In wireless communication, the travelling electromagnetic (EM) wave nature of the signal is not reported comprehensively in the literature. The majority of available work had analyzed the nature of the signal propagating through the channel in time and frequency domains only, whereas the travelling EM wave, actually propagating through the channel, have both time and spatial dependencies. Therefore, a different approach is needed by which both the time and spatial dependencies of EM wave can be examined. Very little work had been reported in the past as well as reported in the literature in which this aspect of EM wave has been analyzed, but the time dependency of frequency and spatial dependency of propagation constant associated with the EM wave had not been included in these available work, which will affect the performance of the channel in practical conditions. Actually these dependencies enable the signal become non-stationary in both time and spatial domains. The FRFT can play a decisive role in analyzing the EM wave with such dependencies due to its inherent characteristic of localizing the non-stationary signals.

5.1 ELECTROMAGNETIC WAVE-PROPAGATION

For an electromagnetic wave, the expression of electric field intensity $n$ for transverse electromagnetic (TEM) mode, which is function of both time and spatial domain variable, is given as-

$$f(x, t) = \cos(k x - \omega t) \quad (5.1.1)$$

The FRFT has already found many applications in the areas of signal processing and optics. While originally formulated in an optical context, the same result equally applicable to electromagnetic waves satisfying the linear wave equation. The far-field diffraction pattern is the Fourier transform of the diffracting object as depicted by one of the central results of diffraction theory. It is possible to further generalize this result by establishing that the field patterns at closer distances are the fractional Fourier transforms of the diffracting object [52, 53, and 110]. As the wave-field propagates, its distribution evolves through fractional transforms of increasing orders. So, the application of FRFT in wave propagation need to be investigated more.
The travelling wave had also been analyzed with the help of dual-domain transform based on Fourier transform (DDT-FT) [36], by transforming the travelling wave expression twice, first from time domain to frequency domain and second in spatial coordinate domain to propagation constant domain, and taking Fourier transform in both cases. The results of DDT-FT had been applied to estimate the transfer function of a wireless communication channel with multiple phase shifts and established that the channel had acted as a band-pass filter in both the frequency and the propagation constant domain.

However, the time dependency of frequency components and spatial dependency of propagation constant had not been considered in [36], e.g., the case when the electromagnetic wave is propagating in a medium having different permittivity for different portions of the medium (viz. in satellite communication, the EM wave has to travel from earth station to satellite through different layers of earth’s atmosphere and space) and also suffers Doppler shift in frequency due to motion of transmitter and/or receiver. This imposes a set of constraint with a requirement that the electromagnetic wave is now required to be analyzed in ‘time-frequency’ plane and ‘spatial coordinate-propagation constant’ plane. The appropriate and perfect solution is the FRFT. Therefore, another type of dual-domain transform is needed to be defined, which is certainly based on the FRFT.

5.1.1 Dual-Domain Transform Based on FT (DDT-FT)

In the class of signals represented by traveling-wave equations, the signal has existence in two domains: the time domain 't' and the spatial domain 'x', as shown in (5.1.1). More specifically, the signal has dependence on two propagation constants: the temporal propagation constant 'ω', and the spatial propagation constant 'k'. So, a transform is needed to map the behavior of the signal in the 'x − t' domain into the 'k − ω' domain. The Fourier transform only maps the time 't' domain into the frequency 'ω' domain and is incapable of performing the 2 × 2 dimensional mapping as required in this case.

The usual Fourier transform of a signal \( f(t) \) is given as –

\[
F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} \, dt
\]  

This expression of FT exists because of the orthogonal nature of the time and frequency variables. From (5.1.1), in the argument of TEM wave equation, it become evident that spatial variable and propagation constant can also be considered as orthogonal quantity, in the same manner as already assumed in the case.
of time and frequency. This approach and assumption initiates the definition of Fourier transform of the
signals that is defined in spatial domain only [34] and such definition is given as -

\[ F(k) = \int_{-\infty}^{\infty} f(x) e^{j k x} \, dx \quad (5.1.3) \]

Based on the (5.1.2) and (5.1.3), the DDT-FT of a function \( f(x, t) \) was defined in [34] and given as –

\[ F(k, \omega) = \iiint_{-\infty}^{\infty} f(x, t) e^{i (k x - \omega t)} \, dx \, dt \quad (5.1.4) \]

Fourier transform measures the correlations between the signal \( f(t) \) and an infinite set of pure
sinusoidal signals in the time domain, while the DDT-FT measures the correlations between the signal
\( f(x, t) \) and an infinite set of pure sinusoidal traveling waves. This is the main difference between the
DDT-FT and the other known extensions to the Fourier transform.

### 5.1.2 Dual-Domain Transform Based on FRFT (DDT-FRFT)

As illustrated above, for the time dependency of frequency and spatial domain variable
dependency of propagation constant, DDT-FT will not able to resolve the problems associated with the
influence introduced on the EM wave when it propagates in such medium. With the beauty of FRFT and
its characteristics, which considers and addresses these dependencies can be utilized to propose and
define a new approach named as dual domain transform based on FRFT (DDT-FRFT). For the
function \( f(x, t) \), the proposed definition of DDT-FRFT can be expressed as –

\[ D_{2}[f(x, t)] = F_{\alpha}(\beta, u) = \iint_{-\infty}^{\infty} f(x, t) \, K_{\alpha}(t, u) \, K_{\alpha}(x, \beta) \, dx \, dt \quad (5.1.5) \]

Where, \( K_{\alpha}(t, u) \) is the kernel of FRFT as given in (1.3.15), \( K_{\alpha}(x, \beta) \) is the analogous kernel defined for
FRFT defined in \( 'x - \beta' \) plane which is given as –

\[ K_{\alpha}(x, \beta) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{\frac{i}{2}(x^2 + \beta^2) \cot \alpha - 2 \beta x \csc \alpha} ; & \text{if } \alpha \neq n \pi \\ \delta(x - \beta) ; & \text{if } \alpha = 2n \pi \\ \delta(x + \beta) ; & \text{if } \alpha = (2n + 1) \pi \end{cases} \quad (5.1.6) \]
And, $F_\alpha(\beta, u)$ represent the DDT-FRFT of EM wave $f(x, t)$. This dual-domain transform based on FRFT, represented as ‘$D_3$’, will be a technique which maps the signal defined in time and spatial ‘$x$’ domain to anywhere in time (spatial coordinate) – frequency (propagation constant) plane.

**5.1.2.1 Shifting property of DDT-FRFT**

The shifting property of DDT-FRFT is needed to be defined to facilitate the application of this transform in channel modeling. Basically, the shifting property of DDT-FRFT is based on the shifting property of FRFT, which is given as

$$
\mathcal{F}(t - \tau) = F(u - \cos \alpha) e^{i \frac{\tau^2}{2} \sin \alpha} \cos \alpha - u \sin \alpha)
$$

(5.1.7)

Where, $F(u)$ is the FRFT of $f(t)$ and $f(t - \tau)$ is the delayed signal of $f(t)$ by delay ‘$\tau$’.

This property illustrates that a shift of function in time domain will result into a shift in fractional Fourier domain along with an introduction of phase in fractional Fourier domain. Based on the definition (5.1.7), the shifting property of DDT-FRFT can be defined.

**Definition:** The DDT-FRFT of a function $f(x, t)$, shifted in time and spatial domains by ‘$\tau$’ and ‘$\delta$’ respectively is given by -

$$
D_3[f(x - \delta, t - \tau)] = e^{i \sin \alpha \left(\frac{\tau}{2} \cos \alpha - ur\right)} e^{i \sin \alpha \left(\frac{\delta^2}{2} \cos \alpha - \beta \delta\right)} F_\alpha(\beta - \delta \cos \alpha, u - \tau \cos \alpha)
$$

(5.1.8)

**Proof:** Considering the LHS of (5.1.8) -

$$
D_3[f(x - \delta, t - \tau)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x - \delta, t - \tau) K_\alpha(t, u) K^*_\alpha(x, \beta) \, dt \, dx
$$

(5.1.9)

The above expression shows that DDT-FRFT performs two times FRFT of signal $f(x - \delta, t - \tau)$, first with respect to variable ‘$t$’ and then with respect to the variable ‘$x$’. So applying the shifting property of FRFT, as shown in (5.1.7), it can be easily shown that –

$$
D_3[f(x - \delta, t - \tau)] = e^{i \sin \alpha \left(\frac{\tau}{2} \cos \alpha - ur\right)} e^{i \sin \alpha \left(\frac{\delta^2}{2} \cos \alpha - \beta \delta\right)} F_\alpha(\beta - \delta \cos \alpha, u - \tau \cos \alpha)
$$

(5.1.10)
5.1.3 Application of DDT-FRFT in Channel Modeling

The shifting property of DDT-FRFT (5.1.8) can be modified as-

\[ D_{\alpha}[f(x - \delta, t - \tau)] = e^{j\phi} e^{j\delta} F_{\alpha}(\beta - \delta \cos \alpha, u - \tau \cos \alpha) \quad (5.1.11) \]

Where,

\[ \phi = \sin \alpha \left( \frac{\tau^2}{2} \cos \alpha - u \tau \right) \quad (5.1.12) \]

And,

\[ \phi = \sin \alpha \left( \frac{\delta^2}{2} \cos \alpha - \beta \delta \right) \quad (5.1.13) \]

By assuming,

\[ \phi_n = \phi + \phi \quad (5.1.14) \]

The output of the channel having ‘N’ multipath, each path with an associated delay ‘\tau’ and associated path ‘\delta’, will be given as-

\[ \frac{0}{p} = \sum_{n=0}^{N-1} A_n e^{j\phi_n} F_{\alpha}(\beta - \delta_n \cos \alpha, u - \tau_n \cos \alpha) \quad (5.1.15) \]

Where, ‘\( A_n \)’ is the amplitude of phase shifted component. From this expression it is visible that the channel imposes following parameters on the signal propagating through it-

- Introduces a gain ‘\( A_n \)’
- A time delay \( \tau_n \) and path delay \( \delta_n \) when transformation is not carried.
- A phase shift ‘\( \phi_n \)’ in FRFT domain, which becomes equal to the phase shift in FT domain for angle parameter \( \alpha = \pi / 2 \).
- A delay \( (\delta_n \cos \alpha, \tau_n \cos \alpha) \) in FRFT domain, which becomes equal to zero at angle parameter \( \alpha = \pi / 2 \).

Since, FRFT converts into FT at angle parameter \( \alpha = \pi / 2 \). Hence, the output of channel is derived for this value of \( \alpha \) and is given as-

\[ \frac{0}{p} = \sum_{n=0}^{N-1} A_n e^{j\phi_n, a=\pi/2} F(k, \omega) \quad (5.1.16) \]
This expression resembles with the output expression given for DDT-FT in [36]. The effects introduced by the channel to the propagating signal are:

- Introduces a gain term ‘$A_n$’
- Introduces a phase shift term in Fourier domain

Here, the output signal which encounters multiple phase shifts in a wireless communication channel can be represented by:

$$A_0 e^{j\phi_0} F(k, \omega) + A_1 e^{j\phi_1} F(k, \omega) + A_2 e^{j\phi_2} F(k, \omega) + \cdots$$  \hspace{1cm} (5.1.17)

Where, $\phi_n$ is the various phase shifts in FT domain. So the transfer function (TF) of the channel can be calculated as:

$$TF = \frac{\text{Channel o/p}}{\text{Channel i/p}} = \frac{A_0 e^{j\phi_0} F(k, \omega)+A_1 e^{j\phi_1} F(k, \omega)+\cdots}{F(k, \omega)}$$  \hspace{1cm} (5.1.18)

By expanding each exponential term as a power series in (5.1.18) and rearranging the expression:

$$TF = (A_0 + A_1 + \cdots) + j(A_0 \phi_0 + A_1 \phi_1 + \cdots) + \frac{j^2}{2} (A_0 \phi_0^2 + A_1 \phi_1^2 + \cdots) + \cdots$$  \hspace{1cm} (5.1.19)

Comparing it with the Fourier transform of a rectangular pulse in time-domain, it can be easily seen that the TF of channel is the transform of pulse-like characteristic in time domain and pulse-like characteristic in spatial domain.

As the FRFT of a rectangular function is calculated from $\alpha = 0$ to $\alpha = \pi/2$, the rectangular function changes its shape from rectangular to sync function. Similarly, the channel response to a signal is analyzed in FRFT domain from $\alpha = 0$ to $\alpha = \pi/2$, it is clear from the discussion above that at $\alpha = 0$, channel imposes time & path delay to the signal; at $\alpha = \pi/2$, channel imposes phase shift term in frequency and propagation constant domain i.e., filtering action will be performed in these domains; and channel will introduce not only phase term but also a delay term in that domain for $0 < \alpha < \pi/2$.

### 5.1.4 Simulation Comparison of DDT-FT & DDT-FRFT

To establish the utility of DDT-FRFT, a case study of chirping in frequency and propagation constant has been undertaken with the help of simulation. Here, for simplicity only the linear variation of
propagation constant with spatial variable is assumed. In this study, the propagation constant ‘k’ and instantaneous frequency ‘ω’, as in (5.1.1), having respective dependencies are given as-

\[
\omega = 2\pi(f_0 + a \ t) \tag{5.1.20}
\]

\[
k = 2\pi(\mu_0 + b \ x) \tag{5.1.21}
\]

For these dependencies of ‘ω’ and ‘k’ on time and spatial variable, the travelling electromagnetic wave has the expression as –

\[
f(x,t) = \cos(2\pi(\mu_0 + b \ x) x - 2\pi(f_0 + a \ t) t) \tag{5.1.22}
\]

The values of \(f_0\), \(a\), \(\mu_0\) and \(b\) are chosen as 2, 10, 1.5 and 20 respectively. The DDT-FRFT has been observed for a duration double of time-period of EM wave in time domain and double of spatial wavelength in spatial domain. First the DDT-FRFT of travelling wave expression is evaluated for \(\alpha = \pi/2\), as it is same as DDT-FT (because DDT-FRFT is converted in DDT-FT at \(\alpha = \pi/2\)), as shown in Figure-5.1. Next, the DDT-FRFT is determined for \(\alpha = 0.2\), as shown in Figure-5.2. The results are shown below-

![Figure-5.1: The magnitude of DDT-FT (DDT-FRFT at \(\alpha = \pi/2\)) of EM wave.](image)

[100]
The simulation results reveal the superiority of DDT-FRFT over DDT-FT especially in noisy conditions. The noise corrupted signal can also be detected by using DDT-FRFT because by applying this proposed transform technique on the EM wave results in an equivalent transformed signal having a clear peak as can be depicted from Figure-5.2 whereas this phenomenon is not observing by DDT-FRFT.

Next section is dedicated to the utility of FRFT for radar signal processing area for the estimation of different parameters associated with the target like – range and velocity. The range of the target from the radar is analyzed in terms of delay associated with received echo signal in comparison of transmitted signal and velocity by which target is moving relative to the radar is calculated from the Doppler shift introduced in the echo signal. So the next section includes the analysis of existing methods with their deficiencies and limitations which enables the introduction of a proposed method.

5.2 RADAR SIGNAL PROCESSING

In RADAR systems the ambiguity function plays an important role in estimating the delay and Doppler shift introduced in received signal. The ambiguity function based on Fourier transform has not been able to resolve the problem of detecting the chirp (used as transmitted signal), which leads to the establishment
of FRFT based ambiguity function. Though, it imposes a large computational complexity in the problems
where only delay and Doppler shift is needed to estimate.

5.2.1 Existing Methods, Ambiguity Function

The estimation of delay and Doppler shift parameters of a particular target, require the evaluation
of cross-ambiguity function between the transmitted and received echo signal. The narrowband limit of
this function has been well studied in both the time and frequency domains and is fundamental to modern
radar and sonar system design. The Fourier transform based ambiguity function is normally used in pulse-
Doppler radar for detecting targets despite the occurrence of phenomenon known as Doppler smearing,
which limits the performance of this method [14].

With the advent of the FRFT, The interpretation of FRFT as rotation in time-frequency plane and
its relation with ambiguity function was introduced in [76]. Subsequently, the ambiguity function has
been defined for both narrowband and wideband, where the discrimination between narrowband and
wideband was made on the basis of comparison of the received signal bandwidth to its center frequency
[18].

In radar signal processing, usually the linear frequency modulated (LFM) signals are used as the
transmitted signals (e.g., in pulse Doppler radar, in FMCW radar etc.), which can also be termed as chirp
signals. As LFM signals are having time dependant instantaneous frequency, the fractional Fourier
transform (FRFT) can very well be utilized in radar signal processing. In recent past many technique
based on FRFT have been introduced for the analysis of LFM signals for delay and Doppler shift
estimation of target [15, 45, 47, 111, 115 and 148]. In all these propositions, if only the delay and Doppler
shift estimation are required then the computation become more complex as either it requires the
computation of cross-ambiguity function of transmitted and received echo or by superimposing some
other technique with FRFT.

Basically cross-ambiguity function is a modified version of the cross-correlation of transmitted
signal and received echo along with a phase term which exhibits the Doppler shift information. If the
signal transmitted is $s(t)$ and received echo is $r(t)$. Then second order cross-ambiguity function based
on Fourier transform can be defined as-

$$A(\tau, u) = \int_{-\infty}^{\infty} s(t) r^*(t + \tau) e^{-j\tau u} \, dt$$

(5.2.1)
Where, \( r^*(t) \) is the complex conjugate of received echo signal \( r(t) \).

LFM signal (chirp signal) along with its phase, instantaneous frequency and chirp rate can be given \[115\] as -

Chirp Signal:  
\[
x(t) = e^{i \frac{2}{2} \pi (a t^2 + b t)}
\] (5.2.2)

Phase:  
\[
\emptyset = 2 \pi (a t^2 + b t)
\] (5.2.3)

Instantaneous frequency:  
\[
f_i = \frac{1}{2 \pi} \frac{d\emptyset}{dt} = 2 a t + b
\] (5.2.4)

Chirp rate:  
\[
2 a
\] (5.2.5)

For transmitted signal \( s(t) \) and received echo signal \( r(t) \), the cross-ambiguity function based on FRFT \[111\] can be defined as-

\[
A(\tau, u) = \int_{-\infty}^{\infty} s(t) r^*(t + \tau) K_\alpha(t, u) \, dt
\] (5.2.6)

Where, \( K_\alpha(t, u) \) is representing the kernel of the FRFT as given in (1.3.15). To reduce the complexity of the existing systems, another method based on FRFT for estimation of delay and Doppler shift is proposed in the next sub-section.

### 5.2.2 Proposed Delay & Doppler Shift Estimation from Echo Signal

Let the transmitted signal is a LFM signal having chirp rate ‘2a’ and center frequency as ‘b’ and time duration ‘T’ is given as-

\[
x(t) = e^{i \frac{2}{2} \pi (a t^2 + b t)} \, rect \left[ \frac{t}{T} \right]
\] (5.2.7)

Where, \( rect \left[ \frac{t}{T} \right] \) represents a rectangular function having duration from \(-T/2\) to \(T/2\)

The received signal from a moving target is given by-

\[
s(t) = x(t - \tau_t) e^{i \frac{2}{2} \pi b (t - \tau_t)}
\] (5.2.8)
Where, \( \tau_t \) is the time-dependent delay due to a target which is moving with a certain velocity, in turn introduces a Doppler component in received phase term and is given as-

\[
\tau_t = \tau + m t
\]  

(5.2.9)

Here ‘m’ is representing the Doppler shift term and is given as-

\[
m = \frac{2v}{c}
\]  

(5.2.10)

Where, \( v \) is the speed of the target and \( c \) is the speed of light in medium between radar and target. By putting this expression of time-dependent delay in expression of received echo-

\[
s(t) = e^{j2\pi \left[ t^2 \left(a (1-m^2) + 2t(b-a \tau) + [a \tau^2 - 2b \tau]\right) \right]} \text{rect} \left[ \frac{t-t_1}{t_2} \right]
\]  

(5.2.11)

The FRFT of received echo indicated by ‘\( S_\alpha(u) \)’ has been derived as -

\[
S_\alpha(u) = \sqrt{1 - \frac{1}{\cot \alpha}}
\]

\[
\int_{t_1}^{t_2} e^{j\frac{2\pi}{2} \left[ t^2 \left(a (1-m^2) + \frac{\cot \alpha}{2} \right) + t(2(1-m) (b-a \tau) - u \csc \alpha a) + \left[a \tau^2 - 2b \tau + u^2 \cot \alpha \frac{a}{2}\right]\right]} dt
\]  

(5.2.12)

The limits of integration \( t_1 \) and \( t_2 \) in above integral are given as-

\[
t_1 = \tau_t - \frac{\tau}{2} \quad \text{and} \quad t_2 = \tau_t + \frac{\tau}{2}
\]  

(5.2.13)

If the maximum of ‘\( S_\alpha(u) \)’ occurs at particular values of \( \alpha \) and \( u \), say \((\alpha_0, u_0)\), then the delay and Doppler component can be determined by using the following relationships –

\[
a(1-m)^2 + \frac{\cot \alpha_0}{2} = 0
\]  

(5.2.14)

And,

\[2(1-m)(b-a \tau) = u_0 \csc \alpha_0 \]  

(5.2.15)

The next sub-section is dedicated to the simulation verification of the analytical approach presented in this sub-section.
5.2.2.1 Simulation verification

For the verification purpose by simulation following values of the parameters related to transmitted and received echo signals are chosen,

- Chirp rate \(2a\) = 4,
- Center frequency \(b\) = 5, and
- Time duration \(T\) = 2.

The assumed value of delay ‘\(\tau\)’ is 1.5 and Doppler parameter ‘\(m\)’ is 3. For these values of \(a\), \(b\), \(m\) and \(\tau\), the values of \(\alpha\) and \(u\) are calculated by using (5.2.14) and (5.2.15) and obtained values are

- \(\alpha = 3.07918\)
- \(u = -0.49896\).

The simulated results are shown below:

\[\text{(a)}\]
The comparison between calculated and simulated values of $\alpha$ and $u$ are shown in Table-5.1 where the similarity between the analytical and simulated results is in conformity of establishment of the proposed method.

Table-5.1: Delay and Doppler parameter associated with received echo

<table>
<thead>
<tr>
<th>Assumed Value</th>
<th>Calculated Value</th>
<th>Simulated Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>$b$</td>
<td>$\tau$</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>1.5</td>
</tr>
</tbody>
</table>

5.3 SUMMARY

In this chapter, a novel dual-domain transform based on fractional Fourier transform (DDT-FRFT) has been proposed. Proposed technique is providing a better solution in a situation where EM wave encounters a Doppler shift due to motion of transmitter and/or receiver along with having different
phase velocity for different mediums in which the wave is propagating. Simulation results shown in section 5.1.4 are in the conformity of this statement, because at angle parameter ‘α’ associated with FRFT having value of ‘π/2’, i.e., the case of Fourier transform, the transformed quantity does not have a clear peak (Figure 5.1), while for α = 0.2, a clear peak is observable in the magnitude response of DDT-FRFT of travelling wave (Figure 5.2). It clearly demonstrate that in noisy conditions, due to a clear peak, the analysis of travelling wave with DDT-FRFT corresponding to α = 0.2 gives better result than DDT-FT. Similarly, the application of the DDT-FRFT is discussed for the channel modeling. Also, it has been established that a multipath wireless channel exhibits pulse-like characteristic in time-domain as well as in spatial-domain, i.e., for α = 0. This also suggests that the channel acts as a band-pass filter in both the frequency domain and propagation constant domain, i.e., for α = π/2. However, the channel introduces a delay in that domain along with the filtering action, for the values of α lie in between these two limits, i.e., α = 0 and α = π/2.

Thereafter, a novel method has been proposed to calculate the delay and Doppler shift parameter associated with a received echo signal in radar communication. Here, delay and Doppler shift parameters are directly estimated by observing the peak in magnitude response of FRFT of received echo signal. Corresponding to this peak the angle parameter value ‘α₀’ and transform domain variable ‘u₀’ are observed. From these observed value of α₀ and u₀, the delay and Doppler shift parameters are estimated as also shown in Table-5.1. The simulation and estimated results shown above are in agreement and consolidating the theory proposed in this article.