Dynamics of cascade three-level system interacting with the classical and quantized field

MIHIR RANJAN NATH\(^1\), SURAJIT SEN\(^1\) and GAUTAM GANGOPADHYAY\(^2\)

\(^1\)Department of Physics, Guru Charan College, Silchar 788 004, India
\(^2\)S N Bose National Centre for Basic Sciences, JD Block, Sector III, Salt Lake City, Kolkata 700 098, India

Email mmath.95@rediffmail.com; ssen55@yahoo.com, gautam@bose.res.in

MS received 4 June 2002, revised 31 May 2003, accepted 31 July 2003

Abstract. We study the exact solutions of the cascade three-level atom interacting with a single mode classical and quantized field with different initial conditions of the atom. For the semiclassical model, it is found that if the atom is initially in the middle level, the time-dependent populations of the upper and lower levels are always equal. This dynamical symmetry exhibited by the classical field is spoiled on quantization of the field mode. To reveal this non-classical effect, a Euler matrix formalism is developed to solve the dressed states of the cascade Jaynes-Cummings model (JCM). Possible modification of such an effect on the collapse and revival phenomenon is also discussed by taking the quantized field in a coherent state.

Keywords. Symmetry breaking, three-level JCM, Euler matrix, collapse revival

PACS Nos 42.50.Ar; 42.50.Ct; 42.50.Dv

1. Introduction

Over the decades, studies of the population inversion of the two, three and multilevel systems have been proved to be an important tool to understand various fundamental aspects of quantum optics [1,2]. Many interesting coherent phenomena are observed if the number of involved levels exceeds two. In particular, the three-level system exhibits a rich class of coherent phenomena such as two-photon coherence [3], double resonance process [4], three-level super-radiance [5], coherent multistep photo-ionization [6], trilevel echoes [7], STIRAP [8], resonance fluorescence [9], quantum jump [10], quantum zero effect [11] etc. [12–16]. From these studies, it is intuitively clear that the atomic initial conditions of the three-level system can generate diverse quantum optical effects which are not usually displayed by a two-level system [17–20]. The idea of the present investigation is to enunciate the three-level system for various initial conditions while taking the field mode to be either classical or quantized. In this paper the three-level system is modelled by the matrices which are spin-one representation of SU(2) group. A dressed-atom approach is developed where the Euler matrix is used to construct the dressed states. We discuss the time
Mihir Ranjan Nath, Surajit Sen and Gautam Gangopadhyay

development of the probabilities both for the semiclassical model and the cascade JCM for various initial conditions and point out the crucial changes. Finally the collapse and revival phenomenon is presented taking the quantized field initially in a coherent state.

The subsequent sections of the paper are organized as follows. To put our treatment in proper perspective, in §2 we have derived the probabilities of three levels taking the field as a classical field. The cascade JCM and its solution in the rotating wave approximation (RWA) is presented in §3. In §4 we have numerically analysed the time-dependent atomic populations and compared with the semiclassical situation by taking the quantized field initially in a number state and in a coherent state. Finally, in conclusion, we highlight the outcome of our paper and make some pertinent remarks.

2. The semiclassical model

The Hamiltonian to describe the semiclassical problem of a cascade three-level system interacting with a single mode classical field is

\[ \mathcal{H} = \hbar \Omega_0 I_z + \frac{\hbar \Omega_1}{\sqrt{2}} (I_+ e^{-i \omega t} + I_- e^{i \omega t}), \]  

(1)

where \( I_\pm \) represent the spin-one representation of \( SU(2) \) matrices corresponding to the cascade three-level system with equal energy gaps (\( \hbar \Omega_0 \)) between the states, namely,

\[ I_+ = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \]  

(2a)

\[ I_- = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \]  

(2b)

\[ I_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \]  

(2c)

\( \hbar \Omega_1 \) is the interaction energy between the three-level system with the classical field mode of frequency, \( \omega \), in RWA. Let the solution of the Schrödinger equation,

\[ i\hbar \frac{\partial \psi}{\partial t} = \mathcal{H} \psi, \]  

(3)

with Hamiltonian (1) is given by

\[ \psi(t) = C_+(t)|+\rangle + C_0(t)|0\rangle + C_-(t)|-\rangle, \]  

(4)

where \( C_+(t), C_0(t) \) and \( C_-(t) \) are the time-dependent normalized amplitudes with the eigenfunctions given by.
Dynamics of cascade three-level system

\[ |+\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad (5a) \]

\[ |0\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad (5b) \]

\[ |\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \quad (5c) \]

We now proceed to calculate the probability amplitudes of the three states. Substituting eq. (4) into eq. (3) and equating the coefficients of \(|+\rangle\), \(|0\rangle\) and \(|\rangle\) from both sides we obtain

\[ i\dot{C}_+(t) = \frac{\omega_0}{\sqrt{2}} |+\rangle \langle 0| C_0(t) + \frac{\omega_1}{\sqrt{2}} |+\rangle \langle \rangle C_0(t), \quad (6a) \]

\[ i\dot{C}_0(t) = \frac{\omega_0}{\sqrt{2}} |0\rangle \langle +| C_+(t) + \frac{\omega_1}{\sqrt{2}} |0\rangle \langle \rangle C_-(t), \quad (6b) \]

\[ i\dot{C}_-(t) = \frac{\omega_1}{\sqrt{2}} |\rangle \langle +| C_+(t) - \frac{\omega_0}{\sqrt{2}} |\rangle \langle \rangle C_-(t), \quad (6c) \]

where the dot represents the derivative with respect to time.

Let the solutions of eqs (6a)-(6c) are of the following form:

\[ C_+(t) = A_+ \exp(is_+t), \quad (7a) \]

\[ C_0(t) = A_0 \exp(is_0t), \quad (7b) \]

\[ C_-(t) = A_- \exp(is_-t), \quad (7c) \]

where \(A_+\) are the time-independent constants to be determined. Plucking back eqs (7a)-(7c) in eqs (6a)-(6c) we obtain

\[ \begin{align*}
(s_0 - \omega + \omega_0)A_+ + \frac{1}{\sqrt{2}} \omega_1 A_0 &= 0, \\
2s_0 A_0 + \frac{1}{\sqrt{2}} \omega (A_+ + A_-) &= 0, \\
(s_0 + \omega - \omega_0)A_- + \frac{1}{\sqrt{2}} \omega_1 A_0 &= 0.
\end{align*} \]

In deriving eqs (8a)-(8c), the time independence of the amplitudes \(A_+, A_-\) and \(A_0\) are ensured by invoking the conditions \(s_+ = s_0 - \omega\) and \(s_- = s_0 + \omega\). The solution of (8) readily yields

\[ Pramana – J. Phys., Vol. 61, No. 6, December 2003 \]
Mihir Ranjan Nath, Surajit Sen and Gautam Gangopadhyay

\[ s_0 = 0 \]  \hspace{1cm} (9a)

\[ s_0 = \pm \sqrt{(\omega - \omega_0)^2 + \omega_i^2} (\equiv \pm \Omega) \]  \hspace{1cm} (9b)

and we have three values of \( s_+ \) and \( s_- \), namely,

\[ s_+^1 = -\omega, \quad s_+^2 = \Omega - \omega, \quad s_+^3 = -\Omega - \omega \]  \hspace{1cm} (10a)

\[ s_-^1 = \omega, \quad s_-^2 = \Omega + \omega, \quad s_-^3 = -\Omega + \omega. \]  \hspace{1cm} (10b)

Using (10), eqs (7) can be written as

\[ C_+(t) = A_+ \exp[-i\omega t] + A_+^b \exp[i(\Omega - \omega)t] + A_+^c \exp[i(-\Omega - \omega)t] \]  \hspace{1cm} (11a)

\[ C_0(t) = A_0^b + A_0^c \exp[i\Omega t] + A_0^d \exp[-i\Omega t] \]  \hspace{1cm} (11b)

\[ C_-(t) = A_-^1 \exp(i\omega t) + A_-^2 \exp[i(\Omega + \omega)t] + A_-^3 \exp[i(-\Omega + \omega)t]. \]  \hspace{1cm} (11c)

where \( A_+ \) are the constants to be calculated from the following initial conditions:

**Case I:** Let us consider at \( t = 0 \), the atom is in the lower level, i.e., \( C_+(0) = 0, C_0(0) = 0, C_-(0) = 1 \). Using eqs (6) and (11), the time-dependent probabilities of the three levels are given by

\[ |C_+(t)|^2 = \frac{\omega_i^4}{\Omega^4} \sin^4 \Omega t/2, \]  \hspace{1cm} (12a)

\[ |C_0(t)|^2 = \frac{\omega_i^2}{2\Omega^4} [4(\omega - \omega_0)^2 \sin^4 \Omega t/2 + \Omega^2 \sin^2 \Omega t], \]  \hspace{1cm} (12b)

\[ |C_-(t)|^2 = \frac{1}{\Omega^2} [(\omega_i^2 \sin^2 \Omega t/2 + \Omega^2 \cos \Omega t)^2 + (\omega - \omega_0)^2 \Omega^2 \sin^2 \Omega t]. \]  \hspace{1cm} (12c)

**Case II:** If we choose the atom initially in the middle level, i.e., \( C_+(0) = 0, C_0(0) = 1, C_-(0) = 0 \), the corresponding probabilities of the levels are given by

\[ |C_+(t)|^2 = \frac{\omega_i^4}{2\Omega^4} [4(\omega - \omega_0)^2 \sin^4 \Omega t/2 + \Omega^2 \sin^2 \Omega t] = |C_-(t)|^2, \]  \hspace{1cm} (13a)

\[ |C_0(t)|^2 = \frac{1}{\Omega^4} \frac{4(\omega - \omega_0)^4}{\Omega^2} \sin^4 \Omega t/2 + \frac{4(\omega - \omega_0)^2}{\Omega^2} \sin^2 \Omega t/2 \cos \Omega t + \cos^2 \Omega t. \]  \hspace{1cm} (13b)

Here we note that, unlike the previous case, the probabilities of the upper and lower levels are equal.

**Case III:** When the atom is initially in the upper level, i.e., \( C_+(0) = 1, C_0(0) = 0, C_-(0) = 0 \), we obtain the following occupation probabilities in the three levels:

\[ 1092 \hspace{1cm} Pramana – J. Phys., Vol. 61, No. 6, December 2003 \]
Dynamics of cascade three-level system

\[
|C_{+}(t)|^2 = \frac{1}{\Omega^2} \left[ (\omega^2 - \Lambda_0 \Delta)^2 \Omega^2 + (\omega - \Delta)^2 \sin^2 \Omega t \right], \\
|C_{0}(t)|^2 = \frac{\alpha^2_0}{2\Omega^2} \left[ 4(\omega - \Delta)^2 \sin^2 \Omega t / 2 + \Omega^2 \sin^2 \Omega t \right], \\
|C_{-}(t)|^2 = \frac{\alpha^2_0}{\Omega^2} \sin^2 \Omega t / 2.
\]

We note that the probability of the middle level for Case III is precisely identical to that of Case I while those of the upper and lower levels are interchanged.

3. Cascade Jaynes–Cummings model

Here we consider the cascade three-level system interacting with a single mode quantized field. The cascade JCM system in the rotating wave approximation [17,18] is described by the Hamiltonian

\[
H = \hbar \omega (a^\dagger a + I_0) + (\Delta I_0 + \hbar \omega (I_0 a + I_0 a^\dagger)),
\]

where \(a^\dagger\) and \(a\) are the creation and annihilation operators, \(g\) the coupling constant and \(\Delta = \hbar (\omega_0 - w)\) the detuning frequency. It is easy to check that both diagonal and interaction parts of the Hamiltonian commute with each other. The eigenfunction of this Hamiltonian is given by

\[
|\psi_{0}(t)\rangle = \sum_{n=0}^{\infty} \left[ C_{n+1}^+ (t) |n+1, -\rangle + C_{0}^+ (t) |n, 0\rangle + C_{-1}^+ (t) |n-1, +\rangle \right].
\]

We note that the Hamiltonian couples the atom-field states \(|n-1, +\rangle, |n, 0\rangle\) and \(|n+1, -\rangle\), where \(n\) represents the number of photons of the field. The interaction part of the Hamiltonian (15) can also be written in the matrix form

\[
H = \begin{pmatrix}
-\Delta & g\hbar \sqrt{n+1} & 0 \\
g\hbar \sqrt{n+1} & 0 & g\hbar \sqrt{n} \\
0 & g\hbar \sqrt{n} & \Delta
\end{pmatrix}.
\]

At resonance (\(\Delta = 0\)), the eigenvalues of the Hamiltonian are given by \(\lambda_+ = g\hbar \sqrt{2n+1}\), \(\lambda_0 = 0\) and \(\lambda_- = -g\hbar \sqrt{2n+1}\) with the corresponding dressed eigenstates

\[
\begin{pmatrix}
|n, 1\rangle \\
|n, 2\rangle \\
|n, 3\rangle
\end{pmatrix} = T
\begin{pmatrix}
|n+1, -\rangle \\
|n, 0\rangle \\
|n-1, +\rangle
\end{pmatrix}.
\]

In eq. (18), the dressed states are constructed by rotating the bare states with the Euler matrix \(T\) parametrized as

\[
T = \begin{pmatrix}
\alpha_{11} & \alpha_{12} & \alpha_{13} \\
\alpha_{21} & \alpha_{22} & \alpha_{23} \\
\alpha_{31} & \alpha_{32} & \alpha_{33}
\end{pmatrix}.
\]
where

\[ a_{11} = \cos \psi \cos \phi - \cos \theta \sin \phi \sin \psi, \]
\[ a_{12} = \cos \psi \sin \phi + \cos \theta \cos \phi \sin \psi, \]
\[ a_{13} = \sin \psi \sin \theta, \]
\[ a_{21} = -\sin \psi \cos \phi - \cos \theta \sin \phi \cos \psi, \]
\[ a_{22} = -\sin \psi \sin \phi + \cos \theta \cos \phi \cos \psi, \]
\[ a_{23} = \cos \psi \sin \theta, \]
\[ a_{31} = \sin \theta \sin \phi, \]
\[ a_{32} = -\sin \theta \cos \phi, \]
\[ a_{33} = \cos \theta. \]

The evaluation of its various elements is presented in the appendix and here we quote the results as follows:

\[ a_{11} = \sqrt{\frac{n+1}{4n+2}}, \quad a_{12} = \frac{1}{\sqrt{2}}, \quad a_{13} = \sqrt{\frac{n}{4n+2}}, \]
\[ a_{21} = -\sqrt{\frac{n}{2n+1}}, \quad a_{22} = 0, \quad a_{23} = \sqrt{\frac{n+1}{2n+1}}, \]
\[ a_{31} = \sqrt{\frac{n+1}{4n+2}}, \quad a_{32} = -\frac{1}{\sqrt{2}}, \quad a_{33} = \sqrt{\frac{n}{4n+2}}. \]

(20)

The time-dependent probability amplitudes of the three levels are given by

\[
\begin{bmatrix}
C_{n+1}^{\phi+1}(t) \\
C_{n}^{\phi}(t) \\
C_{n}^{\phi-1}(t)
\end{bmatrix} = T^{-1} \begin{bmatrix}
e^{-i\Omega t} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & e^{i\Omega t}
\end{bmatrix} T \begin{bmatrix}
C_{n+1}^{\phi+1}(0) \\
C_{n}^{\phi}(0) \\
C_{n}^{\phi-1}(0)
\end{bmatrix},
\]

(21)

where \( \Omega_n = \sqrt{2n+1} \). In the following we consider different initial condition of the atom with the quantized field in a number state \( |n\rangle \).

Case IV: Here we consider that the atom is initially polarized in the lower level and the combined atom-field state is \( |n+1, -\rangle \), i.e., \( C_{n+1}^{\phi+1}(0) = 0, C_{n}^{\phi}(0) = 0, C_{n}^{\phi-1}(0) = 1 \). Using eqs (20) and (21) the time-dependent atomic population of the three levels are given by

\[
|C_{n+1}^{\phi+1}(t)|^2 = \frac{4n(n+1)}{(2n+1)^2} \sin^4 \Omega_{nt}/2, \]

(22a)

\[
|C_{n}^{\phi}(t)|^2 = \frac{(n+1)}{(2n+1)} \sin^2 \Omega_{nt}, \]

(22b)

\[
|C_{n}^{\phi-1}(t)|^2 = 1 - 4 \left[ \frac{n(n+1)}{(2n+1)^2} + \frac{(n+1)^2}{(2n+1)^2} \cos^2 \Omega_{nt}/2 \right] \sin^2 \Omega_{nt}/2. \]

(22c)

Case V: At \( t = 0 \) when the atom is in the middle level and the combined atom-field state is \( |n, 0\rangle \), i.e., \( C_{n}^{\phi+1}(0) = 0, C_{n}^{\phi}(0) = 1, C_{n}^{\phi-1}(0) = 0 \), we find

\[
|C_{n}^{\phi-1}(t)|^2 = \frac{n}{(2n+1)} \sin^2 \Omega_{nt}, \]

(23a)
Dynamics of cascade three-level system

\[ |C'_0(t)|^2 = \cos^2 \Omega t \]  
\[ |C^{n+1}_0(t)|^2 = \frac{n(n+1)}{(2n+1)^2} \sin^2 \Omega t. \]  
\[ |C^{n+1}_0(t)|^2 = \frac{(n+1)}{(2n+1)^2} \sin^2 \Omega t \]  

Case VI: \( C^{n+1}_0(0) = 1, C'_0(0) = 0, C^{n+1}_0(0) = 0 \)

If the atom is initially in the upper level and the atom-field state is \(|n-1, +\rangle\), i.e., \( C^{n-1}_+(0) = 1, C'_0(0) = 0, C^{n-1}_-(0) = 0 \) we obtain the following probabilities:

\[ |C^{n-1}_+(t)|^2 = 1 - 4 \left[ \frac{n(n+1)}{(2n+1)^2} + \frac{n^2}{(2n+1)^2} \cos^2 \Omega t / 2 \right] \sin^2 \Omega t / 2, \]  
\[ |C^0_0(t)|^2 = \frac{n}{(2n+1)^2} \sin^2 \Omega t, \]  
\[ |C^{n+1}_-(t)|^2 = \frac{4n(n+1)}{(2n+1)^2} \sin^4 \Omega t / 2. \]

Finally we note that, at resonance, for large value of \( n \) the probabilities of Case IV, V and VI are identical to Case I, II and III, respectively indicating the validity of the correspondence principle.

4. Numerical results

To explore the physical content, we now proceed to analyse the probabilities of the semi-classical model and the cascade JCM numerically.

For the classical field at resonance, the time evolution of the probabilities \( |C'_+(t)|^2 \) (solid line), \( |C'_0(t)|^2 \) (dashed line) and \( |C'_-(t)|^2 \) (dotted line) corresponding to Case I, II and III, respectively are shown in figure 1. We note that for the cases with atom initially in the lower and upper level, which are displayed in figure 1a and 1c respectively, the probabilities \( |C'_+(t)|^2 \) and \( |C'_-(t)|^2 \) can attain a maximum value equal to unity while \( |C'_0(t)|^2 \) cannot. If we compare these two figures, the time-dependent populations of the lower and upper levels are different by a phase lag corresponding to the initial condition of population. This clearly shows that the probabilities oscillate between the levels \(|+\rangle\) and \(|-\rangle\) alternatively at a Rabi frequency of \( \Omega_1 = \Omega_0 / 2\pi \). On the contrary, the plot of Case II where the atom is initially in the middle level depicted in figure 1b shows that the system oscillates with a Rabi frequency of \( \Omega_1 = \Omega_0 / \pi \) such that the probabilities of \(|+\rangle\) and \(|-\rangle\) states are always equal. When the atom is initially in the middle level, the exactly sinusoidal resonant field interacts with the atom in such a way that the upper and lower levels are dynamically treated on an equal footing. This dynamically symmetrical distribution of population between the upper and lower levels is possible because of the classical field.

For quantized field we consider the time evolution of the probabilities in two different situations of initial condition of the field: (a) when the field is in a number state and (b) when the field is in a coherent state.
Mihir Ranjan Nath, Surajit Sen and Gautam Gangopadhyay

Figure 1. The time evolution of the probabilities of the semiclassical model corresponding to Cases I, II and III. The symmetric pattern of evolution is evident from figures 1a and 1c which are in opposite phase.

Figure 2. The time evolution of the probabilities of the cascade JCM corresponding to Cases IV, V and VI. Figures 2a and 2c depict that the symmetry exhibited by the semiclassical model is spoiled on quantization of the field mode.

(a) For the cascade JCM, the probabilities of Case IV, V and VI are plotted in figure 2 when the field is in a number state with \( n = 1 \) and \( g = 0.1 \). In figure 2a we note that for Case IV, i.e., when the atom is initially in the lower level, the Rabi frequency of oscillation is \( \nu' = \Omega_n / 2 \pi \). However, unlike Case I of the semiclassical model, the probabilities \( |C^\pm |^2 \) never become unity. On the other hand, figure 2b illustrates the probabilities of Case V, i.e., when the atom is initially in the middle level, where the system oscillates with a Rabi frequency \( \nu'' = \Omega_n / \pi \) and once again, in contrast with the corresponding semiclassical situation in Case II, the probabilities of the upper and lower level are not equal. The probabilities of Case VI, i.e., when the atom is initially in the upper level, depicted in figure 2c shows that although it possesses the same Rabi frequency \( \nu'' \), the pattern of oscillation is not out of phase of Case IV. To compare with one can look back the semiclassical interaction where we have shown that in Case I the pattern of oscillation of upper (lower) level population is precisely identical to the lower (upper) level population of Case III.

To understand the implications of such dynamical symmetry breaking qualitatively, various bounds on the probabilities are given (see table 1).

We note that for the semiclassical model, the symmetric evolution of the probabilities results in identical bounds for Cases I and III as shown in table 1. On quantization of the field mode, the bounds corresponding to Cases IV and VI are no longer similar although those for Cases II and V remain the same. At resonance, for large values of \( n \), eqs (22)–(24) of Cases IV, V and VI are precisely identical to eqs (12)–(14) of Case I, II and III, respectively and we recover the same bounds of the semiclassical model.

(b) Finally, we consider that the atom is interacting with the quantized field mode in a coherent state. The coherently averaged probabilities of Cases IV, V and VI are given by
Dynamics of cascade three-level system

Table 1.

<table>
<thead>
<tr>
<th>Case</th>
<th>Semiclassical model</th>
<th>Case</th>
<th>Cascade JCM</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$0 \leq</td>
<td>C_{-}(t)</td>
<td>^2 \leq 1$, $0 \leq</td>
</tr>
<tr>
<td>II</td>
<td>$0 \leq</td>
<td>C_{-}(t)</td>
<td>^2 &lt; 1$, $0 \leq</td>
</tr>
<tr>
<td>III</td>
<td>$0 \leq</td>
<td>C_{-}(t)</td>
<td>^2 &lt; 1$, $0 \leq</td>
</tr>
</tbody>
</table>

\[
\langle P_{+}(t) \rangle = \sum_{n} P_{n} |C_{+}^{n-1}(t)|^2 , \quad (25a)
\]
\[
\langle P_{0}(t) \rangle = \sum_{n} P_{n} |C_{0}^{n}(t)|^2 , \quad (25b)
\]
\[
\langle P_{-}(t) \rangle = \sum_{n} P_{n} |C_{-}^{n+1}(t)|^2 , \quad (25c)
\]

where $P_{n} = \exp[-\bar{n}\bar{n}^{n}/n!$ be the Poisson distribution function and $\bar{n}$ be mean photon number. For all numerical purpose we choose $\bar{n} = 0.1$. We have studied extensively for various values of $\bar{n}$. Figures are given only for $\bar{n} = 50$. Figures 3–5 display the numerical plots of (25) where the collapse and revival of the Rabi oscillation is clearly evident. For low $\bar{n}$ and when the atom is in the middle level the symmetrical values of population of upper and lower levels are not observed until $\bar{n}$ is very high as given in the figures. However, even if $\bar{n} = 50$, the numerical values of the time dependent populations of the upper and lower level are not exactly equal although very close and becomes exactly equal in the limit $\bar{n} \to \infty$. We further note that, if the atom is initially polarized either in the upper or in the lower level, it exhibits similar population oscillation, which is different from the case if it is initially polarized in the middle level. The reproduction of this result analogous to the semiclassical model shows the proximity of the coherent state with large $\bar{n}$ to the classical field.

When the field is quantized, population oscillation depends on the occupation number, $n$, of the field state, for example, $\cos(g \sqrt{2n+1} \tau)$. For a statistical distribution of field state, the spontaneous factor $1$ plays a dominant role when $n$ is low. For an initial number state of the field when $n$ is slightly higher than $1$, the upper and lower levels of the atom are not treated dynamically on an equal footing even when the atom is initially in the middle level. This fine graining of the quantized distribution of photons over the number states $\{|n\}$ generates a complex interference between individual Rabi oscillation corresponding to each $n$ and plays a role until $n$ is very large compared to $1$ and effectively acts as a classical field and thereby the semiclassical situation is satisfied. Note that for an initial vacuum field, i.e., $n = 0$ for the number state and $\bar{n} = 0$ for the coherent state, with the atom initially in the middle level, it cannot go to the upper level at all and the population will oscillate between the lower and middle levels with Rabi frequency $\Omega_{0}$. This asymmetry is still present when the field is in a coherent state with a Poissonian photon distribution with low average photon number, $\bar{n}$, which is generally not symmetric around $\bar{n}$.
distribution is almost symmetric, a Gaussian, around an $\bar{n}$ if $\bar{n}$ is very large which is the case of a classical field. In that situation the upper and lower levels of the atom are treated dynamically on an equal footing and maintains the symmetrical distribution of population in upper and lower levels.

5. Conclusion

We conclude by recapitulating the essential content of our investigation. At the outset we have sculpted the semiclassical model by choosing the spin-one representation of $SU(2)$ group and have calculated the transition probabilities of the three levels. It is shown that at resonance, if the atom is initially polarized in the lower or in the upper level, the various atomic populations oscillate quite differently when it is initially populated in the middle level. When the atom is initially populated in the middle level, the classical field interacts in such a way that the populations of the upper and lower levels are always equal. This
Dynamics of cascade three-level system
dynamically symmetrical populations of the upper and lower levels are destroyed due to
the quantization of the field. To show this quantum behavior, a dressed-atom approach is
presented to solve the cascade JCM. Finally we discuss the restoration of the symmetry
taking the quantized field in a coherent state with large average photon number, the closest
state to the classical state. Although the collapse and revival and some other non-classical
features are well-studied in the context of two-level systems, the above dynamical break­
ing of symmetry due to the quantization of the field has no two-level analog. We hope that
this dynamical behavior in the cascade three-level system should show its signature on the
time-dependent profile of the second-order coherence of the quantized field which will be
discussed elsewhere. The dressed-atom approach developed here may also find its appli­
cation in the V- and A-type three-level systems where the nature of the symmetry should
be different from the cascade system.

Appendix
At resonance, the interaction part of the Hamiltonian of the three-level system is given by
\[
H_{\text{int}} = \begin{pmatrix}
0 & g\sqrt{n+1} & 0 \\
g\sqrt{n+1} & 0 & g\sqrt{n} \\
0 & g\sqrt{n} & 0
\end{pmatrix},
\]
where the eigenvalues are \( \lambda_+ = g\sqrt{2n+1} \), \( \lambda_0 = 0 \) and \( \lambda_- = -g\sqrt{2n+1} \). The Euler ma­
trix \( T \), diagonalizes the Hamiltonian as \( H_0 = \mathbf{T}H_{\text{int}}\mathbf{T}^{-1} \), is given by eq. (19). Using the
trick \( (H_{\text{int}} - \lambda_i)\{X_j\} = 0 \), where \( \{X_j\} \) is the column matrix of \( \mathbf{T} \), corresponding to the
eigenvalue \( \lambda_+ \) we have
\[
\begin{pmatrix}
-g\sqrt{2n+1} & g\sqrt{n+1} & 0 \\
g\sqrt{n+1} & -g\sqrt{2n+1} & g\sqrt{n} \\
0 & g\sqrt{n} & -g\sqrt{2n+1}
\end{pmatrix}
\begin{pmatrix}
\alpha_{11} \\
\alpha_{12} \\
\alpha_{13}
\end{pmatrix} = 0.
\]
These linear equations readily yield
\[
\alpha_{12} = \frac{\sqrt{2n+1}}{\sqrt{n}} \alpha_{13}, \quad \alpha_{12} = \frac{\sqrt{2n+1}}{\sqrt{n+1}} \alpha_{11} \quad \text{and} \quad \alpha_{11} = \frac{\sqrt{n+1}}{\sqrt{n}} \alpha_{13}. \tag{A3}
\]
Using the normalization condition
\[
\alpha_{11}^2 + \alpha_{12}^2 + \alpha_{13}^2 = 1, \tag{A4}
\]
we get \( \alpha_{11} = \sqrt{(n+1)/(4n+2)} \), \( \alpha_{12} = \sqrt{(2n+1)/(4n+2)} = 1/\sqrt{2} \) and \( \alpha_{13} = \sqrt{n/(4n+2)} \). Similarly, corresponding to the eigenvalues \( \lambda_0 \) and \( \lambda_- \) we can obtain other
elements of \( T \), namely,
\[
\alpha_{21} = -\sqrt{\frac{n}{2n+1}}, \quad \alpha_{22} = 0 \quad \text{and} \quad \alpha_{23} = \sqrt{\frac{n+1}{2n+1}}, \tag{A5}
\]
\[
\alpha_{31} = \sqrt{\frac{n+1}{4n+2}}, \quad \alpha_{32} = -\frac{1}{\sqrt{2}} \quad \text{and} \quad \alpha_{33} = \sqrt{\frac{n}{4n+2}}. \tag{A6}
\]
One can now easily read off the Euler’s angles

\[
\sin \theta = \frac{\sqrt{3n + 2}}{4n + 2}, \quad \sin \phi = \frac{\sqrt{n + 1}}{3n + 2} \quad \text{and} \quad \sin \psi = \frac{n}{3n + 2}.
\]

(A7)

Acknowledgements

The authors are thankful to the University Grants Commission, New Delhi for extending partial support. SS is grateful to Prof. George W S Hou for inviting him to the National Taiwan University, Taiwan, where part of the work was carried out.

References


Pramana – J. Phys., Vol. 61, No. 6, December 2003
Effect of field quantization on Rabi oscillation of equidistant cascade four-level system

MIHIR RANJAN NATH, TUSHAR KANTI DEY, SURAJIT SEN and GAUTAM GANGOPADHYAY

1Department of Physics, Guru Charan College, Silchar 788 004, India
2S N Bose National Centre for Basic Sciences, JD Block, Sector III, Salt Lake City, Kolkata 700 098, India
*Corresponding author
E-mail: minath95@rediffmail.com, tkdey54@rediffmail.com,
sen55@yahoo.com, gautam@bose.res.in

MS received 23 February 2007, revised 27 July 2007, accepted 28 August 2008

Abstract. We have exactly solved a model of equidistant cascade four-level system interacting with a single mode radiation field both semiclassically and quantum mechanically by exploiting its similarity with Jaynes–Cummings model. For the classical field, it is shown that the Rabi oscillation of the system initially in the first level (second level) is similar to that of the system when it is initially in the fourth level (third level). We then proceed to solve the quantized version of the model where the dressed state is constructed using a six-parameter four-dimensional matrix and show that the symmetry exhibited in the Rabi oscillation of the system for the semiclassical model is completely destroyed on the quantization of the cavity field. Finally, we have studied the collapse and revival of the system for the cavity field-mode in a coherent state to discuss the restoration of symmetry and its implication is discussed.

Keywords. Rabi oscillation, four-level system; collapse and revival.

PACS Nos 42.50.Aw; 42.50.ct; 42.50.Dv

1. Introduction

Over the decades, the theory of electron spin resonance (ESR) has been regarded as the key model to understand various fundamental aspects of the semiclassical two-level system [1]. Its fully quantized version, namely, the two-level Jaynes–Cummings model (JCM), has also been proven to be a useful theoretical laboratory to address many subtle issues of the light–atom interaction which eventually gives birth to the cavity electrodynamics [1,2]. A natural but non-trivial extension of the JCM is the three-level system and it exhibits a wide variety of quantum-optical phenomena such as two-photon coherence [3], resonance Raman scattering [4], double resonance process [5], population trapping [6], three-level super radiance [7],

141
Mihir Ranyan Nath et al

three-level echoes [8], STIRAP [9], quantum jump [10], quantum zeno effect [11] etc. As a straightforward generalization of the three-level system, the multi-level system interacting with monochromatic laser is also extensively investigated [12–17]. Thus, it is understood that the increase of the number of level leads to the emergence of a plethora of phenomena and the upsurge of ongoing investigations of the four-level system is undoubtedly to predict more phenomena. For example, out of different configurations of the four-level system, the tripod configuration has come into the purview of recent studies particularly because it exhibits the phenomenon of the electromagnetically induced transparency (EIT) [18–25] which also received experimental confirmation [26–29]. Such a system is proposed to generate the non-Abelian phases [30], qubit rotation [31], coherent quantum switching [32], coherent controlling of nonlinear optical properties [33], embedding two qubits [34] etc. These developments lead to the careful scrutiny of all possible configurations of the four-level system including the cascade four-level system which we shall discuss here.

In the recent past, the equidistant cascade four-level system interacting with the semiclassical and quantized field was discussed mainly within the framework of generalized N-level system [35–40]. The other variant of this configuration, often referred to as Tavis–Cummings model, is studied to construct possible controlled unitary gates relevant for the quantum computation [41,42]. However, these treatments are not only devoid of the explicit calculation of the probabilities for all possible initial conditions [43,44], but they also bypass the comparison between the semiclassical and the quantized models which is crucial to discern the exact role of the field quantization on the population oscillation. In this work we have developed a dressed atom approach of calculating the probabilities with all possible initial conditions especially in the spirit of the basic theory of the ESR model and JCM taking the field to be either monochromatic classical or quantized field [1]. This work is the natural extension of our previous works on the equidistant cascade three-level model [44] where it is explicitly shown that the symmetric pattern observed in the population dynamics for the classical field is completely spoilt on the quantization of the cavity mode.

The remaining sections of the paper are organized as follows. In §2 we discuss the equidistant cascade four-level system modeled by the generators of the spin-$\frac{1}{2}$ representation of $SU(2)$ group and then study its Rabi oscillation with different initial conditions taking interacting field to be the classical field. Section 3 deals with the solution of the four-level system taking the cavity field mode to be the quantized mode. In §4 we compare the Rabi oscillation of the system of the semiclassical model with that of the quantized field and discuss the collapse and revival phenomenon and its implications. Finally, in §5 we summarize our results and discuss the outlook of our investigation.

2. The semiclassical cascade four-level system

The Hamiltonian of the equidistant cascade four-level system is given by

\[ H(t) = \hbar \omega_0 J_3 + \hbar \kappa (J_+ \exp(-i\Omega t) + J_- \exp(i\Omega t)), \]

(1)

Rabi oscillation of equidistant cascade four-level system

where \( J_+ \), \( J_- \) and \( J_3 \) are the generators of the spin-\( \frac{3}{2} \) representation of SU(2) group given by

\[
J_+ = \begin{bmatrix}
0 & \sqrt{3} & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & \sqrt{3} \\
0 & 0 & 0 & 0 \\
\end{bmatrix}, \quad J_- = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & \sqrt{3} & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & \sqrt{3} \\
\end{bmatrix},
\]

\[
J_3 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -\frac{\sqrt{3}}{2} & 0 \\
0 & 0 & 0 & -\frac{\sqrt{3}}{2} \\
\end{bmatrix}
\]

In eq (1), \( \hbar \omega_0 \) is the equidistant energy gap between the levels, \( \Omega \) is the frequency of the classical mode and \( \kappa \) is the coupling constant of the light-atom interaction respectively. The time evolution of the system is described by the Schrödinger equation

\[
\frac{\hbar}{i} \frac{\partial \Psi}{\partial t} = H(t)\Psi,
\]

where the above time-dependent Hamiltonian in the matrix form is given by

\[
H(t) = \begin{bmatrix}
\frac{3}{2} \hbar \omega_0 & \sqrt{3} \hbar \kappa \exp[-i\Omega t] & 0 & 0 \\
\sqrt{3} \hbar \kappa \exp[i\Omega t] & \frac{1}{2} \hbar \omega_0 & 2 \hbar \kappa \exp[-i\Omega t] & 0 \\
0 & 2 \hbar \kappa \exp[i\Omega t] & -\frac{1}{2} \hbar \omega_0 & \sqrt{3} \hbar \kappa \exp[-i\Omega t] \\
0 & 0 & \sqrt{3} \hbar \kappa \exp[i\Omega t] & -\frac{3}{2} \hbar \omega_0 \\
\end{bmatrix}.
\]

To find the amplitudes, let the solution of the Schrödinger equation corresponding to this Hamiltonian is given by

\[
\Psi(t) = C_1(t) |1\rangle + C_2(t) |2\rangle + C_3(t) |3\rangle + C_4(t) |4\rangle,
\]

where \( C_1(t), C_2(t), C_3(t) \) and \( C_4(t) \) are the time-dependent normalized amplitudes with basis states

\[
|1\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad |2\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad |3\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad |4\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},
\]

respectively. The Schrödinger equation in eq. (3) can be written as

\[
\hbar \frac{\partial \tilde{\Psi}}{\partial t} = \tilde{H} \tilde{\Psi},
\]

where the time-independent Hamiltonian is given by

\[
\tilde{H} = -\hbar U^\dagger U + U^\dagger H(t)U,
\]
with the unitary operator \( U(t) = e^{-\frac{i}{\hbar} \mathcal{H} t} \). The rotated wave function appearing in eq. (7) is obtained by the unitary transformation

\[
\tilde{\Psi}(t) = U(t) \Psi(t) = e^{-i\frac{\Delta t}{2}} C_1(t) |1\rangle + e^{-i\frac{\Delta t}{2}} C_2(t) |2\rangle + e^{i\frac{\Delta t}{2}} C_3(t) |3\rangle + e^{i\frac{\Delta t}{2}} C_4(t) |4\rangle .
\]

We thus note that the amplitudes are simply modified by a phase term and hence do not contribute to the probabilities. The time-independent Hamiltonian in eq. (8) is given by

\[
\tilde{\mathcal{H}} = \hbar \begin{bmatrix} \frac{3}{2} \Delta & \sqrt{3} \kappa & 0 & 0 \\
\sqrt{3} \kappa & \frac{3}{2} \Delta & 2 \kappa & 0 \\
0 & 2 \kappa & -\frac{3}{2} \Delta & \sqrt{3} \kappa \\
0 & \sqrt{3} \kappa & -\frac{3}{2} \Delta & -\frac{3}{2} \Delta \end{bmatrix}
\]

where \( \Delta = \omega_0 - \Omega \). At resonance (\( \Delta = 0 \)), the eigenvalues of the Hamiltonian are given by \( \lambda_1 = -\lambda_4 = -3\kappa \) and \( \lambda_2 = -\lambda_3 = -\hbar \kappa \) respectively which can also be generated by the transformation

\[
diag(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = T_0 \tilde{\mathcal{H}} T_0^{-1},
\]

where \( T_0 \) is the transformation matrix given by

\[
T_0 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\
a_{21} & a_{22} & a_{23} & a_{24} \\
a_{31} & a_{32} & a_{33} & a_{34} \\
a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}
\]

The different elements of matrix, which preserves the orthogonality, are given by

\[
a_{11} = C_1 C_5 + S_1 S_3 S_4 S_5,
\]

\[
a_{12} = -C_1 S_5 S_6 + S_1 C_3 C_6 + S_1 S_3 S_4 S_6,
\]

\[
a_{13} = S_1 S_3 C_4,
\]

\[
a_{14} = -C_1 S_5 C_6 - S_1 C_3 C_6 + S_1 S_3 S_4 S_5 C_6,
\]

\[
a_{21} = -S_1 C_2 C_6 + (C_1 C_2 S_3 - S_2 C_3) S_4 S_5,
\]

\[
a_{22} = S_1 S_2 S_5 S_6 + (C_1 C_2 C_3 + S_2 S_3) C_6 + (C_1 C_2 C_3 - S_2 C_3) S_4 S_5 S_6,
\]

\[
a_{23} = (C_1 C_2 S_3 - S_2 C_3) C_4,
\]

\[
a_{24} = S_1 C_2 S_5 S_6 - (C_1 C_2 C_3 + S_2 S_3) S_6 + (C_1 C_2 C_3 - S_2 C_3) S_4 S_5 S_6,
\]

\[
a_{31} = -S_1 S_2 C_5 + (C_1 S_2 S_3 + C_2 C_3) S_4 S_5,
\]

\[
a_{32} = S_1 S_2 S_5 S_6 + (C_1 S_2 C_3 - C_2 S_3) C_6 + (C_1 S_2 S_3 + C_2 C_3) S_4 S_5 S_6,
\]

\[
a_{33} = (C_1 S_2 S_3 + C_2 C_3) C_4,
\]

\[
a_{34} = S_1 S_2 S_5 S_6 - (C_1 S_2 C_3 - C_2 S_3) S_6 + (C_1 S_2 S_3 + C_2 C_3) S_4 S_5 S_6,
\]

\[
a_{41} = C_4 S_5.
\]
Rabi oscillation of equidistant cascade four-level system

\[ \alpha_{42} = c_4 c_5 s_6, \]
\[ \alpha_{43} = -s_4, \]
\[ \alpha_{44} = c_4 c_5 c_6. \] (13)

where \( s_i = \sin \theta_i \) and \( c_i = \cos \theta_i \) \((i = 1, 2, 3, 4, 5, 6)\). A straightforward calculation gives various angles to be

\[ \theta_1 = \arccos \left( -\sqrt{\frac{2}{5}} \right), \quad \theta_2 = \frac{3\pi}{4}, \quad \theta_3 = -\frac{\pi}{2}, \]
\[ \theta_4 = -\arcsin \left( \sqrt{\frac{3}{8}} \right), \quad \theta_5 = \arcsin \left( \frac{1}{\sqrt{5}} \right), \quad \theta_6 = \frac{\pi}{3}. \] (14)

The time-dependent probability amplitudes of the four-levels are given by

\[ \begin{bmatrix} C_1(t) \\ C_2(t) \\ C_3(t) \\ C_4(t) \end{bmatrix} = T_\alpha^{-1} \begin{bmatrix} e^{-i\lambda_1 t} & 0 & 0 & 0 \\ 0 & e^{-i\lambda_2 t} & 0 & 0 \\ 0 & 0 & e^{-i\lambda_3 t} & 0 \\ 0 & 0 & 0 & e^{-i\lambda_4 t} \end{bmatrix} T_\alpha \begin{bmatrix} C_1(0) \\ C_2(0) \\ C_3(0) \\ C_4(0) \end{bmatrix}. \] (15)

Later in §4, we proceed to analyze the probabilities of the four levels numerically for four distinct initial conditions, namely, Case I: \( C_1(0) = 1, C_2(0) = 0, C_3(0) = 0, C_4(0) = 0 \), Case II: \( C_1(0) = 0, C_2(0) = 1, C_3(0) = 0, C_4(0) = 0 \), Case III: \( C_1(0) = 0, C_2(0) = 0, C_3(0) = 1, C_4(0) = 0 \) and Case IV: \( C_1(0) = 0, C_2(0) = 0, C_3(0) = 0, C_4(0) = 1 \), respectively.

3. The Jaynes–Cummings model of cascade four-level system

We now consider the equidistant cascade four-level system interacting with a monochromatic quantized cavity field. The Hamiltonian of such a system in the rotating wave approximation (RWA) is given by

\[ H = \hbar(\omega J_3 + \Omega a^\dagger a) + \hbar(\Delta J_3 + g(J_+ a + J_- a^\dagger)). \] (16)

This is an archetype JCM where the Pauli matrices are replaced by the spin-\( \frac{1}{2} \) representation of \( SU(2) \) group. Using the algebra of the \( SU(2) \) group and that of the field mode it is easy to see that the two parts of the Hamiltonian shown in the parenthesis of eq. (16) commute with each other indicating that they have the simultaneous wave function. Let the eigenfunction corresponding to this Hamiltonian is given by

\[ |\Psi_\alpha(t)\rangle = \sum_{n=0}^{\infty} \left[ C_1^{n+2}(t)|n+2, 1\rangle + C_2^{n+1}(t)|n+1, 2\rangle + C_3^n(t)|n, 3\rangle + C_4^{n-1}(t)|n-1, 4\rangle \right], \] (17)

where \( n \) represents the number of photons in the cavity field. The Hamiltonian couples the atom-field states \( |n+2, 1\rangle, |n+1, 2\rangle, |n, 3\rangle, \) and \( |n-1, 4\rangle \) respectively.
At resonance $\Delta = 0$, the interaction part of the Hamiltonian in the matrix form is given by

$$\begin{bmatrix}
0 & \sqrt{3(n+2)} & 0 & 0 \\
\sqrt{3(n+2)} & 0 & 2\sqrt{n+1} & 0 \\
0 & 2\sqrt{n+1} & 0 & \sqrt{3n} \\
0 & 0 & \sqrt{3n} & 0
\end{bmatrix}, \quad (18)$$

with the eigenvalues

$$\begin{align*}
\lambda_{1q} &= -\lambda_{4q} = -gh\sqrt{5(1+n)+b}, \\
\lambda_{2q} &= -\lambda_{3q} = -gh\sqrt{5(1+n)-b},
\end{align*} \quad (19a,b)$$

respectively where $b = \sqrt{23+16n(2+n)}$. The dressed eigenstates are constructed by rotating the bare states as

$$\begin{bmatrix}
|n, 1\rangle \\
|n, 2\rangle \\
|n, 3\rangle \\
|n, 4\rangle
\end{bmatrix} = T_n \begin{bmatrix}
|n+2, 1\rangle \\
|n+1, 2\rangle \\
|n, 3\rangle \\
|n-1, 4\rangle
\end{bmatrix}, \quad (20)$$

where $T_n$ is similar to the aforementioned orthogonal transformation matrix whose different elements are given by

$$\begin{align*}
\alpha_{11} &= -\alpha_{44} = -\frac{(1+b-2n)\sqrt{5+b+5n}}{2\sqrt{3(2+n)(5(5+b)+2n(16+b+8n))}}, \\
\alpha_{21} &= -\alpha_{31} = \frac{(b-1+2n)\sqrt{(5+5n-b)(5+2n+b)}}{12\sqrt{n(n+1)(n+2)b}} \\
\alpha_{12} &= \alpha_{42} = \frac{5+2n+b}{2\sqrt{5(5+b)+2n(16+b+8n)}}, \\
\alpha_{13} &= -\alpha_{43} = -\frac{\sqrt{(1+n)(5+b+5n)}}{\sqrt{5(5+b)+2n(16+b+8n)}}, \\
\alpha_{22} &= \alpha_{32} = -\frac{\sqrt{3n(1+n)}}{\sqrt{b(5+b+2n)}}, \\
\alpha_{14} &= \alpha_{44} = \frac{\sqrt{(b-5-2n)}}{2\sqrt{b}}, \\
\alpha_{23} &= -\alpha_{33} = -\frac{\sqrt{(5+5n-b)(5+2n+b)}}{2\sqrt{3nb}}, \\
\alpha_{24} &= \alpha_{34} = \frac{\sqrt{5+2n+b}}{2\sqrt{b}}.
\end{align*} \quad (21)$$

A straightforward but rigorous calculation gives the explicit expressions of the angle of rotation for the quantized model.
Rabi oscillation of equidistant cascade four-level system

\begin{align*}
\theta_1 &= \arccos \left( \frac{-\alpha_{11}}{\sqrt{1 - \alpha_{12}^2 (1 - \alpha_{11}^2 - \alpha_{13}^2)}} \right), \\
\theta_2 &= \arccos \left( \frac{\alpha_{11} \alpha_{13} \alpha_{23} + (1 - \alpha_{13}^2) \sqrt{(1 - 2 \alpha_{11}^2 - 2 \alpha_{12}^2 - 2 \alpha_{13}^2 - 2 \alpha_{12} \alpha_{13} + \alpha_{12}^2)(1 - 2 \alpha_{11}^2 - 2 \alpha_{13}^2 - 2 \alpha_{12} \alpha_{13} + \alpha_{12}^2)}}{(2 \alpha_{13}^2 - 1) \sqrt{(\alpha_{11}^2 - 1)^2 + \alpha_{13}^2 (\alpha_{13}^2 - 2)}} \right), \\
\theta_3 &= \arcsin \left( \frac{\alpha_{13} \sqrt{\alpha_{11}^2 + \alpha_{13}^2 - 1}}{\sqrt{\alpha_{11}^2 (2 - \alpha_{13}^2) + (1 - \alpha_{13}^2)^2}} \right), \\
\theta_4 &= \arcsin[\alpha_{13}], \\
\theta_5 &= -\arcsin \left( \frac{\alpha_{11}}{\sqrt{1 - \alpha_{11}^2}} \right), \\
\theta_6 &= \arcsin \left( \frac{\alpha_{12}}{\sqrt{1 - \alpha_{11}^2 - \alpha_{13}^2}} \right), \\
\end{align*}

where different elements \( \alpha_{ij} \) appearing in the rotation matrix are defined in eq. (21).

It is easy to see that in the limit \( n \to \infty \), these angles precisely yield those of the semiclassical model given in eq. (14). This clearly shows that our treatment of the quantized model is in conformity with the Bohr correspondence principle and indicates the consistency of our treatment.

The time-dependent probability amplitudes of the four levels are given by

\begin{equation}
\begin{bmatrix}
C_1^{n+2}(t) \\
C_2^{n+1}(t) \\
C_3^{n-1}(t) \\
C_4^{n-1}(t)
\end{bmatrix} = T_n^{-1} \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} T_n \begin{bmatrix}
C_1^{n+2}(0) \\
C_2^{n+1}(0) \\
C_3^{n-1}(0) \\
C_4^{n-1}(0)
\end{bmatrix},
\end{equation}

In the next section we proceed to analyze the probabilities of the four levels for aforesaid initial conditions, namely, Case V: \( C_1^{n+2}(0) = 1, C_2^{n+1}(0) = 0, C_3^{n-1}(0) = 0 \), Case VI: \( C_2^{n+1}(0) = 1, C_3^{n-1}(0) = 0, C_4^{n-1}(0) = 0 \), Case VII: \( C_4^{n-1}(0) = 0, C_2^{n+1}(0) = 0, C_3^{n-1}(0) = 1, C_4^{n-1}(0) = 0 \) and Case VIII: \( C_1^{n+2}(0) = 0, C_2^{n+1}(0) = 0, C_3^{n-1}(0) = 0, C_4^{n-1}(0) = 1 \), respectively and then compare the results with those of the semiclassical model.

4. Numerical results

We are now in position to explore the physical content of our treatment by comparing the probabilities of the semiclassical and quantized cascade four-level system respectively. Figures 1a-d show the plots of the probabilities \( |C_1(t)|^2 \) (level 1, dot-dashed line), \( |C_2(t)|^2 \) (level 2, dotted line), \( |C_3(t)|^2 \) (level 3, dashed line) and \( |C_4(t)|^2 \) (level 4, solid line) for the semiclassical model corresponding to Cases I,
II, III and IV respectively. The comparison of figure 1a (figure 1b) and figure 1d (figure 1c) shows that the pattern of the probability oscillation of Case I (Case II) is similar to that of Case IV (Case III) except the probabilities of level 1 and level 4 and also that of level 2 and level 3 are interchanged. Thus, a regular pattern of the probability oscillation reveals the symmetric behavior of Rabi oscillation for the semiclassical four-level cascade system.

Following ref. [44], in the case of the quantized field, we consider the time evolution of the probabilities for two distinct situations, first, when the field is in a number state representation and then, when the field is in the coherent state representation.

For the number state representation, the Rabi oscillation for Cases V, VI, VII and VIII of the quantized system are shown in figures 2a-d. Here we note that for Case V (Case VI), the oscillation pattern of the system is completely different from that of Case VIII (Case VI). Thus the symmetry observed in the pattern of the population dynamics of the semiclassical model between Case I and Case IV and also between Case II and Case III no longer exists. In other words, for the quantized field, in contrast to the semiclassical case, whether the system initially stays in any one of the four levels, the symmetry of the Rabi oscillation in all cases is completely spoilt. As pointed out earlier, the disappearance of the symmetry is essentially due to the vacuum fluctuation of the quantized cavity mode which survives even at $n = 0$. Recently, we have reported similar breaking pattern in the Rabi oscillation for the equidistant cascade [44] and also for lambda and vee three-level systems [46]. Such breaking is not observed in the case of two-level
Rabi oscillation of equidistant cascade four-level system

![Figure 2. The Rabi oscillation (scaled with $g$) for Cases V, VI, VII and VIII with quantized cavity mode shows the breaking of the aforesaid symmetry between Case I and Case IV and between Case II and Case III, respectively.](image)

Finally, we consider the model interacting with the monochromatic quantized field which is in the coherent state. The coherently averaged probabilities of the system for level-1, level-2, level-3 and level-4 are given by

\[ \langle P_1(t) \rangle = \sum_n w_n |C_1^n(t)|^2 \] (24a)
\[ \langle P_2(t) \rangle = \sum_n w_n |C_2^n(t)|^2 \] (24b)
\[ \langle P_3(t) \rangle = \sum_n w_n |C_3^n(t)|^2 \] (24c)
\[ \langle P_4(t) \rangle = \sum_n w_n |C_4^n(t)|^2 \] (24d)

respectively, where \( w_n = \exp\left[-\frac{n}{\bar{n}}\right] \) is the coherent distribution with \( \bar{n} \) being the mean photon number of the quantized field mode. Figures 3 and 4 display the numerical plots of eq. (24) with \( \bar{n} = 48 \) for Case V, VI, VII and VIII, respectively where the collapse and revival of the Rabi oscillation is clearly evident. The collapse and revival for Case V depicted in figures 3a-d is compared with that of Case VIII shown in figures 3e-h. We note that figures 3a, 3b, 3c and 3d are precisely identical to figures 3h, 3g, 3f and 3e respectively. Similarly, figure 4 compares the collapse and revival of the system for Case VI with that of Case VII, where, similar to the semiclassical model, we note that figures 4a, 4b, 4c and 4d are similar to figures 4h, 4g, 4f and 4e, respectively. Note that a distinct collapse and revival

\[ \text{Pramana - J. Phys., Vol. 70, No. 1, January 2008} \]
Mihr Ranjan Nath et al

Figure 3. Figures 3a–d and figures 3e–h depict the time-dependent collapse and revival phenomenon for Case V and Case VIII respectively. We note that the oscillation pattern of levels 1, 2, 3 and 4 in Case V is similar to that of levels 4, 3, 2 and 1 for Case VIII, respectively.

Figure 4. Figures 4a–d and figures 4e–h depict the collapse and revival phenomenon for Case VI and Case VII, respectively. Here we find that the oscillation pattern of levels 1, 2, 3 and 4 for Case VI is similar to that of levels 4, 3, 2 and 1 for Case VII, respectively.

pattern appears for a coherent cavity field only for a large average photon number. This clearly recovers the symmetric pattern exhibited by the semiclassical four-level system with the classical field mode.

5. Conclusion

This paper examines the behaviour of the oscillation of probability of a cascade four-level system taking the field to be either classical or quantized. The Hamiltonian of the system is constructed from the generators of the spin-$\frac{3}{2}$ representation of the $SU(2)$ group and the probabilities of the four levels are computed for different initial conditions using a generalized Euler angle representation. We argue that the symmetry exhibited in the Rabi oscillation with the classical field is completely destroyed due to the quantum fluctuation of the cavity mode. This symmetry is, however, restored by taking the cavity mode as a coherent state with large average photon number. It is interesting to look for the effect of field quantization on the
Rabi oscillation with other configurations of the four-level system and to scrutinize its nontrivial effect on the various coherent phenomena involving multilevel systems.

Acknowledgement

MRN and TKD thank the University Grants Commission, New Delhi and SS thanks the Department of Science and Technology, New Delhi for partial support. We thank Professor B Bagchi for bringing ref. [45] to our notice. SS is also thankful to S N Bose National Centre for Basic Sciences, Kolkata for supporting his visit to the centre through the associateship program. MRN and SS thank Dr A K Sen for his interest in this problem.

References

Mihir Ranjan Nath et al

[39] F Li, X Li, D L Lin and T F George, Phys Rev A40, 5129 (1989), and references therein
[45] S K Bose and E A Pascoa, Nucl Phys B199, 384 (1980), We have corrected the typos of this reference
[46] M R Nath, S Sen, G Gangopadhyay and A K Sen, Communicated

On the microscopic basis of Newton's law of cooling and beyond

Mihir Ranjan Nath and Surajit Sen
Department of Physics, Guru Charan College, Silchar-788004, India

Gautam Gangopadhyay
S N Bose National Centre for Basic Sciences, JD Block, Sector III, Salt Lake City, Kolkata-700098, India

(Received 9 May 2007; accepted 9 July 2007; published online 4 September 2007)

The microscopic basis of Newton's law of cooling and its modification when the difference in temperature between the system and the surroundings is very large is discussed. When the system of interest is interacting with a small bath, the effect of the dynamical evolution of the bath variables is important to find out its dynamical feedback on the system. As in the usual system-bath approach, however, the bath is finally considered to be in thermal equilibrium and thereby provides an effective generalization of the Born-Markov master equation. It is shown that the cooling at early time is faster than that predicted by Newton's law due to the dynamical feedback of the bath.

© 2007 American Institute of Physics.

[DOI: 10.1063/1.2767624]

I. INTRODUCTION

Classically the phenomenon of cooling of a bulk body may be considered as a process where the flow of heat from the hotter body to a colder environment is governed by the Newton's law of cooling, namely,

$$\frac{dT(t)}{dt} = -\gamma(T(t) - T_R),$$

(1)

with $T(t)$ the instantaneous temperature of the body, $T_R$ the temperature of the environment, and $\gamma$ the characteristic decay constant. The solution of Eq. (1) reads

$$T(t) = T_0 e^{-\gamma t} + T_R(1 - e^{-\gamma t}),$$

(2)

where $T_0$ is the initial temperature of the body at $t=0$ such that $T_0 < T_R$. The classical rate equation does not contain any nonlinear term and it is valid if the difference between the temperature of the system and environment is small.

On the other hand, the cooling of the neutral atoms is generally performed by various coherent laser-cooling techniques, which primarily concern the reduction of the kinetic motion of the center of mass of the trapped atoms. However, if we consider the coherent cooling of the molecules, then the contribution to the thermal energy arises not only from the motion of their center of mass, but also from their rotational or vibrational motion. Thus, for an ensemble of trapped molecules the cooling can arise from the ceasing of the momentum associated with all possible degrees of freedom, although $a$ priori it is difficult to ascertain which of the degrees of freedom contributes most significantly. If we neglect the translational and rotational motion of the trapped molecules, the dissipation of temperature associated with the vibrational degrees of freedom may be formulated quantitatively within the framework of the standard dissipation theory of the damped harmonic oscillator. To formulate the theoretical basis of the vibrational cooling, we assume that a molecule with one or a few modes of vibration as our system of interest, which dissipates its energy into the large number of other modes acting effectively, as the reservoir. In the density matrix formalism of the system-reservoir composite model, the nontrivial weak coupling between the system with the reservoir effectively induces a damping in the system. Consequently, we obtain the Born-Markov master equation of the reduced density matrix $\rho$, where the reservoir oscillators are completely eliminated in terms of the system variables as

$$\frac{d\rho}{dt} = -i\omega_0[a^\dagger a, \rho] - \gamma(a^\dagger a \rho + \rho a^\dagger a - 2a a^\dagger a) - \gamma(1 + \bar{n}_R(T_R))(aa^\dagger \rho + \rho a a^\dagger - 2a^\dagger a),$$

(3)

where the frequency of the system oscillator is $\omega_0$ with $[a, a^\dagger] = 1$ and the average thermal excitation number of the bath is $\bar{n}_R(T_R) = 1/(e^{(\omega_0/k_B T_R)} - 1)$, where $T_R$ is the temperature of the reservoir. Thus, it follows from the master equation that the time evolution of average photon number of the system mode, $\bar{n}_S(t) = \langle a^\dagger a \rangle(dt/dt)$, with frequency $\omega_0$ is described by the rate equation

$$-\frac{d\bar{n}_S(t)}{dt} = \gamma(\bar{n}_S(t) - \bar{n}_R(T_R)).$$

(4)

and its solution is given by

$$\bar{n}_S(t) = \bar{n}_S(0) e^{-\gamma t} + \bar{n}_R(T_R)(1 - e^{-\gamma t}).$$

(5)

Thus, the vibrational cooling corresponds to the feeding of the thermal photon from the system to the reservoir until the system photon number equilibrates with that of the reservoir. This process is known as the thermalization and it is evident from Eq. (5) that, similar to the classical cooling, it occurs after an infinitely large time, namely, $\bar{n}_S(\infty) = \bar{n}_R(T_R)$. We can...
associate the instantaneous average photon number of the system mode with an effective temperature $T(t)$ by using the relation

$$\tilde{n}_s(T(t)) = \frac{1}{e^{\hbar\omega_0/k_BT(t)}} - 1,$$

which in the high temperature limit gives, $\tilde{n}_s(T(t)) = [kT(t)/\hbar\omega_0]$. In the same limit, the initial system photon number and the reservoir photon number are given by $\tilde{n}_s(T(0)) = kT(0)/\hbar\omega_0$ and $\tilde{n}_R(T(0)) = kT_R/\hbar\omega_0$, respectively; putting these values in Eq. (5) we recover Eq. (2). Therefore, the classical Newton’s cooling law in Eq. (1) and (2) can be obtained from the high temperature limiting situation of the Born-Markov master equation in Eqs. (3) and (5). This is valid when the difference in average energy per mode between the system and the reservoir is small, whereby the reservoir does not change with time due to the acceptance of energy from the system of interest.

In the crossroad of various approaches and applications of the system-reservoir composite formalism, the necessity of the finite bandwidth of the reservoir\textsuperscript{5-12} was envisaged right from the beginning, which leads to the possible modification of Eq. (3). The recent experiments involving ultrafast time scale,\textsuperscript{13-15} correlated emission laser (CEL) pulse with adjustable memory time,\textsuperscript{16} experiments on cavity electrodynamics,\textsuperscript{17} etc., have significant impact in the understanding of the models beyond the Born-Markov approximation. However, in all previous studies, the assumption that works at more subtle level is that the photon absorbed by the reservoir from the system cannot bring any dynamical change with it because of the small difference of the average energy between the system and average energy of each degree of freedom of the reservoir. It is therefore of natural interest how the situation changes if we consider that the energy of the system is large enough in comparison to that of each reservoir mode. In this paper we shall show that, due to the large difference of average energy between the system and the reservoir, the flow of the thermal photon of substantial energy from the system to the reservoir effectively leads to the dynamical evolution of the reservoir. Our primary objective is to discuss the vibrational cooling of a system with large energy content by incorporating the aforesaid dynamical evolution of the reservoir.

The remaining sections of the paper are organized as follows. In Sec. II we have developed a formalism to incorporate the evolution of the reservoir and show how it effectively generalizes the Born-Markov master equation beyond leading order of the decay constant. We apply our formalism to the damped harmonic oscillator in Sec. III and show how it affects the thermalization time to make it short. We conclude by highlighting the essence of the paper and discussing its outlook.

II. FORMALISM

The Hamiltonian of a system interacting with the reservoir is given by

$$H = H_S + H_R + V = H_0 + V,$$

where $H_S$, $H_R$ represent the Hamiltonians of the system and reservoir, respectively, and $V$ is the interaction between them. Let $\kappa(t)$ be the joint density matrix of the composite system in the interaction picture (IP); the corresponding evolution equation is

$$\frac{\partial \kappa(t)}{\partial t} = -\frac{i}{\hbar} [V(t), \kappa(t)].$$

The solution of the equation is given by

$$\kappa(t) = \kappa(0) - \frac{i}{\hbar} \int_0^t d\tau' [V(\tau'), \kappa(0)]$$

$$+ \left( \frac{i}{\hbar} \right)^2 \int_0^t d\tau' \int_0^{\tau'} dt'' [V(\tau''), \kappa(t''')] \int_0^{\tau} d\tau^\prime' [V(\tau^\prime'), \kappa(t^\prime')].$$

We consider the interaction Hamiltonian in the IP to be of the following form:

$$V(t) = \hbar \sum_i Q_i(t) F_i(t),$$

where $Q_i(t)$ and $F_i(t)$ are the system and reservoir operators, respectively, in IP. Using the factorization ansatz, namely, $\kappa(0) = s(0)|0\rangle\langle 0|$ and $\kappa(t') = s(t')|f(t')\rangle\langle f(t')|$, respectively, and noting the fact that $Tr_R s(t) = s(t)$, the trace over the reservoir mode in Eq. (9) yields

$$s(t) = s(0) - i \int_0^t dt' \sum_{ij} \{Tr_R [F_i(t') F_j(t') f(t')]$$

$$\times [Q_i(t') Q_j(t') s(t') - Q_i(t') s(t') Q_j(t')]$$

$$- Tr_R [F_i(t') F_j(t') f(t')] [Q_i(t') s(t') Q_j(t')]$$

$$- s(t') Q_i(t') Q_j(t')],$$

where $\langle \cdots \rangle_R = Tr_R [\cdots |0\rangle\langle 0|$ is the average of the reservoir operators. Taking the derivative of Eq. (11) with respect to $t$, we obtain

$$\frac{\partial s(t)}{\partial t} = - \sum_{ij} \int_0^t dt' \{[Q_i Q_j s(t') - Q_i s(t') Q_j]$$

$$\times Tr_R [F_i(t') F_j(t') f(t')]$$

$$- [Q_i s(t') Q_j - s(t') Q_i Q_j]$$

$$\times Tr_R [F_i(t') F_j(t') f(t')] \exp [i(\omega_i t + \omega_j t')].$$

where $\omega_i$ is the characteristic frequencies of the system and the linear term in the reservoir operator vanishes by the symmetry argument. The system oscillator in the IP in $V(t)$ can be expressed in the Schrodinger picture (SP) by using the standard prescription.
\[
Q_i(t) = e^{\frac{i}{\hbar}(\mathcal{H}^{s} - \mathcal{H}^{w})t}Q_i e^{\frac{i}{\hbar}(\mathcal{H}^{s} - \mathcal{H}^{w})t} = \mathcal{P} Q_i e^{-\frac{i}{\hbar}\mathcal{H}^{w}t},
\]
(13)

Now, replacing \( t \) by \( t - \tau \) in Eq. (12) and assuming the Born-Markov approximation, namely, \( s(t - \tau) = s(t) \) for large \( t \), the generalized master equation of the reduced density operator in Schrodinger picture, \( S \), is obtained as
\[
\frac{dS}{dt} = -\frac{i}{\hbar} [H_s, S] - \sum_{i,j} \{ [Q_iS, Q_j - S, Q_i]W_{ij}^r [t] - [Q_iS, Q_j - S, Q_j]W_{ij}^r [t] \} \delta (\omega_i^r + \omega_j^r),
\]
(14)

where
\[
W_{ij}^r [t] = \int_0^t d\tau e^{i\omega_j^r \tau} Tr_s [F_j (t - \tau) F_i (t - \tau)],
\]
(15a)
\[
W_{ij}^r [t] = \int_0^t d\tau e^{i\omega_i^r \tau} Tr_s [F_i (t - \tau) F_j (t - \tau)],
\]
(15b)

which are to be calculated in different situations.

To include the evolution of the reservoir in this scenario, using \( \text{Tr}_s = f(t) \) along with Eq. (10), the trace of Eq. (9) over the system yields
\[
\mathcal{W}_{ij}^r [t] = -i \sum_I \int_0^t d\tau e^{i\omega_i^r \tau} \int_0^{t-\tau} dt (Q_i (t_1) F_j (t - \tau) F_i (t_1))_R - (\langle F_i (t_1) F_j (t - \tau) F_i (t) \rangle)_R
\]
\[- \sum_{I,J} \int_0^t d\tau e^{i\omega_i^r \tau} \int_0^{t-\tau} dt_1 \int_0^{t_1} dt_2 (\langle F_i (t_1) F_j (t - \tau) F_i (t_2) \rangle)_R - (\langle F_i (t_1) F_j (t - \tau) F_i (t) \rangle)_R)
\]
\[\times (Q_i (t_2) F_j (t_1) F_i (t_1) - \langle F_i (t_1) F_i (t_1) F_j (t - \tau) \rangle)_R)
\]
\[\times (Q_i (t_2) Q_j (t_1)_R) \delta \{\omega_i^r + \omega_j^r\} \delta \{t - t_1 - t_2\},
\]
(16)

where \( \langle \cdots \rangle_R = \text{Tr}_{r} \{ \cdots \} \) represents the average of the system operator which depends on the initial population distribution of the system. In deriving Eq. (16), unlike the previous case, we use the ansatz \( s(t) = s(0) \) to terminate the series beyond the second order in interaction. Plugging Eq. (16) back into Eq. (15a), we find
\[
\mathcal{W}_{ij}^r [t] = \mathcal{W}_{ij}^0 [t] + \mathcal{W}_{ij}^r [t] + \cdots,
\]
(17)

where \( \mathcal{W}_{ij}^0 [t] = \int_0^t d\tau e^{i\omega_i^r \tau} (F_i (t) F_j (t - \tau))_R \) is the usual lowest order reservoir correlator. In Eq. (17) the term next to lowest order arises due to the correlation among the system oscillators, and it is given by

\[
\mathcal{W}_{ij}^r [t] = \int_0^t d\tau e^{i\omega_i^r \tau} \int_0^{t-\tau} dt (Q_i (t_1) F_j (t - \tau) F_i (t_1))_R - \langle F_i (t_1) F_j (t - \tau) F_i (t) \rangle)_R
\]
\[\times (Q_i (t_2) F_j (t_1) F_i (t_1) - \langle F_i (t_1) F_i (t_1) F_j (t - \tau) \rangle)_R)
\]
\[\times (Q_i (t_2) Q_j (t_1)_R) \delta \{\omega_i^r + \omega_j^r\} \delta \{t - t_1 - t_2\},
\]
(18)

It is customary to write the reservoir Hamiltonian in the following form:
\[
H_R = \sum_k \hbar \omega_k \left( b_k^\dagger b_k + \frac{1}{2} \right),
\]
(19)

where \( \omega_k \) is the frequency of the reservoir modes. The time-dependent reservoir operators in the IP appearing in Eq. (18) can be expressed in the SP as
\[
F_1 (t) = \sum_p \kappa_p e^{i\omega_p (t) t} b_p e^{-i\omega_p t},
\]
(20a)
\[
F_2 (t) = \sum_q \kappa_q e^{i\omega_q (t) t} b_q^\dagger e^{-i\omega_q t},
\]
(20b)

where \( \omega_i \) (\( i=p,q \)) is the angular frequency of the reservoir oscillators mode and \( \kappa_i \) the coupling constants, respectively.

From Eq. (18) we now proceed to evaluate the spectral density function \( \mathcal{W}_{ij}^r [t] \) for \( i=1 \) and \( j=2, \)
\[
\mathcal{W}_{12}^r [t] = \sum_{I,J} \int_0^t d\tau e^{i\omega_i^r \tau} \int_0^{t-\tau} dt_1 \int_0^{t_1} dt_2 \langle (F_i (t_1) F_i (t_1) F_j (t - \tau))_R
\times (Q_i (t_2) Q_j (t_1)_R) \delta \{\omega_i^r + \omega_j^r\} \delta \{t - t_1 - t_2\},
\]
(21)

where, once again, the linear term in the system operators is dropped by the symmetry argument. Throughout the treatment we assume that the reservoir is in a thermal distribution, and thus only the diagonal terms will survive. Substituting Eqs. (13), (20a), and (20b) in Eq. (21), we obtain (for details, see the Appendix)
\[
\hat{W}_{\text{res}}[t] = \sum_{r=1}^{\infty} \frac{1}{\hbar} \sum_{\omega_r} \left| \frac{\kappa_r}{\hbar} \right|^2 e^{i(\omega_r-\omega) t} \int_0^t dt_1 e^{i(\omega_r-\omega) t_1} \int_0^{t_1} dt_2 e^{i(\omega_r-\omega) t_2} \\
\times \left[ (2 + \bar{n}_\text{R}(\omega_r, T_R) + \bar{n}_\text{R}(\omega_r, T_R)) (Q_1 Q_2)_5 - (\bar{n}_\text{R}(\omega_r, T_R) + \bar{n}_\text{R}(\omega_r, T_R)) (Q_2 Q_2)_5 \right],
\]

(22)

where \( \bar{n}_\text{R}(\omega_r, T_R) \) is the average thermal photon number of the reservoir and we have considered the system frequency to be \( \omega_1^2 = -\omega_2^2 = \omega_0 \) for convenience without losing generality. Thus, we note that in Eq. (22), the evolution of the reservoir arises due to the correlation functions of the system operators. Finally, converting the sum over modes into the frequency space integrals, we find

\[
\hat{W}_{\text{res}}[t] = \int_0^\infty d\omega_r D(\omega_r) \left| \frac{\kappa_r}{\hbar} \right|^2 \int_0^\infty d\omega_r D(\omega_r) \left| \frac{\kappa_r}{\hbar} \right|^2 e^{i(\omega_r-\omega) t} \int_0^t dt_1 e^{i(\omega_r-\omega) t_1} \int_0^{t_1} dt_2 e^{i(\omega_r-\omega) t_2} \\
\times \left[ (2 + \bar{n}_\text{R}(\omega_r, T_R) + \bar{n}_\text{R}(\omega_r, T_R)) (Q_1 Q_2)_5 - (\bar{n}_\text{R}(\omega_r, T_R) + \bar{n}_\text{R}(\omega_r, T_R)) (Q_2 Q_2)_5 \right],
\]

(23)

where \( D(\omega_r) \) and \( D(\omega_r) \) are the density of states. Proceeding in the similar way, we can show that \( \hat{W}_{\text{res}}[t] = \hat{W}_{\text{res}}[t] \). The time development of the reservoir for any simple system can be calculated from Eq. (23).

### III. APPLICATION TO SIMPLE HARMONIC OSCILLATOR

The vibrational cooling may be modeled by a harmonic oscillator interacting with large number of the reservoir modes which undergo damping. The free Hamiltonian and interaction term of such composite system are given by

\[
H_S = \hbar \omega_0 \left( a^\dagger a + 1 \right),
\]

(24a)

\[
V = \hbar \sum_k \left( \kappa_k a^\dagger b_k + \kappa_k^* a b_k \right),
\]

(24b)

respectively, with the generic reservoir Hamiltonian given by Eq. (19). In Eqs. (24a) and (24b), the system operators are in the Schrödinger picture i.e., \( Q_1 = a^\dagger \) and \( Q_2 = a \). Taking \( \bar{n}_S(t) = \langle a^\dagger a(t) \rangle_S \) to be the average photon number of the system in time \( t \) and the upper limits of integration as \( t, t-\tau, \tau \rightarrow \infty \), Eq. (23) becomes

\[
\hat{W}_{\text{res}}[t] = 2\pi^2 D^2(\omega_0) \left| \frac{\kappa_r}{\hbar} \right|^2 \bar{n}_S(t) - \bar{n}_R(T_R),
\]

(25)

the difference between the instantaneous average excitation number of the system and the thermal average photon number of the reservoir. In deriving Eq. (25) we have neglected the principal parts which correspond to a small Lamb shift. Substituting Eqs. (25) in Eq. (17) (with \( i = 1 \) and \( j = 2 \)) and plucking back the resulting equation in Eq. (15a) and (15b), we obtain a generalized Born-Markov master equation of the damped harmonic oscillator,

\[
dS(t) = -\frac{i}{\hbar} \left[ H_S, S \right] - \frac{\gamma_1(t)}{2} \left[ a^\dagger a S - 2a Sa^\dagger + Sa^\dagger a \right] - \frac{\gamma_2(t)}{2} \left[ aa^\dagger S - 2a^\dagger Sa + Saa^\dagger \right],
\]

(26)

where

\[
\gamma_1(t) = \gamma (1 + \bar{n}_R(T_R)) + \gamma^2 (\bar{n}_S(t) - \bar{n}_R(T_R)) t,
\]

(27a)

\[
\gamma_2(t) = \gamma (\bar{n}_S(T_R) + \bar{n}_R(0) - \bar{n}_R(T_R)) t,
\]

(27b)

with \( \gamma = 2\pi |\kappa(\omega)|^2 D(\omega_0) \) the decay constant. Thus, a linear time-dependent term appearing beyond the leading order of the decay constant becomes important if \( \langle \bar{n}_S(t) - \bar{n}_R(T_R) \rangle \gg 0 \). If the initial average energy of the system is much more than the thermal average excitation of the bath, then the time-dependent decay rate \( \gamma_1(t) \) and \( \gamma_2(t) \) appreciably affects the decay dynamics in an early time.

To address the notion of thermalization in our scenario, we need to calculate the time evolution of \( \bar{n}_S(t) = \langle a^\dagger a(t) \rangle_S \) from Eq. (26), which is governed by the rate equation

\[
-\frac{d\bar{n}_S(t)}{dt} = \gamma (\bar{n}_S(t) - \bar{n}_R(T_R))(1 + \gamma t),
\]

(28)

and its solution reads

\[
\bar{n}_S(t) = \bar{n}_R(T_R) + \bar{n}_R(0) - \bar{n}_R(T_R) e^{-\gamma t(1+\gamma t/2)}.
\]

(29)

Comparing Eq. (5) with Eq. (29), we note that in the latter case the decay rate is time dependent and thermalization becomes faster due to the incorporation of the dynamical evolution of the reservoir.

Here, we note that in the high temperature approximation, Eq. (28) leads to the modified Newton’s law of cooling as

\[
-\frac{dT(t)}{dt} = \gamma (T(t) - T_R) + \gamma^2 (T(t) - T_R) t,
\]

(30)

where the term beyond the leading order of the decay constant becomes significant if \( T_0 \gg T_R \). Figure 1 shows the comparison of the plots of Eq. (2) with Eq. (30), where we note that the thermalization occurs at a faster rate. Cooling in early time is much faster than predicted by Newton’s law. As a first-order correction the theory is valid up to \( t \ll \gamma^{-1} \); for a time longer than \( t \gg \gamma^{-1} \), Newton’s exponential law should be considered for thermalization. Cooling time for reaching from 2000 °C to 800 °C is almost two-thirds in the modified dynamics of that in the Newton’s cooling law. In comparison to Newton’s law, where the time required to bring the tem-
the hospitality of S N Bose National Centre for Basic Sciences, Kolkata, where part of the work was carried out.

APPENDIX: FOUR-POINT CORRELATION FUNCTIONS

In this appendix we shall derive the four-point reservoir correlation functions appearing in Eq. (21). The four-point reservoir correlators can be expressed in terms of two-point correlators by using the identity,

\[ \langle 0'000,0\rangle = \langle 0'00,0\rangle \langle 0001\rangle \]

(A1)

Using Eq. (A1) let us calculate the term in the square bracket of Eq. (21) with \( i=1, j=2 \) and \( l, m \) runs from 1 to 2.

\[
\sum_{l,m} \left[ \langle F_F(t)F_F(t')F_F(t_1)F_F(t_2)\rangle_R - \langle F_F(t)F_F(t')F_F(t_1)F_F(t_2)\rangle_R \langle Q_{00} \rangle \right]
\]

To obtain the above equation we have neglected the off-diagonal terms, since the reservoir is assumed to be a thermal one. Substituting Eqs. (13), (20a), and (20b) in Eq. (A2), the straightforward simplification yields

\[
\sum_{l,m} \left[ \langle F_F(t)F_F(t')F_F(t_1)F_F(t_2)\rangle_R - \langle F_F(t)F_F(t')F_F(t_1)F_F(t_2)\rangle_R \langle Q_{00} \rangle \right]
\]

where \( t' \) is replaced by \( t - \tau \). Finally, substituting Eq. (A3) in Eq. (21), we obtain Eq. (22).

2C. Cohen-Tannoudji, Phys Rep. 219, 153 (1992), and references therein
10Sec, for example, G Car, Nanostructures and Nanomaterials, Synthesis, Properties and Application (Imperial College Press, London, 2004), and references therein
The half-thermalization time is given by $t_{1/2} = \ln 2/\gamma$, for the modified case the half-thermalization time is given by $t_{1/2} = (\sqrt{1 + 2 \ln 2 - 1})/\gamma$. Therefore, in the modified case also $t_{1/2}$ is independent of $(T(0) - T_R)$. From Eq (26), the corresponding master equation of the diagonal elements of the density matrix can be given by the loss-gain equation of population for the stepladder model of a harmonic oscillator as

$$\frac{dS_{ii}(t)}{dt} = \sum_{j \neq i} \{W(i|j)S_{jj}(t) - W(j|i)S_{ii}(t)\}. \quad (31)$$

Here, the transition rates connecting only the neighboring levels are specifically given by

$$W(i + 1|i) = (i + 1)\gamma(1 + [\bar{n}_R + (\bar{n}_R(t) - \bar{n}_R)\gamma]),$$

$$W(i + 1|i) = i\gamma[\bar{n}_R + (\bar{n}_R - \bar{n}_P)\gamma], \quad (32)$$

which means that the $(i+1)$th to $i$th state transition rate is time dependent, and vice versa. On top of the usual temperature-dependent rate, it depends on $\gamma$ and on the difference in temperature of the reservoir from that of the instantaneous temperature of the system, $(\bar{n}_R(t) - \bar{n}_R)$. Usually we consider $\gamma < 1$ for the first-order perturbative effect. A direct consequence of the loss-gain-type master equation shows that the bath-induced forward and backward rates are modified by a factor of $\bar{n}_R(\bar{n}_R(t) - \bar{n}_R)\gamma$ instead of $\bar{n}_R$. As the modification arises due to the system-induced dynamics of the reservoir, which is considered as a first-order effect, we can safely assume that the rate is primarily governed by the factor $\bar{n}_R$ and therefore we can consider $(\bar{n}_R(t) - \bar{n}_R)\gamma t \leq \bar{n}_R$. This amounts to the fact that the initial system temperature should not be arbitrarily high compared to the temperature of the reservoir. Otherwise, a strong nonequilibrium evolution of the bath will produce a nonlinear coupled dynamical system and bath variables, which is an immensely difficult problem to tackle to provide any tangible physical result.

IV. CONCLUSION

In this paper we have developed the quantum theory of cooling of a system with large energy content when the reservoir has also a dynamical evolution rather than thermal equilibrium. It is explicitly demonstrated for such system that the thermal equilibrium is attained much faster than in comparison to the case of exponential decay when the reservoir is at equilibrium. Our study reveals that, the larger the initial photon number content of the system, the faster the rate of cooling. Possible modification of the Newton's classical law of cooling beyond the leading order of the decay constant is pointed out, and it is shown that the higher order term becomes significant if the difference between the average energy per mode of the system and the reservoir is considerable. The analysis is strictly valid for a very short time, $t < \gamma^{-1}$, and initial temperature or average energy of the system is not too much higher compared to the reservoir as the modification in the theory arises due to first-order perturbation effect. We have also considered a repeated neglect of off-diagonal terms corresponding to bath degrees of freedom arising from the dynamics where only the diagonal elements of the bath density are modified in time. A faster thermalization or faster cooling is qualitatively understandable as the bath is interacting with the system more actively instead of passively waiting in its equilibrium distribution in the usual approach.

To treat a finite size of the bath, one immediate choice is to restrict the number of modes in the bath. This is equivalent to a pronounced recurrence of population of the vibrational states due to the back and forth exchange of energy between the modes of the system and bath. However, in our approach we have assumed the fact that the system experiences a feedback due to the dynamical evolution of the bath, but ultimately the bath is assumed to be in thermal equilibrium. We have calculated the two-point and four-point correlation functions of the bath variables by averaging over the thermal equilibrium distribution instead of a nonequilibrium distribution of the bath. The population decay is non-exponential due to this, which has simple dependence on the difference between the average energy of each mode from the time-dependent state of the system to the bath at equilibrium. A more systematic approach to treat finite size of the bath is under consideration and will be published elsewhere.

In the midst of several currently interesting coherent cooling mechanisms of atoms and molecules induced by laser, this incoherent mechanism of vibrational cooling may be found worthwhile because of the huge availability of nanomaterials which can support a large number of degrees of freedom. It can effectively act as a bath as well as a finite quantum system to recouple energy with a subsystem of interest which is composed of a single or a few modes of vibration. Other associated aspects of the system-reservoir formalism require careful scrutiny in the light of the dynamical evolution of the reservoir proposed here.

ACKNOWLEDGMENTS

M.R. N thanks the University Grants Commission and S. S thanks the Department of Science and Technology, New Delhi for financial support. We thank Professor D.S. Ray for many fruitful discussions. M. R. N. and S. S. also thank Dr. A. K. Sen for his interest in this problem, and acknowledge...
Dynamical symmetry breaking of lambda and vee-type three-level systems on quantization of the field modes

Mihir Ranjan Nath¹ and Surajit Sen²
Department of Physics
Guru Charan College
Silchar 788004, India

Asoke Kumar Sen³
Department of Physics
Assam University
Silchar 788011, India

Gautam Gangopadhyay⁴
S N Bose National Centre for Basic Sciences
JD Block, Sector III
Salt Lake City, Kolkata 700098, India

Abstract

We develop a scheme to construct the Hamiltonians of the lambda, vee and cascade type of three-level configurations using the generators of $SU(3)$ group. It turns out that this approach provides a well defined selection rule to give different Hamiltonians for each configurations. The lambda and vee type configurations are exactly solved with different initial conditions while taking the two-mode classical and quantized fields. For the classical field, it is shown that the Rabi oscillation of the lambda model is similar to that of the vee model and the dynamics of the vee model can be recovered from lambda model and vice versa simply by inversion. We then proceed to solve the quantized version of both models introducing a novel Euler matrix formalism. It is shown that this dynamical symmetry exhibited in the Rabi oscillation of two configurations for the semiclassical models is completely destroyed on quantization of the field modes. The symmetry can be restored within the quantized models when the field modes are both in the coherent states with large average photon number which is depicted through the collapse and revival of the Rabi oscillations.

¹mrnath,95@rediffmail.com
²ssen55@yahoo.com
³asokesen@sanarchet.in
⁴gautam@bose.res.in
I. Introduction

Quantum Optics gives birth to many novel proposals which are within reach of present-day ingenious experiments performed with intense narrow-band tunable laser and high-Q superconducting cavity [1]. Major thrust in the atomic, molecular and optical experiments primarily involves the coherent manipulation of the quantum states which may be useful to verify several interesting results of quantum information theory and also the experimental realization of the quantum computer [2,3]. The actual number of the quantum mechanical states of atoms involved in the interaction with light is of much importance in these days since many coherent effects are due to the level structure of the atom. It is well-known that the two-level system and its quantized version, namely, the Jaynes-Cummings model (JCM), have been proved to be an useful theoretical laboratory to understand many subtle issues of the cavity electrodynamics [4,5]. The two-level system is modeled using the Pauli’s spin matrices - the spin-half representation of SU(2) group, where apart from the level number, the spectrum is designated by the photon number as the quantum number. A natural but non-trivial extension of the JCM is the three-level system and it exhibits plethora of optical phenomena such as, two-photon coherence [6], resonance Raman scattering [7], double resonance process [8], population trapping [9], three-level super radiance [10], three-level echoes [11], STIRAP [12], quantum jump [13], quantum zeno effect [14], Electromagnetically Induced Transparency [15,16] etc. There are three distinct schemes of the three-level configurations which are classified as the lambda, vee and cascade systems respectively. The Hamiltonians of these configurations are generally modeled by two two-level systems coupled by the two modes of cavity fields of different frequencies [17,18]. Although these Hamiltonians succeed in revealing several phenomena [19,20], however, their ad hoc construction subsides the underlying symmetry and its role in the population dynamics of these systems. The connection between the SU(N) symmetry and the N-level system in general, was investigated extensively in recent past [21-27]. These studies not only mimic the possible connection between quantum optics with the octet symmetry, well-known paradigm of particle physics, but for \( N = 3 \), it also reveals several interesting results such as the realization of the eight dimensional Bloch equation, existence of non-linear constants [18,22], population transfer via continuum [28], dynamical aspects of three-level system in absence of dissipation [29] etc. However, inspite of these progress, a general formalism as well as the \textit{ab initio} solutions of all three configurations are yet to be developed for the reasons mentioned below.

The generic model Hamiltonian of a three-level configuration with three well-defined energy levels can be represented by the hermitian matrix

\[
H = \begin{bmatrix}
\Delta_3 & h_{32} & h_{31} \\
h_{32} & \Delta_2 & h_{21} \\
h_{31} & h_{21} & \Delta_1
\end{bmatrix},
\]

(1)

where \( h_{ij}(i, j = 1, 2, 3) \) be the matrix element of specific transition and \( \Delta_i \) be the detuning which vanishes at resonance. We note that from Eq.(1), the lambda system, which corresponds to the transition \( 1 \leftrightarrow 3 \leftrightarrow 2 \) shown in Fig.1a, can be described by the Hamiltonian with elements \( h_{21} = 0, h_{32} \neq 0 \) and \( h_{31} \neq 0 \). Similarly the vee model, characterized by the transition \( 3 \leftrightarrow 1 \leftrightarrow 2 \) shown in Fig.1b, corresponds to the elements \( h_{21} \neq 0, h_{32} = 0 \) and \( h_{31} \neq 0 \) and
for the cascade model we have transition $1 \leftrightarrow 2 \leftrightarrow 3$, we have $h_{21} \neq 0$, $h_{32} \neq 0$ and $h_{31} = 0$ respectively. Thus we have distinct Hamiltonian for three different configurations which can be read off from Eq.(1) shown in Fig.1. This definition, however, differs from the proposal advocated by Hioe and Eberly, who argued the order of the energy levels to be $E_1 < E_3 < E_2$ for the lambda system, $E_2 < E_3 < E_1$ for the vee system and $E_1 < E_2 < E_3$ for the cascade system respectively [18,21,22]. In their scheme, the level-2 is always be the intermediary level which becomes the upper, lower and middle level to generate the lambda, vee and cascade configurations respectively. It is worth noting that, if we follow their scheme, these energy conditions map all three three-level configurations to a unique cascade Hamiltonian described by the matrix with elements $h_{12} \neq 0$, $h_{23} \neq 0$ and $h_{13} = 0$ in Eq.(1). Thus because of the similar structure of the model Hamiltonian, if we start formulating the solutions of the lambda, vee and cascade configurations, then it would led to same spectral feature. Furthermore, due to the same reason, the eight dimensional Bloch equation always remains same for all three models [18,22]. Both of these consequences go against the usual notion because wide range of coherent phenomena mentioned above arises essentially due to different class of the three-level configurations. Thus it is worth pursing to formulate a comprehensive approach, where we have distinct Hamiltonian for three configurations without altering the second level for each model.

The problem of preparing multilevel atoms using one or more laser pulses is of considerable importance from experimental point of view. Thus the completeness of the study of the three-level systems requires the exact solution of these models to find the probability amplitudes of all levels, the effect of the field quantization on the population oscillation and, most importantly, the observation of the collapse and revival effect. In recent past, the three-level systems and its several ramifications were extensively covered in a general framework of the $SU(N)$ group having $N$-levels [21-27,30,31]. Also, the semiclassical model [24,32,33] and its fully quantized version [23,34,35] are studied, but to our knowledge, the pursuit of the exact solutions of different three-level systems in the spirit of the theory of Electron Spin Resonance (ESR) model and JCM, are still to be facilitated analytically.

In a recent paper, we have studied the exact solutions of the equidistant cascade system interacting with the single mode classical and quantized field with different initial conditions [36]. It is shown that for the semiclassical model the Rabi oscillation exhibits a symmetric pattern of evolution, which is destroyed on quantization of the cavity field. We also show that this symmetry is restored by taking the cavity mode to be the coherent state indicating the proximity of the coherent state to the classical field. We have further studied the equidistant cascade four-level system and obtain similar conclusions [37]. To extend above studies for the lambda and vee models we note that the vee configuration can be obtained from the lambda configuration simply by inversion. However, it is worth noting that, the lambda configuration is associated with processes such as STIRAP [12], EIT [15,16] etc, while the vee configuration corresponds to the phenomena such as quantum jump [13], quantum zeno effect [14], quantum beat [3] etc indicating that both the processes are fundamentally different. It is therefore natural to examine the inversion symmetry between the models by comparing their Rabi oscillations and study the effect of the field quantization on that symmetry. The comparison shows that the inversion symmetry exhibited by the semiclassical models is completely spoiled on quantization.
of the cavity modes indicating the non-trivial role of the vacuum fluctuation in the symmetry breaking.

The remaining part of the paper is organized as follows. In Section-II, we discuss the basic tenets of the \( SU(3) \) group necessary to develop the Hamiltonian of all possible three-level configurations. Section-III deals with the solution of the lambda model taking the two field modes as the classical fields and then in Section IV we proceed to solve the corresponding quantized version of the model using a novel Euler matrix formalism. Section-V and VI we present similar calculation for the vee model taking the mode fields to be first classical and then quantized respectively. In Section-VII we compare the population dynamics in both models and discuss its implications. Finally in Section-VIII we conclude our results.

II. The Models

The most general Hamiltonian of a typical three-level configuration is given by Eq.(1) which contains several non-zero matrix elements showing all possible allowed transitions. To show how the \( SU(3) \) symmetry group provides a definite scheme of selection rule which forbids any one of the three transitions to give the Hamiltonian of a specific model, let us briefly recall the tenets of \( SU(3) \) group described by the Gell-Mann matrices, namely,

\[
\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \lambda_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \lambda_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]

\[
\lambda_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \lambda_5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \quad \lambda_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},
\]

\[
\lambda_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}. \tag{2}
\]

These matrices follow the following commutation and anti-commutation relations

\[
[\lambda_i, \lambda_j] = 2i f_{ijk} \lambda_k, \quad \{\lambda_i, \lambda_j\} = \frac{4}{3} \delta_{ij} + 2 d_{ijk} \lambda_k, \tag{3}
\]

respectively, where \( d_{ijk} \) and \( f_{ijk} \) (\( i, j = 1, 2, ..8 \)) represent completely symmetric and completely antisymmetric structure constants which characterizes \( SU(3) \) group [39]. It is customary to define the shift operators \( T, U \) and \( V \) spin as

\[
T_{\pm} = \frac{1}{2}(\lambda_1 \pm i \lambda_2), \quad U_{\pm} = \frac{1}{2}(\lambda_5 \pm i \lambda_7), \quad V_{\pm} = \frac{1}{2}(\lambda_4 \pm i \lambda_8). \tag{4}
\]

They satisfy the closed algebra

\[
[U_{+}, U_{-}] = U_3, \quad [V_{+}, V_{-}] = V_3, \quad [T_{+}, T_{-}] = T_3, \quad [T_3, T_{\pm}] = \mp 2 T_{\pm}, \quad [T_3, V_{\pm}] = \pm V_{\pm}, \quad [V_3, T_{\pm}] = \pm T_{\pm}, \quad [V_3, V_{\pm}] = \pm 2 V_{\pm}, \tag{5}
\]

4
\[
[U_\pm, T_\pm] = \pm T_\pm, \quad [U_\pm, U_\pm] = \pm 2U_\pm, \quad [U_\pm, V_\pm] = \pm V_\pm,
\]
\[
[T_+, V_-] = -U_-, \quad [T_+, U_+] = V_+, \quad [U_+, V_-] = T_-,
\]
\[
[T_-, U_+] = U_+, \quad [T_-, U_-] = -V_-, \quad [U_-, V_+] = -T_+,
\]

where the diagonal terms are \( T_3 = \lambda_3, \ U_3 = (\sqrt{3}\lambda_3 - \lambda_3)/2 \) and \( V_3 = (\sqrt{3}\lambda_3 + \lambda_3)/2 \), respectively.

The Hamiltonian of the semiclassical lambda model is given by

\[
H^L = H^L_1 + H^L_1,
\]

where the unperturbed and interaction parts including the detuning terms are given by

\[
H^L_1 = \hbar(\Omega_1 - \omega_1 - \omega_2)V_3 + \hbar(\Omega_2 - \omega_1 - \omega_2)T_3,
\]

and

\[
H^L_1 = \hbar(\Delta^L_1 V_3 + \Delta^L_1 T_3) + \hbar\kappa_1 (V_+ \exp(-i\Omega_1 t) + V_- \exp(i\Omega_1 t)) + \hbar \kappa_2 (T_+ \exp(-i\Omega_2 t) + T_- \exp(i\Omega_2 t)),
\]

respectively. In Eq.(6), \( \Omega_i \) (\( i = 1,2 \)) are the external frequencies of the bi-chromatic field, \( \kappa_i \) are the coupling parameters and \( \hbar \omega_1 (= -E_1), \hbar \omega_2 (= -E_2), \hbar (\omega_2 + \omega_1)(= E_3) \) be the respective energies of the three levels. \( \Delta^L_1 = (2\omega_1 + \omega_2 - \Omega_1) \) and \( \Delta^L_2 = (\omega_1 + 2\omega_2 - \Omega_2) \) represent the respective detuning from the bi-chromatic external frequencies as shown in Fig.1.

Proceeding in the same way, the semiclassical vee type three-level system can be written as

\[
H^V = H^V_1 + H^V_1,
\]

where

\[
H^V_1 = \hbar(\Omega_1 - \omega_1 - \omega_2)V_3 + \hbar(\Omega_2 - \omega_1 - \omega_2)U_3,
\]

and

\[
H^V_1 = \hbar(\Delta^V_1 V_3 + \Delta^V_1 U_3) + \hbar \kappa_1 (V_+ \exp(-i\Omega_1 t) + V_- \exp(i\Omega_1 t)) + \hbar \kappa_2 (U_+ \exp(-i\Omega_2 t) + U_- \exp(i\Omega_2 t))
\]

where \( \Delta^V_1 = (2\omega_1 + \omega_2 - \Omega_1) \) and \( \Delta^V_2 = (2\omega_2 + \omega_1 - \Omega_2) \) be the detuning shown in Fig.2.

Similarly the semiclassical cascade three-level model is given by

\[
H^C = H^C_1 + H^C_1,
\]

where

\[
H^C_1 = \hbar(\Omega_1 + \omega_2 - \omega_1)U_3 + \hbar(\Omega_2 + \omega_1 - \omega_2)T_3,
\]

and

\[
H^C_1 = \hbar(\Delta^C_1 U_3 + \Delta^C_1 T_3) + \hbar \kappa_1 (U_+ \exp(-i\Omega_1 t) + U_- \exp(i\Omega_1 t)) + \hbar \kappa_2 (T_+ \exp(-i\Omega_2 t) + T_- \exp(i\Omega_2 t))
\]
respectively with respective detuning \( \Delta_1^\pm = (2\omega_1 - \omega_2 - \Omega_1) \) and \( \Delta_2^\pm = (2\omega_2 - \omega_1 - \Omega_2) \).

Taking the fields to be the quantized cavity fields, in the rotating wave approximation, the Hamiltonian of the quantized lambda configuration is given by

\[
H^\Lambda = H^\Lambda_f + H^\Lambda_i,
\]

where,

\[
H^\Lambda_f = \hbar\Omega_2 - \omega_1 - \omega_2}\mathcal{T}_3 + \frac{2}{j=1} \Omega_j a^\dagger_ia_j, \quad (9a)
\]

\[
H^\Lambda_i = \hbar\Delta_1^\Lambda V_3 + \hbar\Delta_2^\Lambda T_3 + \hbar g_1(V_{+a_1} + V_{-a_1}) + \hbar g_2(T_{+a_2} + T_{-a_2}), \quad (9b)
\]

where \( a_i^\dagger \) and \( a_i \) (\( i = 1, 2 \)) be the creation and annihilation operators of the cavity modes, \( g_i \) be the coupling constants and \( \Omega_i \) be the mode frequencies. Proceeding in the similar pattern, the Hamiltonian of the quantized vee system is given by

\[
H^V = H^V_f + H^V_i, \quad (10a)
\]

where,

\[
H^V_f = \hbar\Omega_2 - \omega_1 - \omega_2\mathcal{T}_3 + \frac{2}{j=1} \Omega_j a^\dagger_ia_j, \quad (10b)
\]

\[
H^V_i = \hbar\Delta_1^V V_3 + \hbar\Delta_2^V T_3 + \hbar g_1(V_{+a_1} + V_{-a_1}) + \hbar g_2(U_{+a_2} + U_{-a_2}), \quad (10c)
\]

respectively. Similarly the Hamiltonian of the quantized cascade system reads

\[
H^\Xi = H^\Xi_f + H^\Xi_i, \quad (11a)
\]

where

\[
H^\Xi_f = \hbar\Omega_2 - \omega_1 - \omega_2\mathcal{T}_3 + \frac{2}{j=1} \Omega_j a^\dagger_ia_j, \quad (11b)
\]

\[
H^\Xi_i = \hbar\Delta_1^\Xi U_3 + \hbar\Delta_2^\Xi T_3 + \hbar g_1(U_{+a_1} + U_{-a_1}) + \hbar g_2(T_{+a_2} + T_{-a_2}). \quad (11c)
\]

Using the algebra given in Eq.(5) and that of field operators, it is easy to check that \([H_f, H^n_f] = 0\) for \( \Delta_i^\Lambda = -\Delta_i^V \) (\( i = \Lambda \), and \( V \)) for the lambda and vee model and \( \Delta_i^\Xi = \Delta_i^\Xi \) for the cascade model which are identified as the two photon resonance condition and equal detuning conditions, respectively [18,21,22,24,26]. This ensures that each piece of the Hamiltonian has the simultaneous eigen functions. Thus we note that, unlike Ref.[18,21,22], precise formulation of the aforementioned three-level configurations require the use of a subset of Gell-Mann \( \lambda_i \) matrices rather than the use of all matrices. We now proceed to solve the lambda and vee configurations for the classical and the quantized field separately.

III. The semiclassical lambda system

At zero detuning the Hamiltonian of the lambda type three-level system is given by
\[
H^A = \begin{pmatrix}
\hbar (\omega_1 + \omega_2) & \hbar \kappa_2 \exp[-i\Omega_2 t] & \hbar \kappa_1 \exp[-i\Omega_1 t] \\
\hbar \kappa_2 \exp[i\Omega_2 t] & -\hbar \omega_2 & 0 \\
\hbar \kappa_1 \exp[i\Omega_1 t] & 0 & -\hbar \omega_1
\end{pmatrix}.
\] (12)

The solution of the Schrödinger equation corresponding to Hamiltonian (12) is given by

\[
\Psi(t) = C_1(t) |1\rangle + C_2(t) |2\rangle + C_3(t) |3\rangle
\] (13)

where \(C_1(t), C_2(t)\) and \(C_3(t)\) be the time-dependent normalized amplitudes of the lower, middle and upper levels with the respective basis states,

\[
|1\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad |2\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad |3\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},
\] (14)

respectively. We now proceed to calculate the probability amplitudes of the three states. Substituting Eq.(13) in Schrödinger equation and equating the coefficients of \(|2\rangle\), \(|3\rangle\) and \(|1\rangle\) from both sides we obtain

\[
i \frac{\partial C_1}{\partial t} = (\omega_2 + \omega_1)C_3 + \kappa_1 \exp(-i\Omega_1 t)C_1 + \kappa_2 \exp(-i\Omega_2 t)C_2,
\] (15a)

\[
i \frac{\partial C_2}{\partial t} = -\omega_2 C_2 + \kappa_2 \exp(i\Omega_2 t)C_3,
\] (15b)

\[
i \frac{\partial C_3}{\partial t} = -\omega_1 C_1 + \kappa_1 \exp(i\Omega_1 t)C_3.
\] (15c)

Let the solutions of Eqs.(15a-c) are of the following form,

\[
C_1 = A_1 \exp(iS_1 t),
\] (16a)

\[
C_2 = A_2 \exp(iS_2 t),
\] (16b)

\[
C_3 = A_3 \exp(iS_3 t),
\] (16c)

where \(A_1, A_2, A_3\) are the time independent constants to be determined. Putting Eqs.(16a-c) in Eqs.(15a-c) we obtain

\[
(S_3 + \omega_2 + \omega_1)A_3 + \kappa_2 A_2 + \kappa_1 A_1 = 0,
\] (17a)

\[
(S_3 + \Omega_2 - \omega_2)A_2 + \kappa_2 A_3 = 0,
\] (17b)

\[
(S_3 + \Omega_1 - \omega_1)A_1 + \kappa_1 A_3 = 0.
\] (17c)

In deriving Eqs.(17), the time independence of the amplitudes \(A_3, A_2\) and \(A_1\) are ensured by invoking the conditions \(S_2 = S_3 + \Omega_2\) and \(S_1 = S_3 + \Omega_1\). At resonance, we have \(\Delta_1^A = 0 = -\Delta_2^A\)

i.e, \((2\omega_2 + \omega_1) - \Omega_2 = 0 = (\omega_2 + 2\omega_1) - \Omega_1\) and the solution of Eq.(17) yields

\[
S_3 = -(\omega_2 + \omega_1) \pm \Delta,
\] (18a)

\[
S_3 = -(\omega_2 + \omega_1).
\] (18b)

where \( \Delta = \sqrt{\kappa_1^2 + \kappa_2^2} \) and we have three values of \( S_2 \) and \( S_1 \) namely

\[
S_2^1 = \omega_2, S_2^{2,3} = \omega_2 \pm \Delta, \quad (19a)
\]

\[
S_1^1 = \omega_1, S_1^{2,3} = \omega_1 \pm \Delta. \quad (19b)
\]

Using Eqs.(18) and (19), Eq.(16) can be written as

\[
C_3(t) = A_3^1 \exp(-i(\omega_2 + \omega_1)t)
\]

\[
+ A_3^2 \exp(i(-(\omega_2 + \omega_1) + \Delta)t) + A_3^3 \exp(i(-(\omega_2 + \omega_1) - \Delta)t), \quad (20a)
\]

\[
C_2(t) = A_2^1 \exp(i\omega_2 t) + A_2^2 \exp(i(\omega_2 + \Delta)t) + A_2^3 (i(\omega_2 + \Delta)t), \quad (20b)
\]

\[
C_1(t) = A_1^1 \exp(i\omega_1 t) + A_1^2 \exp(i(\omega_1 + \Delta)t) + A_1^3 (i(\omega_1 - \Delta)t), \quad (20c)
\]

where \( A_r \)'s are the constants which can be calculated from the following initial conditions:

Case-I: At \( t = 0 \) let the atom is in level-1, i.e. \( C_1(0) = 1, C_2(0) = 0, C_3(0) = 0 \). Using Eqns (15) and (20), the corresponding time-dependent probabilities of the three levels are

\[
|C_3(t)|^2 = \frac{\alpha_1^2}{\Delta^2} \sin^2 \Delta t, \quad (21a)
\]

\[
|C_2(t)|^2 = \frac{\alpha_2^2}{\Delta^2} \sin^2 \Delta t/2, \quad (21b)
\]

\[
|C_1(t)|^2 = \frac{1}{\Delta^2} (\kappa_1^2 + \kappa_1^2 \cos \Delta t)^2. \quad (21c)
\]

Case-II: If the atom is initially in level-2, i.e. \( C_1(0) = 0, C_2(0) = 1 \) and \( C_3(0) = 0 \), the probabilities of the three states are

\[
|C_3(t)|^2 = \frac{\alpha_2^2}{\Delta^2} \sin^2 \Delta t, \quad (22a)
\]

\[
|C_2(t)|^2 = \frac{1}{\Delta^2} (\kappa_2^2 + \kappa_2^2 \cos \Delta t)^2, \quad (22b)
\]

\[
|C_1(t)|^2 = \frac{4 \alpha_1^2 \kappa_2^2}{\Delta^4} \sin^2 \Delta t/2. \quad (22c)
\]

Case-III: When the atom is initially in level-3, i.e. \( C_1(0) = 0, C_2(0) = 0 \) and \( C_3(0) = 1 \), the time evolution of the probabilities of the three states are

\[
|C_3(t)|^2 = \cos^2 \Delta t. \quad (23a)
\]

\[
|C_2(t)|^2 = \frac{\alpha_1^2}{\Delta^2} \sin^2 \Delta t, \quad (23b)
\]

\[
|C_1(t)|^2 = \frac{\alpha_1^2}{\Delta^2} \sin^2 \Delta t. \quad (23c)
\]

We now proceed to solve the quantized version of the above model.

IV. The quantized lambda system

We now consider the three-level lambda system interacting with a bi-chromatic quantized fields described by the Hamiltonian Eq.(9). At zero detuning the solution of the Hamiltonian is given by
\[ |\Psi_A(t)\rangle = \sum_{n,m=0}^{\infty} [C_1^{n-1,m+1}(t) |n-1,m+1,1\rangle + C_2^n(t) |n,m,2\rangle + C_3^{n-1,m}(t) |n-1,m,3\rangle], \] 

where \( n \) and \( m \) represent the photon number corresponding to two modes of the bi-chromatic fields. This interaction Hamiltonian that couples the atom-field states \( |n-1,m,3\rangle \), \( |n,m,2\rangle \) and \( |n-1,m+1,1\rangle \) and forms the lambda configuration shown in Fig.1 is given by

\[ H_{IJ}^J = h \begin{bmatrix} 0 & g_2 \sqrt{n} & g_1 \sqrt{m+1} \\ g_2 \sqrt{n} & 0 & 0 \\ g_1 \sqrt{m+1} & 0 & 0 \end{bmatrix}. \] (25)

The eigenvalues of the Hamiltonian are given by \( \lambda_{\pm} = \pm \hbar \sqrt{ng_2^2 + (m+1)g_1^2} \) \((= \pm \hbar \Omega_{nm})\) and \( \lambda_0 = 0 (= \Omega_0) \), respectively with the corresponding dressed eigenstates

\[ \begin{bmatrix} |nm, 3\rangle \\ |nm, 2\rangle \\ |nm, 1\rangle \end{bmatrix} = T_{n,m}(g_1, g_2) \begin{bmatrix} |n-1,m, 3\rangle \\ |n,m, 2\rangle \\ |n-1,m+1, 1\rangle \end{bmatrix}. \] (26)

In Eq.(26), the dressed states are constructed by rotating the bare states with the Euler matrix given by

\[ T_{n,m}(g_1, g_2) = \begin{bmatrix} C_1 c_2 - c_1 s_2 s_3 & C_3 s_2 - c_1 c_2 s_3 & s_1 s_2 \\ -s_3 c_2 - c_1 s_2 c_3 & -s_3 s_2 + c_1 c_2 c_3 & c_1 s_2 \\ s_1 s_2 & -s_1 c_2 & c_1 \end{bmatrix} \] (27)

where \( s_i = \sin \theta_i \) and \( c_i = \cos \theta_i \) \((i = 1, 2, 3)\). The elements of the matrix are found to

\[ T_{n,m}(g_1, g_2) = \begin{bmatrix} \frac{1}{\sqrt{2}} & g_2 \sqrt{n} & g_1 \sqrt{m+1} \\ 0 & g_1 \sqrt{\frac{n}{2(n g_2^2 + (m+1) g_1^2)}} & g_2 \sqrt{\frac{m+1}{2(n g_2^2 + (m+1) g_1^2)}} \\ -\frac{1}{\sqrt{2}} & g_2 \sqrt{\frac{m+1}{2(n g_2^2 + (m+1) g_1^2)}} & g_1 \sqrt{\frac{n}{2(n g_2^2 + (m+1) g_1^2)}} \end{bmatrix}. \] (28)

with corresponding Euler angles,

\[ \theta_1 = \arccos\left[-\frac{\sqrt{1+mg_1^2}}{\sqrt{(1+m)g_1^2+2n g_2^2}}\right], \quad \theta_2 = -\arccos\left[-\frac{\sqrt{ng_2}}{\sqrt{(1+m)g_1^2+2n g_2^2}}\right], \quad \theta_3 = \arccos\left(-\frac{\sqrt{2n g_2}}{\sqrt{(1+m)g_1^2+2n g_2^2}}\right). \] (29)

The time-dependent probability amplitudes of the three levels are given by

\[ \begin{bmatrix} C_1^{n-1,m+1}(t) \\ C_2^n(t) \\ C_3^{n-1,m}(t) \end{bmatrix} = T_{n,m}^{-1}(g_1, g_2) \begin{bmatrix} C_1^{n-1,m+1}(0) \\ C_2^n(0) \\ C_3^{n-1,m}(0) \end{bmatrix} \] (30)

Now similar to the semiclassical model the probabilities corresponding to different initial conditions are:

Case-IV: When the atom is initially in level-1, i.e., \( C_1^{n-1,m+1}(0) = 1, C_2^n(0) = 0 \) and \( C_3^{n-1,m}(0) = 0 \), the time-dependent atomic populations of the three states are given by
\[ |C_{3}^{n-1,m}(t)|^2 = \frac{(m+1)g_{1}^2}{\Omega_{nm}^2} \sin^2 \Omega_{nm} t, \quad (31a) \]
\[ |C_{2}^{n,m}(t)|^2 = 4\beta_{2}^2 \sin^2 \Omega_{nm} t, \quad (31b) \]
\[ |C_{1}^{n-1,m+1}(t)|^2 = \frac{1}{\Omega_{nm}^2} \left[ n^2 g_{2}^2 + (m+1)g_{1}^2 \cos \Omega_{nm} t \right]^2. \quad (31c) \]

Case-V: When the atom is initially in level-2, i.e., \( C_{1}^{n-1,m+1} = 0 \), \( C_{2}^{n,m} = 1 \) and \( C_{3}^{n-1,m} = 0 \), the probabilities of three states are
\[ |C_{3}^{n-1,m}(t)|^2 = \frac{ng_{2}^2}{\Omega_{nm}^2} \sin^2 \Omega_{nm} t, \quad (32a) \]
\[ |C_{2}^{n,m}(t)|^2 = \frac{1}{\Omega_{nm}^2} \left[ (m+1)g_{1}^2 + ng_{2}^2 \cos \Omega_{nm} t \right]^2, \quad (32b) \]
\[ |C_{1}^{n-1,m+1}(t)|^2 = 4\beta_{2}^2 \sin^2 \Omega_{nm} t. \quad (32c) \]

Case-VI: If the atom is initially in level-3, then we have \( C_{1}^{n-1,m+1} = 0 \), \( C_{2}^{n,m} = 0 \) and \( C_{3}^{n-1,m+1} = 1 \) and the corresponding probabilities are
\[ |C_{3}^{n-1,m}(t)|^2 = \cos^2 \Omega_{nm} t, \quad (33a) \]
\[ |C_{2}^{n,m}(t)|^2 = \frac{ng_{2}^2}{\Omega_{nm}^2} \sin^2 \Omega_{nm} t, \quad (33b) \]
\[ |C_{1}^{n-1,m+1}(t)|^2 = \frac{(m+1)g_{1}^2}{\Omega_{nm}^2} \sin^2 \Omega_{nm} t. \quad (33c) \]

We now proceed to evaluate the population oscillations of different levels of the vee system with similar initial conditions.

V. The semiclassical vee system

At zero detuning, the Hamiltonian of the semiclassical three-level vee system interacting with two-mode classical fields is given by
\[ H^{V} = \begin{bmatrix} \hbar \omega_1 & 0 & \hbar \kappa_1 \exp[-i\Omega_1 t] \\ 0 & \hbar \omega_2 & \hbar \kappa_2 \exp[-i\Omega_2 t] \\ \hbar \kappa_1 \exp[i\Omega_1 t] & \hbar \kappa_2 \exp[i\Omega_2 t] & -\hbar(\omega_1 + \omega_2) \end{bmatrix}. \quad (34) \]

Let the solution of the Schrödinger equation corresponding to Eq.(34) is given by
\[ \Psi(t) = C_{1}(t) |1\rangle + C_{2}(t) |2\rangle + C_{3}(t) |3\rangle, \quad (35) \]
where \( C_{1}(t) \), \( C_{2}(t) \) and \( C_{3}(t) \) are the time-dependent normalized amplitudes with the basis vectors defined in Eqs.(13). To calculate the probability amplitudes of three states, substituting Eq.(35) into the Schrödinger equation we obtain
\[ \frac{\partial C_{3}}{\partial t} = \omega_1 C_3 + \kappa_1 \exp(-i\Omega_1 t) C_1, \quad (36a) \]
\[ \frac{\partial C_{2}}{\partial t} = \omega_2 C_2 + \kappa_2 \exp(-i\Omega_2 t) C_1, \quad (36b) \]
\[ i \frac{d C_1}{dt} = - (\omega_1 + \omega_2) C_1 + \kappa_2 \exp(i \Omega_2 t) C_2 + \kappa_1 \exp(i \Omega_1 t) C_3. \]  

(36c)

Let the solutions of Eqs.(36) are of the following form:

\[ C_3(t) = A_3 \exp(i S_3 t), \]  

(37a)

\[ C_2(t) = A_2 \exp(i S_2 t), \]  

(37b)

\[ C_1(t) = A_1 \exp(i S_1 t), \]  

(37c)

where \( A_i \)-s are the time independent constants to be determined from the boundary conditions.

From Eq.(36) and Eq.(37) we obtain

\[ (S_1 - \Omega_1 + \omega_1) A_3 + \kappa_1 A_1 = 0, \]  

(38a)

\[ (S_1 - \Omega_2 + \omega_2) A_2 + \kappa_2 A_1 = 0, \]  

(38b)

\[ (S_1 - \omega_2 - \omega_1) A_1 + \kappa_2 A_2 + \kappa_1 A_3 = 0. \]  

(38c)

In deriving Eqs.(38), the time independence of the amplitudes \( A_3, A_2 \) and \( A_1 \) are ensured by invoking the conditions \( S_2 = S_1 - \Omega_2 \) and \( S_3 = S_1 - \Omega_1 \). At resonance, \( \Delta^V = 0 = -\Delta^V \) i.e. \( (2 \omega_2 + \omega_1) - \Omega_2 = 0 = (\omega_2 + 2 \omega_1) - \Omega_1 \) and the solutions of Eq.(38) are given by

\[ S_1 = (\omega_1 + \omega_2) \]  

(39a)

\[ S_1 = (\omega_1 + \omega_2) \pm \Delta \]  

(39b)

and we have three values of \( S_2 \) and \( S_3 \)

\[ S_2^1 = -\omega_2, S_2^{2,3} = -\omega_2 \pm \Delta \]  

(40a)

\[ S_3^1 = -\omega_1, S_3^{2,3} = -\omega_1 \pm \Delta. \]  

(40b)

Using Eqs.(39) and (40), Eqs. (37) can be written as

\[ C_3(t) = A_1^2 \exp(-i \omega_1 t) + A_3^2 \exp(-i(\omega_1 + \Delta) t) + A_3^3 (-i(\omega_1 - \Delta) t), \]  

(41a)

\[ C_2(t) = A_2^1 \exp(-i \omega_2 t) + A_2^2 \exp(-i(\omega_2 + \Delta) t) + A_2^3 (-i(\omega_2 - \Delta) t), \]  

(41b)

\[ C_1(t) = A_1^1 \exp(i(\omega_2 + \omega_1) t) + A_1^2 \exp(i((\omega_2 + \omega_1) + \Delta) t) + A_1^3 \exp(i((\omega_2 + \omega_1) - \Delta) t), \]  

(41c)

where \( A_i \)-s are the constants which are calculated below from the various initial conditions.

Case-I: Let us consider initially at \( t = 0 \), the atom is in level-1, i.e, \( C_1(0) = 1, C_2(0) = 0 \) and \( C_3(0) = 0 \). Using Eqs. (36) and (41), the time dependent probabilities of the three levels are given by

\[ |C_3(t)|^2 = \frac{\kappa_2^2}{\Delta^2} \sin^2 \Delta t, \]  

(42a)
$|C_2(t)|^2 = \frac{\kappa_2^2}{\Delta^2} \sin^2 \Delta t,$  
(42b)

$|C_1(t)|^2 = \cos^2 \Delta t.$  
(42c)

Case-II: If the atom is initially in level-2, i.e., $C_1(0) = 0$, $C_2(0) = 1$ and $C_3(0) = 0$, the corresponding probabilities of the states are given by

$|C_3(t)|^2 = 4 \frac{\kappa_2^2 \kappa_3^2}{\Delta^2} \sin^4 \Delta t/2,$  
(43a)

$|C_2(t)|^2 = \frac{1}{\Delta^2} (\kappa_2^2 + \kappa_3^2 \cos \Delta t)^2,$  
(43b)

$|C_1(t)|^2 = \frac{\kappa_3^2}{\Delta^2} \sin^2 \Delta t.$  
(43c)

Case-III: When the atom is initially in level-3, i.e, $C_1(0) = 0$, $C_2(0) = 0$ and $C_3(0) = 1$, we obtain the occupation probabilities of the three states as follows:

$|C_3(t)|^2 = \frac{1}{\Delta^2} (\kappa_2^2 + \kappa_3^2 \cos \Delta t)^2,$  
(44a)

$|C_2(t)|^2 = 4 \frac{\kappa_2^2 \kappa_3^2}{\Delta^2} \sin^4 \Delta t/2,$  
(44b)

$|C_1(t)|^2 = \frac{\kappa_3^2}{\Delta^2} \sin^2 \Delta t.$  
(44c)

VI. The quantized vee system

The eigenfunction of the quantized vee system described by the Hamiltonian in Eq.(10) is given by

$$|\Psi_{\text{v}}(t)\rangle = \sum_{n,m=0}^{\infty} [C_1^{n+1,m}(t) |n+1,m,1\rangle + C_2^{n,m}(t) |n,m,2\rangle + C_3^{n+1,m-1}(t) |n+1,m-1,3\rangle].$$  
(45)

Once again we note that the Hamiltonian couples the atom-field states $|n+1,m,1\rangle$, $|n,m,2\rangle$ and $|n+1,m-1,3\rangle$ forming vee configuration depicted in Fig.2. The interaction part of the Hamiltonian (45) can also be expressed in the matrix form

$$H_{\text{int}} = \hbar \begin{pmatrix} 0 & 0 & g_1 \sqrt{n+1} \\ 0 & 0 & g_2 \sqrt{n+1} \\ g_1 \sqrt{n} & g_2 \sqrt{n+1} & 0 \end{pmatrix},$$  
(46)

and the corresponding eigenvalues are $\lambda_{\pm} = \pm \hbar \sqrt{m g_1^2 + (n+1) g_2^2}$ ($= \pm \hbar \Omega_{nm}$) and $\lambda_0 = 0$ respectively. The dressed eigenstate is given by

$$\begin{pmatrix} |nm,3\rangle \\ |nm,2\rangle \\ |nm,1\rangle \end{pmatrix} = T_{n,m} \begin{pmatrix} |n+1,m-1,3\rangle \\ |n,m,2\rangle \\ |n+1,m,1\rangle \end{pmatrix},$$  
(47)

the rotation matrix is found to be
\[ T_{n,m} = \begin{bmatrix} g_1 \sqrt{\frac{m_1}{2(n+1)g_2^2 + mg_1^2}} & g_2 \sqrt{\frac{n+1}{2(n+1)g_2^2 + mg_1^2}} & 1 \sqrt{2} \\ -g_2 \sqrt{\frac{n+1}{(n+1)g_2^2 + mg_1^2}} & g_1 \sqrt{\frac{m_1}{(n+1)g_2^2 + mg_1^2}} & 0 \sqrt{2} \\ -g_1 \sqrt{\frac{m_1}{2(n+1)g_2^2 + mg_1^2}} & -g_2 \sqrt{\frac{n+1}{2(n+1)g_2^2 + mg_1^2}} & 1 \sqrt{2} \end{bmatrix} \]  

(48)

The straightforward evaluation yields the various Euler angles are

\[ \theta_1 = -\frac{\pi}{4}, \quad \theta_2 = \arccos\left(-\frac{\sqrt{n+lg_2}}{\sqrt{mg_1^2 + (1+n)g_2^2}}\right), \quad \theta_3 = -\frac{\pi}{2}. \]

The time-dependent probability amplitudes of the three levels are given by

\[
\begin{bmatrix}
C_{3}^{n+1,m-1}(t) \\
C_{2}^{n,m}(t) \\
C_{1}^{n+1,m}(t)
\end{bmatrix} = T_{n,m}^{-1}
\begin{bmatrix}
\cos t \Omega_{nm} \\
0 \\
0
\end{bmatrix} T_{n,m}
\begin{bmatrix}
C_{3}^{n+1,m-1}(0) \\
C_{2}^{n,m}(0) \\
C_{1}^{n+1,m}(0)
\end{bmatrix}.
\]

(50)

Once again we proceed to calculate the probabilities for different initial conditions.

Case-IV: Here we consider initially the atom is in level-1 i.e, \( C_{1}^{n+1,m} = 1, C_{2}^{n,m} = 0 \) and \( C_{3}^{n+1,m-1} = 0 \). Using Eqs.(49) and (50) the time-dependent probabilities of the three levels are given by

\[
|C_{3}^{n+1,m-1}(t)|^2 = \frac{mg_1^2}{\Omega_{nm}^2} \sin^2 \Omega_{nm} t,
\]

(51a)

\[
|C_{2}^{n,m}(t)|^2 = \frac{(n+1)g_2^2}{\Omega_{nm}^2} \sin^2 \Omega_{nm} t,
\]

(51b)

\[
|C_{1}^{n+1,m}(t)|^2 = \cos^2 \Omega_{nm} t.
\]

(51c)

Case-V: If the atom is initially in level-2 i.e, \( C_{3}^{n+1,m-1} = 0, C_{2}^{n,m} = 1 \) and \( C_{1}^{n+1,m} = 0 \), then corresponding probabilities are

\[
|C_{3}^{n+1,m-1}(t)|^2 = \frac{g_2^2 (n+1)(m)}{\Omega_{nm}^2} \sin^4 \Omega_{nm} t/2,
\]

(52a)

\[
|C_{2}^{n,m}(t)|^2 = \frac{1}{\Omega_{nm}^2} [mg_1^2 + (n+1)g_2^2 \cos \Omega_{nm} t]^2,
\]

(52b)

\[
|C_{1}^{n+1,m}(t)|^2 = \frac{g_2^2 (n+1)}{\Omega_{nm}^2} \sin^2 \Omega_{nm} t.
\]

(52c)

Case-VI: Finally if the atom is initially in level-3 i.e, \( C_{1}^{n+1,m} = 0, C_{2}^{n,m} = 0 \) and \( C_{3}^{n+1,m-1} = 1 \), then

\[
|C_{3}^{n+1,m-1}(t)|^2 = \frac{1}{\Omega_{nm}^2} [mg_1^2 \cos \Omega_{nm} t + (n+1)g_2^2]^2,
\]

(53a)

\[
|C_{2}^{n,m}(t)|^2 = \frac{g_2^2 (n+1)(m)}{\Omega_{nm}^2} \sin^4 \Omega_{nm} t/2,
\]

(53b)

\[
|C_{1}^{n+1,m}(t)|^2 = \frac{mg_1^2}{\Omega_{nm}^2} \sin^2 \Omega_{nm} t.
\]

(53c)

Finally we note that for large values of \( n \) and \( m \), Case-IV, V and VI become identical to Case-I, II and III, respectively. This precisely shows the validity of the Bohr’s correspondence principle indicating the consistency of our approach.
VII. Numerical results and discussion

Before going to show the numerical plots of the semiclassical and quantized lambda and vee systems, we first consider their analytical results. If we compare Case-I, II, III of both cases, we find that the probabilities in Case-I (Case-III) of lambda system is the same as in Case-III (Case-I) of vee system except the populations of 1st and 3rd levels are interchanged. See Eqs.(21 & 44) and Eqs.(23 & 42) for detailed comparison. Also Case-II respective models are similar which is evident by comparing Eqs.(22 & 43). In contrast, for the quantized model, Case-IV (Case-VI) of the lambda system is no longer same as in Case-VI (Case-IV) of the vee system. This breaking of symmetry is evident by comparing the analytical results, Eqs.(31 & 53), Eqs.(32 & 52) and Eqs.(32 & 51) respectively. Unlike previous case, also Case-V both the models are distinct which is evident from Eqs.(22 & 43).

In what follows we compare the probabilities of the semiclassical and quantized lambda and vee systems respectively. Fig.3 and 4 show the plots of the probabilities $|C_1(t)|^2$ (dotted line), $|C_2(t)|^2$ (dashed line) and $|C_3(t)|^2$ (solid line) for the semiclassical lambda and vee models when the atom is initially at level-1 (Case-I), level-2 (Case-II) and level-3 (Case-III) respectively. The comparison of the plots shows that the pattern of the probability oscillation of the lambda system for Case-I shown in Fig.3a (Case-III in Fig.3c) is similar to that of Case-III shown in Fig.4c (Case-I in Fig.4a) of the vee system. More particularly we note that in all cases the oscillation of level-2 remains unchanged, while the oscillation of level-3 (level-1) of the lambda system for Case-I is identical to that of level-1 (level-3) of the vee system for Case-III. Furthermore, comparison of Fig.3b and Fig.4b for Case-II shows that the time evolution of the probabilities of level-2 of both systems also remains similar while those of level-3 and level-1 are interchanged. From the behaviour of the probability curve we can conclude that the lambda and vee configurations are essentially identical to each other as we can obtain one configuration from another simply by the inversion followed by the interchange of probabilities.

For the quantized field, we first consider the time evolution of the probabilities taking the field is in a number state representation. In the number state representation, the vacuum Rabi oscillation corresponding to Case-IV, V and VI of the lambda and vee systems are shown in Fig.5 and Fig.6 respectively. We note that, unlike previous case, the Rabi oscillation for Case-IV shown in Fig.5a (Case-VI shown in Fig.5c) for the lambda model is no longer similar to Case-VI shown in Fig.6c (Case-IV shown in Fig.6a) for the vee model. Furthermore, we note that for Case-V, the oscillation patterns of Fig.5b is completely different from that of Fig.6b. In a word, for the quantized field, in contrast to the semiclassical case, the symmetry in the pattern of the vacuum Rabi oscillation in all cases is completely spoiled irrespective of the fact whether the system stays initially in any one of the three levels.

The quantum origin of the breaking of the symmetric pattern of the Rabi oscillation is the following. We note that due to the appearance of the terms like $(n+1)$ or $(m+1)$, several elements in the probabilities given by Eqs.(31,32,33) for the lambda system and Eqs.(51,52,53) for the vee are non zero even at $m = 0$ and $n = 0$. We argue that the vacuum Rabi oscillation interferes with the probability oscillations of various levels and spoils their symmetric structure. Thus as a consequence of the vacuum fluctuation, the symmetry of probability amplitudes of
the dressed states of both models formed by the coherent superposition of the bare states is also lost. In the other word, the invertibility between the lambda and vee models exhibited for the classical field disappears as the direct consequence of the quantization of the cavity modes.

Finally we consider the lambda and vee models interacting with the bi-chromatic quantized fields which are in the coherent state. The coherently averaged probabilities of level-3, level-2 and level-1 are given by

$$\langle P_3(t) \rangle_{\Lambda} = \sum_{n,m} W_n W_m |C_3^{n-1,m}(t)|^2, \quad (54a)$$

$$\langle P_2(t) \rangle_{\Lambda} = \sum_{n,m} W_n W_m |C_2^{n,m}(t)|^2, \quad (54b)$$

$$\langle P_1(t) \rangle_{\Lambda} = \sum_{n,m} W_n W_m |C_1^{n-1,m+1}(t)|^2, \quad (54c)$$

for the lambda system and

$$\langle P_3(t) \rangle_{V} = \sum_{n,m} W_n W_m |C_3^{n+1,m-1}(t)|^2, \quad (55a)$$

$$\langle P_2(t) \rangle_{V} = \sum_{n,m} W_n W_m |C_2^{n,m}(t)|^2, \quad (55b)$$

$$\langle P_1(t) \rangle_{V} = \sum_{n,m} W_n W_m |C_1^{n+1,m}(t)|^2, \quad (55c)$$

for the vee system, where $W_n = \frac{1}{\sqrt{n!}} \exp\left[-\bar{n}\right] \bar{n}^n$ and $W_m = \frac{1}{\sqrt{m!}} \exp\left[-\bar{m}\right] \bar{m}^m$ with $\bar{n}$ and $\bar{m}$ be the mean photon numbers of the two quantized modes, respectively. Fig.7-9 display the numerical plots of Eq.(54) and Eq.(55) for Case-IV, V and VI respectively where the collapse and revival of the Rabi oscillation is clearly evident for large average photon numbers in both the fields. We note that in all cases, the collapse and revival of level-2 of both the systems are identical to each other. Furthermore, we note that the collapse and revival for lambda system initially in level-1 shown in Fig.7a, Fig.7b and Fig.7c (level-3 shown in Fig.9a, Fig.9b and Fig.9c) is the same as that of the vee system if it is initially in level-3 shown in Fig.7f, Fig.7e and Fig.7d (level-1 shown in Fig.9f, Fig.9e and Fig.9d) respectively. On the other hand, if the system is initially in level-2, the collapse and revival of the lambda systems shown in Fig.8a, Fig.8b and Fig.8c are identical to Fig.8f, Fig.8e and Fig.8d respectively for the vee system. This is precisely the situation what we obtained in case of the semiclassical model. Thus the symmetry broken in the case of the quantized model is restored back again indicating that the coherent state with large average photon number is very close to the classical state where the effect of field population in the vacuum level is almost zero. It is needless to say that the coherent state with very low average photon number in the field modes can not show the symmetric dynamics in lambda and vee systems.

VIII. Conclusion
This paper presents the explicit construction of the Hamiltonians of the lambda, vee and cascade type of three-level configurations from the Gell-Mann matrices of $SU(3)$ group and compares the exact solutions of the first two models with different initial conditions. It is shown that the Hamiltonians of different configurations of the three-level systems are different. We emphasize that there is a conceptual difference between our treatment and the existing approach by Hioe and Eberly [18,21,22]. These authors advocate the existence of different energy conditions which effectively leads to same cascade Hamiltonian ($h_{21} \neq 0$, $h_{32} \neq 0$ and $h_{31} = 0$ in Eq.(1)) having similar spectral feature irrespective of the configuration. We justify our approach by noting the fact that the two-photon condition and the equal detuning condition is a natural outcome of our analysis. For the lambda and vee models, the transition probabilities of the three levels for different initial conditions are calculated while taking the atom interacting with the bi-chromatic classical and quantized field respectively. It is shown that due to the vacuum fluctuation, the inversion symmetry exhibited by the semiclassical models is completely destroyed. In other words, the dynamics for the semiclassical lambda system can be completely obtained from the knowledge of the vee system and vice versa while such recovery is not possible if the field modes are quantized. The symmetry is restored again when the field modes are in the coherent state with large average photon number. Such breaking of the symmetric pattern of the quantum Rabi oscillation is not observed in case of the two-level Jaynes-Cummings model and therefore it is essentially a nontrivial feature of the multi-level systems which is manifested if the number of levels exceeds two. This investigation is a part of our sequel studies of the symmetry breaking effect for the equidistant cascade three-level and equidistant cascade four-level systems respectively [36,37]. Following the scheme of constructing of the model Hamiltonians, it is easy to show that we have different eight dimensional Bloch equations and non-linear constants for different configurations of the three-level systems and these issues will be considered elsewhere [40]. The breaking of the inversion symmetry of the lambda and vee models as a direct effect of the field quantization is an intricate issue especially in context with future cavity experiments with the multilevel systems.

Acknowledgement

MRN thanks University Grants Commission and SS thanks Department of Science and Technology, New Delhi for partial financial support. We thank Dr T K Dey for discussions. SS is also thankful to S N Bose National Centre for Basic Sciences, Kolkata, for supporting his visit to the centre through the Visiting Associateship program.

References


[30] F Li, X Li, D L Lin and T F George, Phys Rev A40 5129 (1989) and references therein
[40] M R Nath, S Sen and G Gangopadhyay, (In preparation)
![Lambda type transition](image1)

Fig. 1: Lambda type transition

![Vee type transition](image2)

Fig. 2: Vee type transition

[Fig. 3]: The time evolution of the probabilities $|C_1(t)|^2$ (dotted line), $|C_2(t)|^2$ (dashed line) and $|C_3(t)|^2$ (solid line) of the semiclassical lambda system for Case-I, II and III respectively with values $\kappa_1 = .2$, $\kappa_2 = .1$.

[Fig. 4]: The time variation of the probabilities $|C_1(t)|^2$ (dotted line), $|C_2(t)|^2$ (dashed line) and $|C_3(t)|^2$ (solid line) of the semiclassical vee system for Case-I, II and III respectively with above values of $\kappa_1$, $\kappa_2$. 

19
[Fig.5]: The Rabi oscillation of the quantized lambda system when the fields are in the number state for Case-I, II and III, respectively with $g_1 = 2$, $g_2 = 1$, $n = 1$, $m = 1$.

[Fig.6]: The Rabi oscillation of the quantized vee system when the fields are in the number states for Case-I, II and III, respectively for same values of $g_1$, $g_2$, $n$, $m$.

[Fig.7]: Figs.7a-c display the time-dependent collapse and revival phenomenon of level-3, level-2 and level-1 of the lambda system for Case-IV, while Figs.7d-f show that of the level-3, level-2 and level-1 respectively for of Case-VI of the vee system taking the field modes are in coherent states with $\bar{n} = 30$ and $\bar{m} = 20$. 
[Fig.8]: Figs.8a-c display the time-dependent of collapse and revival of level-3, level-2 and level-1 of the lambda system for Case-V while Figs.8d-f show that of level-3, level-2 and level-1 of the vee system for Case-V with the same values of $\tilde{n}$ and $\tilde{m}$. as in Fig.7.

[Fig.9]: Figs.9a-c display the time-dependent of collapse and revival of level-3, level-2 and level-1 of the lambda system for Case-VI while Figs.9d-f show that for level-3, level-2 and level-1 respectively for the vee system for Case-IV with the same values of $\tilde{n}$ and $\tilde{m}$ as in Fig.7.