_3 + N \). Now \( q_1 + N = q_3 + N = q_2 + N \in M/N(Q_2) \). So \( M/N(Q_1) \subseteq M/N(Q_2) \). Similarly, \( M/N(Q_2) \subseteq M/N(Q_1) \).
Example 2.6. The monoid $M = (\mathbb{Z}_6, +)$ is a ternary semimodule over $(\mathbb{Z}_6^-, +, \cdot)$. Clearly, $N = \{\bar{0}, \bar{2}, \bar{4}\}$ is a partitioning ternary subsemimodule of $M$ with respect to three sets $Q_1 = \{\bar{0}, \bar{1}\}$, $Q_2 = \{\bar{0}, \bar{3}\}$, $Q_3 = \{\bar{0}, \bar{5}\}$ where $M/N(Q_1) = \{\bar{0} + N, \bar{1} + N\} = \{\{\bar{0}, \bar{2}, \bar{4}\}, \{\bar{1}, \bar{3}, \bar{5}\}\}$, $M/N(Q_2) = \{\bar{0} + N, \bar{3} + N\} = \{\{\bar{0}, \bar{2}, \bar{4}\}, \{\bar{1}, \bar{3}, \bar{5}\}\}$, $M/N(Q_3) = \{\bar{0} + N, \bar{5} + N\} = \{\{\bar{0}, \bar{2}, \bar{4}\}, \{\bar{1}, \bar{3}, \bar{5}\}\}$. Here $M/N(Q_1), M/N(Q_2)$ and $M/N(Q_3)$ are equal as sets. But $M/N(Q_1), M/N(Q_2)$ and $M/N(Q_3)$ considered as $(\mathbb{Z}_6^-, +, \cdot)$-ternary semimodules are not pairwise equal because $\bar{1} + N \notin M/N(Q_2)$ and $\bar{3} + N \notin M/N(Q_3)$ as $\bar{1} \notin Q_2$ and $\bar{3} \notin Q_3$. Also $\bar{3} + N \in M/N(Q_2)$ but $\bar{3} + N \notin M/N(Q_3)$ as $\bar{3} \notin Q_3$.

2.3 Maximal ternary semimodule homomorphisms

In this section, we introduce the concept of a maximal ternary $R$-semimodule homomorphism.

Definition 2.7. A ternary $R$-semimodule homomorphism $f : M \to M'$ will be called maximal if for each $a \in f(M)$ there exists a unique $q_a \in f^{-1}\{\{a\}\}$ such that $x + \ker f \subseteq q_a + \ker f$ for each $x \in f^{-1}\{\{a\}\}$ where $\ker f = \{x \in M : f(x) = 0_{M'}\}$.

Clearly, every $R$-module homomorphism and every maximal $R$-semimodule homomorphism is a maximal ternary $R$-semimodule homomorphism.

Lemma 2.8. If $f : M \to M'$ is a one-one ternary $R$-semimodule homomorphism, then $f$ is maximal.

Proof. Let $a = f(m) \in f(M)$ where $m \in M$. If $x \in f^{-1}\{\{a\}\}$, then $f(x) = a = f(m)$. Since $f$ is one-one, $x = m = q_a$ say. Now $x + \ker f \subseteq q_a + \ker f$ where $q_a = m \in f^{-1}\{\{a\}\}$ is a unique element as $f$ is one-one. \qed
The converse of Lemma 2.8 is not true.

**Example 2.9.** Let \( M = (\mathbb{Z}_6^+, +) \) and \( R = (\mathbb{Z}_6^+, +, \cdot) \). Then \( M \) and \( M' \) are ternary \( R \)-semimodules. Clearly, \( f : M \to M' \) defined by \( f(x) = r \) where \( x \equiv r \mod 6 \), \( 0 \leq r \leq 5 \), is an onto ternary \( R \)-semimodule homomorphism and \( \ker f = \{0, 6, 12, 18, \ldots \} \). For any \( \overline{a} \in M' \), there exists a unique \( q_a = a \in f^{-1}(\{\overline{a}\}) \) such that \( x + \ker f \subseteq q_a + \ker f \) for all \( x \in f^{-1}(\{\overline{a}\}) \). Hence \( f \) is a maximal ternary \( R \)-semimodule homomorphism. Clearly, \( f \) is not one-one.

P. J. Allen [1, Lemma 14, Lemma 15 and Theorem 16] has proved the results for semirings. These results have been extended for semimodules over semirings by J. N. Chaudhari and D. R. Bonde [19] and for ternary semirings by J. N. Chaudhari and K. J. Ingale [24]. However we extend the following Lemma 2.10, Lemma 2.13 and Theorem 2.14 for ternary semimodules over ternary semirings.

**Lemma 2.10.** If \( f : M \to M' \) is a maximal ternary \( R \)-semimodule homomorphism, then \( \ker f \) is a partitioning ternary subsemimodule of \( M \).

**Proof.** Since \( f \) is maximal, for each \( a \in M' \) there exists a unique \( q_a \in f^{-1}(\{a\}) \) such that \( x + \ker f \subseteq q_a + \ker f \) for each \( x \in f^{-1}(\{a\}) \). Take \( Q = \{q_a : a \in M'\} \). Clearly, \( \cup \{q_a + \ker f : q_a \in Q\} \subseteq M \). On the other hand, if \( m \in M \), then \( f(m) \in M' \). Now \( m \in f^{-1}([f(m)]) \) implies \( m \in m + \ker f \subseteq q_{f(m)} + \ker f \). Hence \( M \subseteq \cup \{q_a + \ker f : q_a \in Q\} \).

Now for \( q_a, q_b \in Q \), suppose that \( (q_a + \ker f) \cap (q_b + \ker f) \neq \phi \). Let \( q_a + k = q_b + k' \) for some \( k, k' \in \ker f \). Now \( a = f(q_a) + f(k) = f(q_a + k) = f(q_b + k') = f(q_b) + f(k') = b \).

Hence \( q_a = q_b \). Thus, \( \ker f \) is a partitioning ternary subsemimodule of \( M \). \( \square \)

The converse of the above Lemma 2.10 is not true.
Example 2.11. Consider the idempotent monoids $M = (\mathbb{Z}_0^-, \min)$ and $M' = (\{0, -1\}, \min)$. By Example 1.15 (7), $M, M'$ are ternary $R$-semimodules, where $R = (\mathbb{Z}_0^+, +, \cdot)$.
Then $f : M \to M'$ defined by

$$f(x) = \begin{cases} 
0 & \text{if } x \geq -5 \\
-1 & \text{if } x < -5 
\end{cases}$$

is an onto ternary $R$-semimodule homomorphism. Also $\ker f = \{0, -1, -2, -3, -4, -5\}$ is a partitioning ternary subsemimodule of $M$ with $Q = \{0, -6, -7, \cdots\}$. For $-1 \in M'$ there cannot exists any $q_{-1} \in f^{-1}(\{-1\})$ such that $x + \ker f \subseteq q_{-1} + \ker f$ for all $x \in f^{-1}(\{-1\})$. So $f$ is not a maximal ternary $R$-semimodule homomorphism.

Example 2.12. Let $M, M', R$ and $f$ be defined as in Example 2.9. Then $f$ is a maximal ternary $R$-semimodule homomorphism and clearly, $\ker f = \{0, 6, 12, 18, \cdots\}$ is a partitioning ternary subsemimodule of $M$.

Lemma 2.13. Let $M, M', f$ and $Q$ be as stated in Lemma 2.10. Let $q_a, q_b, q_c \in Q$ and $r_1, r_2 \in R$, then

(i) If $q_a + q_b + \ker f \subseteq q_c + \ker f$, then $a + b = c$.

(ii) If $r_1 r_2 q_a + \ker f \subseteq r_1 r_2 q_c + \ker f$, then $r_1 r_2 a = r_1 r_2 c$.

Proof. (i) Since $q_a + q_b \in q_a + q_b + \ker f \subseteq q_c + \ker f$, $q_a + q_b = q_c + k$ for some $k \in \ker f$.
Now $a + b = f(q_a) + f(q_b) = f(q_a + q_b) = f(q_c + k) = f(q_c) + f(k) = c$.

(ii) Since $r_1 r_2 q_a \in r_1 r_2 q_a + \ker f \subseteq r_1 r_2 q_c + \ker f$, $r_1 r_2 q_a = r_1 r_2 q_c + k'$ for some $k' \in \ker f$. Now $r_1 r_2 a = r_1 r_2 f(q_a) = f(r_1 r_2 q_a) = f(r_1 r_2 q_c + k') = f(r_1 r_2 q_c) + f(k') = r_1 r_2 f(q_c) = r_1 r_2 c$. □
Theorem 2.14. If \( f : M \rightarrow M' \) is an one-one or maximal ternary \( R \)-semimodule homomorphism, then \( M/\ker f(Q) \cong \text{Im}f \) where \( Q \) is as stated in Lemma 2.10.

Proof. If \( f : M \rightarrow M' \) is one-one, then by Lemma 2.8, \( f \) is maximal. If \( f \) is maximal, then by Lemma 2.10, \( \ker f \) is a partitioning ternary subsemimodule of \( M \). Define \( \overline{f} : M/\ker f(Q) \rightarrow \text{Im}f \) by \( \overline{f}(q_a + \ker f) = f(q_a) = a \) for each \( q_a \in Q \). If \( q_a + \ker f, q_b + \ker f \in M/\ker f(Q) \), then \( \overline{f}(q_a + \ker f) = \overline{f}(q_b + \ker f) \iff a = b \iff q_a + \ker f = q_b + \ker f \).

Hence \( \overline{f} \) is well defined and one-one.

For \( q_a + \ker f, q_b + \ker f \in M/\ker f(Q), r_1, r_2 \in R \), consider

(i) \( \overline{f}((q_a + \ker f) \oplus (q_b + \ker f)) = \overline{f}(q_c + \ker f) = c \) where \( q_c \) is a unique element in \( Q \) such that \( q_a + q_b + \ker f \subseteq q_c + \ker f \). By Lemma 2.13, \( a + b = c \). Now \( \overline{f}(q_a + \ker f) + \overline{f}(q_b + \ker f) = a + b = c = \overline{f}(q_c + \ker f) = \overline{f}((q_a + \ker f) \oplus (q_b + \ker f)) \).

(ii) \( \overline{f}(r_1 \odot r_2 \odot (q_a + \ker f)) = \overline{f}(q_d + \ker f) = d \) where \( q_d \) is a unique element in \( Q \) such that \( r_1r_2q_a + \ker f \subseteq q_d + \ker f \). By Lemma 2.13, \( r_1r_2a = d \). Therefore \( r_1 \odot r_2 \odot \overline{f}(q_a + \ker f) = r_1r_2a = d = \overline{f}(r_1 \odot r_2 \odot (q_a + \ker f)) \). Hence \( \overline{f} \) is a ternary \( R \)-semimodule isomorphism. Thus, \( M/\ker f(Q) \cong \text{Im}f \).

\[ \square \]

2.4 Steady ternary semimodule homomorphisms

In this section, we generalize the concept of steady \( R \)-semimodule homomorphism [21] to ternary \( R \)-semimodule homomorphism and hence obtain a relation between maximal and steady ternary \( R \)-semimodule homomorphisms.

Let \( M \) be a ternary \( R \)-semimodule. An equivalence relation \( \rho \) on \( M \) is said to be a ternary \( R \)-congruence relation if \( apb \) implies \( (a + c)\rho(b + c) \) for all \( c \in M \)
and \((r_1r_2a)\rho(r_1r_2b)\) for all \(r_1, r_2 \in R\). If \(f : M \to M'\) is a ternary \(R\)-semimodule homomorphism, then \(f\) induces a ternary \(R\)-congruence relation \(\equiv_f\) on \(M\) given by \(m \equiv_f m' \iff f(m) = f(m')\) where \(m, m' \in M\). If \(N\) is a ternary subsemimodule of a ternary \(R\)-semimodule \(M\), then \(N\) induces a ternary \(R\)-congruence relation \(\equiv_N\) on \(M\) given by \(m \equiv_N m' \iff \) there exist \(n, n' \in N\) such that \(m + n = m' + n'\) where \(m, m' \in M\). If \(f : M \to M'\) is a ternary \(R\)-semimodule homomorphism, then clearly \(m \equiv_{kerf} m'\) implies \(m \equiv_f m'\) where \(m, m' \in M\). Following example shows that the converse implication is not true.

**Example 2.15.** Let \(M, M', R\) and \(f\) be as stated in Example 2.11. Then \(f\) is an onto ternary \(R\)-semimodule homomorphism with \(kerf = \{0, -1, -2, -3, -4, -5\}\). Here \(-6 \equiv_f -7\) but \(-6 \equiv_{kerf} -7\) is not true.

**Definition 2.16.** A ternary \(R\)-semimodule homomorphism \(f : M \to M'\) will be called steady if \(\equiv_f\) coincides with \(\equiv_{kerf}\).

**Remark 2.4.1.** A steady ternary homomorphism \(f\) is one-one if and only if \(kerf = \{0_M\}\).

**Lemma 2.17.** If \(f : M \to M'\) is a maximal ternary \(R\)-semimodule homomorphism, then \(f\) is steady.

**Proof.** Let \(m \equiv_f m'\) where \(m, m' \in M\). Therefore \(f(m) = f(m') = n\) say. Since \(f\) is a maximal ternary \(R\)-semimodule homomorphism, there exists a unique \(q_n \in f^{-1}(\{n\})\) such that \(x + kerf \subseteq q_n + kerf\) for all \(x \in f^{-1}(\{n\})\). Now \(m, m' \in f^{-1}(\{n\})\) implies \(m = q_n + k_1\) and \(m' = q_n + k_2\) for some \(k_1, k_2 \in kerf\). Then \(m + k_2 = q_n + k_1 + k_2 = m' + k_1\). Now \(m \equiv_{kerf} m'\). Hence \(f\) is steady. \(\square\)

The converse of Lemma 2.17 is not true.
Example 2.18. Consider the ternary semiring $A = (\mathbb{Z}^-, gcd^-, lcm^-)$ as defined in Example 1.4(5). Then $A$ is a ternary semiring and hence $A$ is a ternary semimodule over the ternary semiring $R = (\mathbb{Z}^+, +, \cdot)$. Clearly, $f : A \to A$ defined by

$$f(a) = \begin{cases} 
0 & \text{if } a \text{ is even} \\
-1 & \text{if } a \text{ is odd,}
\end{cases}$$

is a ternary $R$-semimodule homomorphism. If $f$ is a maximal ternary $R$-semimodule homomorphism, then by Lemma 2.10, $ker f = \{0, -2, -4, ...\}$ is a partitioning ternary subsemimodule of $A$ and hence a partitioning ideal of $A$. But $ker f$ is not a partitioning ideal of $A$, because $-3, -5 \in Q = A \setminus ker f \cup \{0\}$ and $gcd^-\{-3, ker f\} = -1 = gcd^-\{-5, ker f\} \Rightarrow gcd^-\{-3, ker f\} \cap gcd^-\{-5, ker f\} \neq \phi$, but $-3 \neq -5$. Hence $f$ is not a maximal ternary $R$-semimodule homomorphism. If $m, m' \in A$ and $m \equiv_f m'$, then $f(m) = f(m')$. Clearly $gcd^-\{m, -2\} = gcd^-\{m', -2\}$ where $-2 \in ker f$. So $m \equiv_{ker f} m'$. Hence $f$ is a steady ternary $R$-semimodule homomorphism.

Theorem 2.19. Let $f : M \to M'$ be a ternary $R$-semimodule homomorphism. Then the following statements are equivalent.

1) $f$ is one-one;

2) $ker f = \{0\}$ and $f$ is maximal;

3) $ker f = \{0\}$ and $f$ is steady.

Proof. (1)$\Rightarrow$(2) Clearly $ker f = \{0\}$ and by Lemma 2.8, $f$ is maximal.

(2)$\Rightarrow$(3) It follows from Lemma 2.17.

(3)$\Rightarrow$(1) If $m, m' \in M$ and $f(m) = f(m')$, then $m \equiv_f m'$. Since $f$ is steady, $m + k_1 = m' + k_2$ for some $k_1, k_2 \in ker f = \{0\}$. Hence $f$ is one-one. \qed
**Theorem 2.20.** If $f : M \to M$ is an onto steady ternary $R$-semimodule homomorphism, then $f^k$ is steady for $k \geq 1$.

**Proof.** Clearly, $f^k$ is an onto ternary $R$-semimodule homomorphism for $k \geq 1$. To show $f^k$ is steady. We use induction on $k$. If $k = 1$, then result is given. Now suppose that $f^k$ is steady. Let $m, m' \in M$ such that $m \equiv f^{k+1} m'$. Then $f^{k+1}(m) = f^{k+1}(m') \Rightarrow f^k(f(m)) = f^k(f(m')) \Rightarrow f(m) \equiv f_{k} f(m') \Rightarrow f(m) \equiv \ker f^k f(m')$, as $f^k$ is steady. So there exist $x, x' \in \ker f^k$ such that $f(m) + x = f(m') + x'$. Since $f$ is an onto, there exist $y, y' \in M$ such that $f(y) = x$ and $f(y') = x'$. So $f(m) + f(y) = f(m') + f(y') \Rightarrow f(m + y) = f(m' + y') \Rightarrow m + y \equiv f m' + y' \Rightarrow m + y \equiv \ker f m' + y'$, as $f$ is steady. Therefore there exist $z, z' \in \ker f$ such that $m + y + z = m' + y' + z' \ldots (1)$. Now $f^{k+1}(y + z) = f^{k+1}(y) + f^{k+1}(z) = f^k(f(y)) + f^k(f(z)) = f^k(x) + f^k(0) = 0 + 0 = 0$. So $y + z \in \ker f^{k+1}$. Similarly, $y' + z' \in \ker f^{k+1}$. Hence from $(1)$, $m \equiv \ker f^{k+1} m'$. So $f^{k+1}$ is steady. Thus by mathematical induction $f^k$ is steady for $k \geq 1$. \qed