Chapter-3

Bayes inferences on Marshall–Olkin Generalized Exponential Distribution

3.1 Introduction

Various methods have been proposed and studied in literature to introduce additional parameters to expand families of distributions for added flexibility or to construct covariate models. In particular, where the one parameter exponential family of distributions is not sufficiently broad, a number of wider families such as the gamma, Weibull and Gompertz-Makeham distributions are in common use. The introduction of the new parameter may thus lead to the accelerated life model and proportional hazards model etc. With this intention Marshall-Olkin (1997) has introduced a new method of adding a parameter to a family of distributions with applications to the Exponential and Weibull families.

In life testing experiment as a preventive maintenance equipment is subject to some checkup involving repair or replacement of its part or parts and overhauling during the observational period of its useful life. The intervened exponential distribution (IED) introduced by Muralidharan and Patel (2004) is one such model based on exponential distribution to improve the reliability of the component under a preventive maintenance policy.

If $F(x)$ is the survival function corresponds to the random variable $X$ then the generalized family of distribution will have a survival functions
\[
\tilde{G}(x; \alpha) = \frac{\alpha F(x)}{1 - (1 - \alpha) F(x)} ; -\infty < x < \infty, \ 0 < \alpha < \infty
\]  

(3.1)

In particular, when \( \tilde{F}(x) = \exp(-\lambda x) \), then a two parameter Marshall-Olkin generalized exponential (MOGE) model will have density

\[
f(x; \alpha, \lambda) = \frac{\alpha \lambda e^{\lambda x}}{[e^{\lambda x} - (1 - \alpha)]^2} ; \ x > 0; \ \alpha > 0, \lambda > 0
\]  

(3.2)

For \( \alpha=1 \), the model (1.2) becomes the usual one-parameter exponential distribution. Like in Muralidharan and Patel (2004), the parameter \( \alpha \) may be called the intervention or induced parameter. For different values of \( \alpha \), (3.2) takes different shapes. Therefore, it also can be used for analyzing skewed data. Interestingly unlike GE, Weibull or Gamma when the PDF is unimodal, here the pdf does not start at zero. Therefore, if the data consists of high early observations, then this might be a very useful model. It may have also increasing and decreasing hazard functions. The algorithm to generate MOGE sample is given in the following algorithm.

**Algorithm- 3.1**

**Step-1:** Input \( n \), \( \alpha \) and \( \lambda \).

**Step-2:** Generate \( u_t \sim U(0,1) \)

**Step-3:** \( x_t = \frac{1}{\lambda} \ln \left( \frac{\alpha}{1 - u_t} + (1 - \alpha) \right) \)

**Step-4:** Repeat step-2 and step-3 for \( t=1,2,\ldots,n \)

![Fig 3.1 Probability density function of MOGE for \( \lambda=1 \)](image)
Fig 3.2 Reliability function of MOGE for $\lambda=1$

Fig 3.3 Hazard rate function of MOGE for $\lambda=1$

Fig 3.4 Reverse Hazard rate function of MOGE for $\lambda=1$
It is observed that the mean of the above distribution is \( \frac{\alpha \ln \alpha}{\lambda (1 - \alpha)} \). The other moments does not have any closed form expressions. In particular, if \( \mu \) is the mean and \( \sigma \) is the standard deviation of \( X \), then the coefficient of variation \( \sigma / \mu \) is less than 1 for \( \alpha > 1 \) and is greater than 1 for \( \alpha < 1 \). For approximate expressions of moments and stability theorems see Marshall and Olkin (1997). The hazard rate of (3.2) is

\[
h(x) = \frac{\lambda e^{\lambda x}}{e^{\lambda x} - (1 - \alpha)} \tag{3.3}\]

Note that for \( \alpha = 1 \), the hazard function is constant, for \( 0 < \alpha \leq 1 \), \( h(x) \) is decreasing in \( x \) and for \( \alpha \geq 1 \), \( h(x) \) is increasing in \( x \). See also Figure 3.3 for details.

The reversed hazard function becomes quite popular in the recent time. The reversed hazard function for the MOGE is

\[
r(x; \alpha, \lambda) = \frac{f(x; \alpha, \lambda)}{F(x; \alpha, \lambda)} = \frac{\alpha \lambda e^{\lambda x}}{(e^{\lambda x} - 1)(e^{\lambda x} - (1 - \alpha))} \tag{3.4}\]

It is observed that for \( \alpha = 0.5, 1.0, 2.0 \) the reversed hazard function is a decreasing function of \( x \), see Figure 3.4

Although, (3.1) is a modified form of exponential distribution, it is not a member of exponential family of distributions. But, still it enjoys the memory less property as the conditional survival function satisfies

\[
\Pr(X > x + t \mid X > x) \begin{cases} 
\leq \Pr(X > t), \alpha \geq 1 \\
\geq \Pr(X > t), 0 < \alpha \leq 1
\end{cases}.
\]

In this Chapter the objective is to provide some inference problems associated with the parameters involved in the model. In the Section-3.2 we discuss different method of point estimation of the parameters and fisher information’s. Section 3.3 deals with various methods to find Confidence Intervals. Section 3.4 deals with the testing of hypothesis. In Section 3.5 Simulation study has been done.
We have considered numerical example in Section 3.6 and the last section is on stress-strength context.

3.2 Estimation

Below we provide various estimation procedures for MOGE parameters

3.2.1. Maximum Likelihood Estimation

If \( X_1, X_2, \ldots, X_n \) are the random sample of size \( n \) from (3.2), then the log likelihood function is given by

\[
\text{Ln } L = n \ln(\alpha \lambda) + \lambda \sum_{i=1}^{n} x_i - 2 \sum_{i=1}^{n} \ln[e^{\lambda x_i} - (1 - \alpha)]
\]  

and the maximum likelihood (ML) equations are

\[
\frac{\partial \ln L}{\partial \alpha} = 0 \Rightarrow \frac{n}{\alpha} - 2 \sum_{i=1}^{n} \frac{1}{e^{\lambda x_i} - (1 - \alpha)} = 0 \tag{3.6}
\]

and

\[
\frac{\partial \ln L}{\partial \lambda} = 0 \Rightarrow \frac{n}{\lambda} + \sum_{i=1}^{n} x_i - 2 \sum_{i=1}^{n} \frac{x_i e^{\lambda x_i}}{[e^{\lambda x_i} - (1 - \alpha)]} = 0 \tag{3.7}
\]

Here both the equations are nonlinear and should be solved simultaneously. We found from our experience that for a faster convergence and an efficient solution one needs a good starting value. The initial solution can be found in the following way: If all higher powers of \( x \) are ignored then \( \ln(1-x) = -x \). Accordingly, (3.5) becomes

\[
\ln L = n \ln(\alpha \lambda) - \lambda \sum_{i=1}^{n} x_i + 2(1 - \alpha) \sum_{i=1}^{n} e^{-\lambda x_i}
\]

Then the likelihood equations are

\[
\frac{\partial \ln L}{\partial \alpha} = 0 \Rightarrow \frac{n}{\alpha} - 2 \sum_{i=1}^{n} e^{-\lambda x_i} = 0 \tag{3.8}
\]
and

\[
\frac{\partial \ln L}{\partial \lambda} = 0 \Rightarrow \frac{n}{\lambda} = \sum_{i=1}^{n} x_i - 2(1 - \alpha) \sum_{i=1}^{n} x_i e^{-\lambda x_i}
\] (3.9)

Equation (3.8) yields \(\hat{\alpha} = \frac{n}{2 \sum_{i=1}^{n} e^{-\lambda x_i}}\), which on substitution in (3.9) yields the likelihood equation for \(\lambda\) as

\[
\frac{n}{\lambda} - \sum_{i=1}^{n} x_i - 2 \sum_{i=1}^{n} x_i e^{-\lambda x_i} = 0
\] (3.10)

(3.10) is now an equation in terms of \(\lambda\) and can be solved easily to get the estimate \(\hat{\lambda}\). Use this \(\hat{\lambda}\) to get the estimate of \(\hat{\alpha}\). We call these estimates as the first order ML estimates. The second order ML estimates are obtained by using the first order estimates as the initial solutions for equations (3.6) and (3.7).

The asymptotic variance-covariance matrix is obtained by inverting the Fisher information matrix. The elements of Fisher information matrices are

\[
\begin{bmatrix}
i_{\alpha\alpha} & i_{\alpha\lambda} \\i_{\lambda\alpha} & i_{\lambda\lambda}
\end{bmatrix} = \begin{bmatrix} 
-\frac{\partial^2 \ln L}{\partial \alpha^2} |_{\hat{\alpha}, \hat{\lambda}} & -\frac{\partial^2 \ln L}{\partial \alpha \partial \lambda} |_{\hat{\alpha}, \hat{\lambda}} \\-\frac{\partial^2 \ln L}{\partial \lambda \partial \alpha} |_{\hat{\alpha}, \hat{\lambda}} & -\frac{\partial^2 \ln L}{\partial \lambda^2} |_{\hat{\alpha}, \hat{\lambda}}
\end{bmatrix}
\] (3.11)

where

\[
\frac{\partial^2 \ln L}{\partial \alpha^2} = -\frac{n}{\alpha^2} + 2 \sum_{i=1}^{n} \frac{1}{[e^{\lambda x_i} - (1 - \alpha)]^2}
\]

\[
\frac{\partial^2 \ln L}{\partial \alpha \partial \lambda} = \sum_{i=1}^{n} x_i e^{\lambda x_i}
\]

\[
\frac{\partial^2 \ln L}{\partial \lambda^2} = -\frac{n}{\lambda^2} + 2(1 - \alpha) \sum_{i=1}^{n} \frac{x_i^2 e^{\lambda x_i}}{[e^{\lambda x_i} - (1 - \alpha)]^2}
\]
By inverting the above information matrix, one can obtain the variance-covariance matrix which can then be used for constructing confidence intervals for the parameters.

### 3.2.2. Least Square estimation

In this section we provide the regression based method estimators of the unknown parameters, which was originally suggested by Swain, Venkatraman and Wilson (1988) to estimate the parameters of beta distribution. Suppose $X_1, X_2, \ldots, X_n$ is a random sample of size $n$ from a distribution function $F(.)$ and suppose $X_{(i)}; i=1,2,\ldots,n$ denotes the ordered sample. The proposed method uses the distribution $F(X_{(i)})$. For a sample of size $n$, we have

$$E(F(X_{(j)})) = \frac{j}{n+1}, \quad V(F(X_{(j)})) = \frac{j(n-j+1)}{(n+1)^2 (n+2)}$$

and

$$\text{Cov}(F(X_{(j)}), F(X_{(k)})) = \frac{j(n-k+1)}{(n+1)^2 (n+2)}; \quad \text{for } j < k,$$

See Johnson, Kotz and Balakrishnan (1995). Using the expectations and the variances, least square estimator can be obtained by minimizing

$$\sum_{j=1}^{n} \left( F(X_{(j)}) - \frac{j}{n+1} \right)^2$$

with respect to the unknown parameters. Therefore in case of Marshall-Olkin distribution the least squares estimators of $\alpha$ and $\lambda$ can be obtained by minimizing

$$\sum_{i=1}^{n} \left[ 1 - \frac{\alpha}{e^{\lambda X_{(i)}} - (1-\alpha)} - \frac{j}{n+1} \right]^2$$

with respect to $\alpha$ and $\lambda$. Letting

$$Q = \sum_{i=1}^{n} \left[ 1 - \frac{\alpha}{e^{\lambda X_{(i)}} - (1-\alpha)} - \frac{j}{n+1} \right]^2$$

(3.12)
we get

\[
\frac{\partial Q}{\partial \alpha} = 2 \sum_{j=1}^{n} \left[ \left( 1 - \frac{\alpha}{e^{\lambda x(j)} - (1 - \alpha)} \right) - \frac{j}{n+1} \right] \frac{1 - e^{\lambda x(j)}}{(e^{\lambda x(j)} - (1 - \alpha))^2} \]  

(3.13)

and

\[
\frac{\partial Q}{\partial \lambda} = 2 \sum_{j=1}^{n} \left[ \left( 1 - \frac{\alpha}{e^{\lambda x(j)} - (1 - \alpha)} \right) - \frac{j}{n+1} \right] \frac{x(j) \alpha e^{\lambda x(j)}}{(e^{\lambda x(j)} - (1 - \alpha))^2} \]  

(3.14)

Equating this two equations equals to zero we get two non-linear equations which can be solve numerically using Newton-Raphson Method.

### 3.2.3. Percentile estimation

The Marshall – Olkin distribution has the explicit distribution function, therefore in this case the unknown parameters \( \alpha \) and \( \lambda \) can be estimated by equating the sample percentile points with the population percentile points and it is known as the percentile method. This method was originally explored by Kao (1958, 1959). Here we consider the case when both the parameters are unknown. The distribution function of MOGE can be written as

\[
F(x; \alpha, \lambda) = 1.0 - \frac{\alpha}{\exp(\lambda x) - (1 - \alpha)}
\]

Therefore

\[
x = \frac{1}{\lambda} \left[ \ln \left( \frac{\alpha}{1.0 - U} + 1 - \alpha \right) \right]
\]

If \( p_i \) denotes an estimate of \( F(x(i); \alpha, \lambda) \), then the percentile estimators of \( \alpha \) and \( \lambda \) can be obtained by minimizing

\[
L_1 = \sum_{i=1}^{n} \left( x(i) - \frac{1}{\lambda} \ln \left( \frac{\alpha}{1 - p_i} + 1 - \alpha \right) \right)^2
\]

(3.15)
With respect of \( \alpha \) and \( \lambda \). Here \( x_{(i)} \)'s are ordered sample and the minimization has to be performed iteratively. It is possible to use several estimators of \( p_i \)'s. For example \( p_i = i/(n+1) \) is the most used estimator as it is an unbiased estimator of \( F(x_{(i)}; \alpha, \lambda) \).

Differentiating \( L_1 \) with respect to \( \alpha \) and equating with zero we get

\[
\frac{\partial L_1}{\partial \alpha} = \sum_{i=1}^{n} \left( x_{(i)} - \ln(z) \right) \frac{p_i}{\lambda z_i (1 - p_i)} = 0
\]  
(3.16)

\[
\frac{\partial L_1}{\partial \lambda} = 0 \Rightarrow \lambda = \frac{\sum_{i=1}^{n} (\ln(z_i))^2}{\sum_{i=1}^{n} x_{(i)} \ln(z)}
\]  
(3.17)

where \( z_i = \frac{\alpha}{1 - p_i} + (1 - \alpha) \)

### 3.2.4. Bayesian Estimation

Here we consider the Bayes estimation of the unknown parameters, when both are unknown. Here we assume gamma prior for \( \alpha \) and \( \lambda \).

\[
\begin{align*}
\pi_1(\lambda) & \propto \lambda^{b-1} e^{-a\lambda} \\
\pi_2(\alpha) & \propto \alpha^{d-1} e^{-c\alpha}
\end{align*}
\]  
(3.18, 3.19)

If \( X_1, X_2, \ldots, X_n \) are the random sample of size \( n \) from (3.2), then the likelihood function is given by

\[
L(x, \alpha, \lambda) = \frac{\alpha^n \lambda^\lambda \sum_{i=1}^{n} x_i}{\prod_{i=1}^{n} [e^{\lambda x_i} - (1 - \alpha)]^2}
\]  
(3.20)

the joint posterior density function of \( \alpha \) and \( \lambda \) can be written as

\[
\Pi(\alpha, \lambda) = \frac{L(x, \alpha, \lambda) \pi_1(\lambda) \pi_2(\alpha)}{\int \int L(x, \alpha, \lambda) \pi_1(\lambda) \pi_2(\alpha) d\alpha d\lambda}
\]  
(3.21)
Therefore, the Bayes estimator of any function of $\alpha$ and $\lambda$, say $g(\alpha, \lambda)$ under the squared error loss function is

$$E_{\alpha, \lambda | x}(g(\alpha, \lambda)) = \frac{\int_{0}^{\infty} \int_{0}^{\infty} g(\alpha, \lambda) L(x, \alpha, \lambda) \pi_1(\lambda) \pi_2(\alpha) d\alpha d\lambda}{\int_{0}^{\infty} \int_{0}^{\infty} L(x, \alpha, \lambda) \pi_1(\lambda) \pi_2(\alpha) d\alpha d\lambda} \quad (3.22)$$

It is not possible to compute (3.22) analytically in this case. Therefore we use MCMC methods to find the Bayes estimator of $\alpha$ and $\lambda$. The conditional distribution of $\alpha$ given $\lambda$ is

$$\Pi_1(\alpha | \lambda, x) \propto \frac{\alpha^{n+d-1} e^{-c\alpha}}{\prod_{i=1}^{n} [e^{\lambda x_i} - (1 - \alpha)]^2} \quad (3.23)$$

and the conditional distribution of $\lambda$ given $\alpha$ is

$$\Pi_2(\lambda | \alpha, x) \propto \frac{\lambda^{n+b-1} e^{-\lambda(a - \sum_{i=1}^{n} x_i)}}{\prod_{i=1}^{n} [e^{\lambda x_i} - (1 - \alpha)]^2} \quad (3.24)$$
The kernel (3.23) and (3.24) may be easy to work with and the estimates can be obtained directly from the posterior densities. In Table 3.1, we present the Bayes estimate and their corresponding Credibility intervals. Fig 3.5 and fig 3.6 represents the posterior pdf of \( \alpha \) and \( \lambda \). To compare performance of Bayes estimation, we have plotted exact pdf with predictive pdf in fig 3.7 and exact hazard function with predicted hazard function in fig 3.8.

![Fig 3.6 Posterior density function of \( \lambda \) (\( \alpha = 10 \))](image)

![Fig 3.7 Predicted density function and Actual density function(\( \alpha=10.0 \) and \( \lambda=1.5 \))](image)
Fig 3.8  Predicted Hazard function and Actual hazard function(α=10.0 and λ=1.5)

Table 3.1 Bayes Estimates and Bayesian Credible Intervals

<table>
<thead>
<tr>
<th>Actual Values</th>
<th>Posterior Mean</th>
<th>Posterior Median</th>
<th>Bayesian Credible Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>α=2.0</td>
<td>1.4740</td>
<td>1.1254</td>
<td>(0.00, 3.6630)</td>
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<tr>
<td>λ=10.0</td>
<td>8.5629</td>
<td>7.8264</td>
<td>(0.00, 15.6046)</td>
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<tr>
<td>α=10.0</td>
<td>12.2568</td>
<td>12.7426</td>
<td>(0.00, 30.0773)</td>
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<tr>
<td>λ=1.5</td>
<td>1.4894</td>
<td>1.2203</td>
<td>(0.00, 3.9080)</td>
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<td>α=5</td>
<td>3.9877</td>
<td>3.2522</td>
<td>(0.00, 8.2435)</td>
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<td>λ=5</td>
<td>5.0023</td>
<td>4.0844</td>
<td>(0.00, 10.7218)</td>
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<tr>
<td>α=5</td>
<td>3.7077</td>
<td>3.0513</td>
<td>(0.00, 8.4249)</td>
</tr>
<tr>
<td>λ=0.5</td>
<td>0.3298</td>
<td>0.2693</td>
<td>(0.00, 0.8739)</td>
</tr>
</tbody>
</table>

3.3. Confidence interval estimation

Here, we explore the use of likelihoods for the construction of Confidence Intervals for (α, λ). The large sample confidence intervals may be obtained using the ML estimates discussed above. Here we present other Confidence Intervals.
3.3.1. Profile likelihood

The profile likelihood for a single parameter summarizes the sample information for that parameter and provides likelihood confidence intervals. Therefore the profile likelihood for $\alpha$ is $R(\alpha) = \max_{\lambda} \left[ \frac{L(\alpha, \lambda)}{\hat{L}(\alpha, \lambda)} \right]$. Then the interval over which $R(\alpha) > \exp\left[-\chi^2_{1-p;1}/2\right]$ is an approximate 100(1-p) % confidence interval for $\alpha$. Values of $\alpha$ with high profile likelihood are more plausible than those with lower values of profile likelihood. Similarly the profile likelihood for $\lambda$ is $R(\lambda) = \max_{\alpha} \left[ \frac{L(\alpha, \lambda)}{\hat{L}(\alpha, \lambda)} \right]$. For more references, we refer to Meeker and Escobar (1998). In figures 3.9 and 3.10, we present the profile likelihood plot for $\alpha$ and $\lambda$. For both the plots we assumed the parameter values as $\alpha=2$ and $\lambda=10$. The ML estimates for $\alpha$ and $\lambda$ are 1.94 and 11.31 respectively.

![Profile likelihood for $\alpha$](image)
3.3.2. Bootstrap confidence interval

A more appropriate and reasonable method of constructing confidence interval is the bootstrap based confidence interval. When used properly, bootstrap intervals can be expected to be more accurate than the normal-approximation methods and competitive with the likelihood-based methods. The bootstrap approximation of \( Z_\alpha \) can be obtained by using the bootstrap samples to compute 10,000 values of \( Z_{\alpha_j} = \frac{\hat{\alpha}_j - \hat{\alpha}}{\hat{SE}(\hat{\alpha}_j)} \), where \( \hat{\alpha}_j \) is the jth bootstrap estimate of \( \alpha \) and \( \hat{SE}(\alpha_j) \) is the corresponding standard error of the estimate. Then an approximate 100(1-p)% confidence interval for \( \alpha \) based on the assumption that the simulated distribution of \( Z_{\alpha_j} \), provides a good approximation to the distribution of \( Z_{\hat{\alpha}} \) is

\[
\left[ \alpha \sim \hat{\alpha} \right] = \left[ \alpha - Z_{\alpha_j} \hat{SE}(\hat{\alpha}), \alpha - Z_{\alpha_j} \hat{SE}(\hat{\alpha}) \right]
\]

(3.25)
Similarly, one can construct the bootstrap confidence interval for $\lambda$. In Table 3.2 we present a summary of estimates and 95% confidence intervals for various combinations of the parameters of the model.

### 3. 4. Testing of hypothesis

Since the parameters $\alpha$ and $\lambda$ are not orthogonal to each other and since the family is not a member of the exponential family of distributions, it is difficult to suggest uniformly most powerful (UMP) tests. Moreover, it is also very difficult to obtain the sufficient statistics for the parameters even if one is known. Hence, the large sample tests for $\alpha$ and $\lambda$ are obtained as

(i). To test $H_0 : \alpha \geq \alpha_0$ against $H_1 : \alpha < \alpha_0$, we reject $H_0$ if

$$\frac{\sqrt{n}(\hat{\alpha} - \alpha_0)}{\sqrt{\text{Var}(\hat{\alpha})}} < \xi_{\alpha}$$

(ii). To test $H_0 : \lambda \geq \lambda_0$ against $H_1 : \lambda < \lambda_0$, we reject $H_0$ if

$$\frac{\sqrt{n}(\hat{\lambda} - \lambda_0)}{\sqrt{\text{Var}(\hat{\lambda})}} < \xi_{\lambda},$$

where $\xi_{\alpha}$ is the standard normal percentile point. Since the parameter of interest is the intervention parameter $\alpha$, a test for $H_0 : \alpha = 1$ against $H_1 : \alpha \neq 1$ is important and we use likelihood ratio (LR) test. If $\Lambda = \frac{L_0(1, \hat{\lambda})}{L_1(\alpha, \hat{\lambda})}$ is the ratio of conditional likelihood to unconditional likelihood, then we reject $H_0$ if $-2\ln(\Lambda) \geq \chi^2_{1, \alpha}$. The power of the test for $\lambda=10$ and different values of $n$ and $\alpha$ for 5% significance levels are given in Table 3.3. It is observed that the power of the test is increasing as the value of $\alpha$ and sample size increases. As discussed earlier since a large value of $\alpha$ is recommended, we report the power estimates for $\alpha > 1$ only.
Table 3.2. Estimates and confidence intervals

<table>
<thead>
<tr>
<th>Actual</th>
<th>1st order MLE</th>
<th>2nd order (MLE)</th>
<th>Bootstrap estimates</th>
<th>Likelihood based confidence interval</th>
<th>Bootstrap confidence interval</th>
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<tr>
<td>Values</td>
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<td>$\alpha=2.0$</td>
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<td>1.94</td>
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<td>$\alpha=10.0$</td>
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<td>10.4668</td>
<td>(5.545,19.340)</td>
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<tr>
<td></td>
<td></td>
<td>0.7755</td>
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<td>(2.535,9.067)</td>
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<td></td>
<td></td>
<td>3.1134</td>
<td>5.4815</td>
<td>5.5603</td>
<td>(4.343,7.345)</td>
</tr>
<tr>
<td>$\alpha=5$</td>
<td>$\lambda=0.5$</td>
<td>1.2119</td>
<td>4.8047</td>
<td>5.3034</td>
<td>(2.598,9.235)</td>
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<tr>
<td></td>
<td></td>
<td>0.3107</td>
<td>0.5482</td>
<td>0.5584</td>
<td>(0.416,0.688)</td>
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Table 3.3. Power of the test for 5% significance level

<table>
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<tr>
<th>n</th>
<th>$\alpha = 1.2$</th>
<th>$\alpha = 1.5$</th>
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<tr>
<td>10</td>
<td>0.0232</td>
<td>0.0518</td>
<td>0.0768</td>
<td>0.1156</td>
<td>0.2343</td>
</tr>
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<td>20</td>
<td>0.0266</td>
<td>0.0526</td>
<td>0.0992</td>
<td>0.1943</td>
<td>0.3322</td>
</tr>
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<td>30</td>
<td>0.0435</td>
<td>0.0897</td>
<td>0.2345</td>
<td>0.4563</td>
<td>0.6578</td>
</tr>
<tr>
<td>50</td>
<td>0.0692</td>
<td>0.1524</td>
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<tr>
<td>80</td>
<td>0.0982</td>
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<td>0.5172</td>
<td>0.8608</td>
<td>0.9908</td>
</tr>
<tr>
<td>100</td>
<td>0.1106</td>
<td>0.2806</td>
<td>0.5846</td>
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<td>0.9982</td>
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<tr>
<td>120</td>
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<td>0.5643</td>
<td>0.8765</td>
<td>0.9876</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

3.5. Simulation study

We perform extensive simulations to compare the performances of the different methods for different sample sizes and for different parametric values. All the computations are performed in Turbo-C.

We consider different sample sizes ranging from very small to large. We report Average estimates 10,000 replications. Note that it will give the accuracy in the order $\pm (10000)^{-0.5} = \pm 0.01$ (Karian and Dudewicz, 1999). Therefore, we report all the results up to four decimal places.

We present the results of the different methods when both the parameters are unknown. The $\hat{\alpha}_{MLE}, \hat{\lambda}_{MLE}, \hat{\alpha}_{LSE}, \hat{\lambda}_{LSE}, \hat{\alpha}_{PCE}$ can be obtained by solving Non-
linear equations (3.6), (3.7), (3.13),(3.14), (3.16). \( \hat{\lambda}_{PCE} \) can be obtained using (3.17). The estimates are presented in Table 3.4. For each method as sample size increases estimated goes nearer to actual value. For different sample size MLE and Bayesian give consistent result compare to LSE and PCE.

3. 6. Numerical example

In this section we consider two data sets for which we conclude whether the proposed distribution is suitable or not. The first data set is based on Harter (1962) data which may be directly read from Bain and Engelhardt (1991). For this data set we obtained MLE \( \hat{\alpha} = 1.03 \) and \( \hat{\lambda} = 0.012 \) with \(-2\ln(\Lambda) = 0.8019\). Since the value of \(-2\ln(\Lambda) < 3.9414\), we accept the hypothesis and conclude that an exponential distribution is suitable for this data set. Further we also obtained the estimates of the correlation coefficient between the parameters as 0.27 which shows the parameters are more or less orthogonal. The Bayes estimate of this data set are \( \hat{\alpha} = 1.0174 \) and \( \hat{\lambda} = 0.0426 \). We plot the pdf and hazard function of the Harter dataset by taking ML estimates and Bayes estimates of \( \alpha \) and \( \lambda \).

Fig. 3.11  Probability density function of Harter data
Table 3.4. Estimates and confidence intervals

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<tr>
<th></th>
<th>Actual Values</th>
<th>MLE</th>
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<th>PCE</th>
<th>Bayesian</th>
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<td>0.3569</td>
<td>0.2541</td>
<td>0.4874</td>
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</table>

Fig. 3.12 Hazard function of Harter data
The second data set is based on the 23 Ball bearing fatigue data analyzed by Lawless (1982). For reader’s convenience we reproduce the data set as follows: 17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.40, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04 and 173.40. The ML estimates of the parameters for this data set are \( \hat{\alpha} = 4.611 \) and \( \hat{\lambda} = 0.00302 \), while Bayes estimates of the parameters are \( \hat{\alpha} = 6.3734 \) and \( \hat{\alpha} = 0.054 \) respectively. For testing \( H_0 : \alpha = 1 \) against \( H_1 : \alpha \neq 1 \), the likelihood ratio test value is obtained as 10.6671 which are much larger than the tabulated value of chi-square and hence the hypothesis is rejected. Further the estimated correlation coefficient between the parameters is 0.75722 which establishes that the parameters are not orthogonal. Hence a MOGE distribution is a suitable model for the above data set. Fig 3.13 shows probability density function of the data corresponding to ML and Bayes estimates of the parameters. The graph of Hazard function for ML and Bayes estimates of the Ball bearing data are represented in fig 3.14.

![Fig. 3.13 Probability density function of Ball bearing data](image-url)
3.7 Inference for Stress-Strength parameter

In this section inference for $R = P(Y < X)$ is considered when $X$ and $Y$ are independent Marshall–Olkin generalized exponential distribution. The estimation of $R$ is very common in the statistical literature. For example, if $X$ is the strength of a component which is subject to a stress $Y$, then $R$ is a measure of system performance and arises in the context of mechanical reliability of a system. The system fails if and only if at any time the applied stress is greater than its strength. Such situations are seen in many engineering fields like structures, deterioration of rocket motors, static fatigue of ceramic components, fatigue failure of aircraft structures and the ageing of concrete pressure vessels etc. Some other examples are the following:

- The receptors of a communication system operates only if it is stimulated by a source where magnitude $X$ is greater than a random lower threshold $Y$ for the system. In this $R$ is the probability that the receptor operates.

- If $X$ and $Y$ are failure observations on the stability of an engineering design, then $R$ would be the predictive probability that $Y < X$. Similarly, if $X$ and $Y$ represent lifetimes of two electric devices, then $R$ is the probability that one fails before the other.
• If $Y$ represents the diameter of a shaft and $X$ represents the diameter of a bearing that is to be mounted on the shaft, then $R$ is the probability that the bearing fits without interferences.

• If $Y$ represents the maximum chamber pressure generated by ignition of a solid propellant and $X$ represents the strength of the rocket chamber, then $R$ is the probability of successful firing of the engine.

Again the stress-strength relationship raises many interest in the estimation of the functional $R$ because of the following reasons: suppose, the stress may be expensive to sample, such as might be the case in missile flights, but the physical characteristics of the missile system (strengths) such as the propulsive force, angles of elevation, changes in atmospheric condition and so on may all have known distribution. The other instance is that suppose $Y$ is the response for control group, and $X$ refers to a treatment group, then $R$ is the measure of the effect of the treatment etc. We obtain the maximum likelihood estimator (MLE) of $R$, assuming common scale parameters and assuming common shape parameters. Two bootstrap confidence intervals of $R$ are also proposed.

It may be mentioned here that related problems have been widely used in the statistical literature. The MLE of $P(Y<X)$, when $X$ and $Y$ have bivariate exponential distribution, has been considered by Awad et al. (1981). Church and Harris (1970), Downtown (1973), Govidarajulu (1967), Woodward and Kelley (1977) and Owen, Craswell and Hanson (1977) considered the estimation of $P(Y < X)$, when $X$ and $Y$ are normally distributed. Similar problem for the multivariate normal distribution has been considered by Gupta and Gupta (1990). Kelley, Kelley and Schucany (1976), Sathe and Shah (1981), Tong (1974, 1977) considered the estimation of $P(Y<X)$, when $X$ and $Y$ are independent gamma random variables. Ahmad, Fakhry and Jaheen (1997) and Surles and Padgett (2001, 1998) considered the estimation of $P(Y < X)$, where $X$ and $Y$ are Burr Type X random variables.
3.7.1 Maximum Likelihood Estimator of $R$

**Case – I:** Let $X \sim \text{MOGE} (\alpha, \lambda)$, $Y \sim \text{MOGE} (\alpha, \kappa)$, where $X$ and $Y$ are independently distributed. Then,

$$R = P(Y < X) = \int_0^\infty \int_0^\infty \frac{\alpha \lambda e^{\lambda x}}{[e^{\lambda x} - (1 - \alpha)]^2} \frac{\alpha \kappa e^{\kappa y}}{[e^{\kappa y} - (1 - \alpha)]^2} \, dy \, dx$$

$$= 1.0 - \alpha^2 \int_0^\infty \frac{\lambda e^{\lambda x}}{[e^{\lambda x} - (1 - \alpha)]^2} \frac{1}{e^{\lambda x} - (1 - \alpha)} \, dx$$

$$= 1.0 - \alpha^2 \int_0^\infty \frac{1}{\alpha t^2 ((t + (1 - \alpha))^{\lambda/\lambda} - (1 - \alpha))} \, dt \quad (3.26)$$

where $t = e^{\lambda x} - (1 - \alpha)$. To compute the MLE of $R$, first we obtain the MLE of $\alpha$, $\lambda$ and $\kappa$. Suppose $X_1, X_2, ..., X_n$ is a random sample from $\text{MOGE} (\alpha, \lambda)$ and $Y_1, Y_2, ..., Y_m$ is a random sample from $\text{MOGE} (\alpha, \kappa)$. Therefore, the likelihood function of the observed samples is

$$L(x, \alpha, \lambda, \kappa) = \frac{\alpha^n \lambda^n e^{\lambda \sum x_i}}{\prod_{i=1}^n \left(e^{\lambda x_i} - (1 - \alpha)\right)^2} \frac{\alpha^m \kappa^m e^{\lambda \sum y_i}}{\prod_{i=1}^m \left(e^{\kappa y_i} - (1 - \alpha)\right)^2} \quad (3.27)$$

If $l = \log L$, then log-likelihood function becomes

$$l(\alpha, \lambda, \kappa) = n \ln(\alpha) + m \ln(\alpha) + n \ln(\lambda) + m \ln(\kappa) + \lambda \sum x_i + \kappa \sum y_i - 2 \sum_{i=1}^n \ln[e^{\lambda x_i} - (1 - \alpha)] - 2 \sum_{i=1}^m \ln[e^{\kappa y_i} - (1 - \alpha)]$$

The MLE’s of $\alpha$, $\lambda$ and $\kappa$ say $\hat{\alpha}$, $\hat{\lambda}$ and $\hat{\kappa}$ respectively, can be obtained as the solutions of

$$\frac{\partial l}{\partial \alpha} = 0 \Rightarrow \frac{n + m}{\alpha} - 2 \sum_{i=1}^n \frac{1}{e^{\lambda x_i} - (1 - \alpha)} - 2 \sum_{i=1}^m \frac{1}{e^{\kappa y_i} - (1 - \alpha)} = 0 \quad (3.28)$$
\[
\frac{\partial l}{\partial \lambda} = 0 \Rightarrow \frac{n}{\lambda} + \sum_{i=1}^{n} x_i - 2 \sum_{i=1}^{n} \frac{x_i e^{\lambda x_i}}{e^{\lambda x_i} - (1 - \alpha)} = 0 \quad (3.29)
\]

\[
\frac{\partial l}{\partial \kappa} = 0 \Rightarrow \frac{m}{\kappa} + \sum_{i=1}^{m} y_i - 2 \sum_{i=1}^{m} \frac{y_i e^{\kappa y_i}}{e^{\kappa y_i} - (1 - \alpha)} = 0 \quad (3.30)
\]

**Case – II:** Let \( X \sim \text{MOGE} (\alpha, \lambda), Y \sim \text{MOGE} (\beta, \lambda) \), where \( X \) and \( Y \) are independently distributed. Therefore,

\[
R = P(Y < X) = \int_{0}^{\infty} \int_{0}^{\infty} \frac{\alpha \lambda e^\lambda x}{[e^\lambda x - (1 - \alpha)]^2} \frac{\beta \lambda e^\lambda y}{[e^\lambda y - (1 - \beta)]^2} dy \, dx
\]

\[
= 1.0 - \alpha \beta \int_{0}^{\infty} \frac{\lambda e^\lambda x}{[e^\lambda x - (1 - \alpha)]^2} \frac{1}{e^\lambda x - (1 - \beta)} \, dx
\]

\[
= 1.0 - \alpha \beta \int_{0}^{\infty} \frac{1}{t \left( \alpha + (\alpha - \beta) \right)} \, dt
\]

Where \( t = e^\lambda x - (1 - \alpha) \)

\[
= 1.0 - \frac{\beta (-\alpha + \alpha \ln(\alpha) - \alpha \ln(\beta))}{(\alpha - \beta)^2} \quad (3.31)
\]

Here the MLE’s of \( \alpha, \beta \) and \( \lambda \) say \( \hat{\alpha}, \hat{\beta}, \text{ and } \hat{\lambda} \) respectively, are the solutions of

\[
\frac{\partial l}{\partial \alpha} = 0 \Rightarrow \frac{n}{\alpha} - 2 \sum_{i=1}^{n} \frac{1}{e^{\lambda x_i} - (1 - \alpha)} = 0 \quad (3.32)
\]

\[
\frac{\partial l}{\partial \beta} = 0 \Rightarrow \frac{m}{\beta} - 2 \sum_{i=1}^{m} \frac{1}{e^{\lambda y_i} - (1 - \beta)} = 0 \quad (3.33)
\]

\[
\frac{\partial l}{\partial \lambda} = 0 \Rightarrow \frac{n + m}{\lambda} + \sum_{i=1}^{n} x_i + \sum_{i=1}^{m} y_i - 2 \sum_{i=1}^{n} \frac{x_i e^{\lambda x_i}}{e^{\lambda x_i} - (1 - \alpha)} - 2 \sum_{i=1}^{m} \frac{y_i e^{\lambda y_i}}{e^{\lambda y_i} - (1 - \beta)} = 0
\]

\[
(3.34)
\]

### 3.7.2 Numerical Experiment and Discussion

In this section we present some results based on Monte Carlo simulations to compare the performance of the different methods mainly for small sample sizes. All
computations were done in “C-program”, different programs were written for calculating R in different cases.

In Case – I, \( X \sim \text{MOGE} (\alpha, \lambda), \ Y \sim \text{MOGE} (\alpha, \kappa) \), where \( X \) and \( Y \) are independently distributed, we consider two cases separately to draw inference on \( R \), namely when (i) \( \alpha \) is unknown and (ii) \( \alpha \) is known. Similarly for Case-II \( X \sim \text{MOGE} (\alpha , \lambda), \ Y \sim \text{MOGE} (\beta, \lambda) \), where \( X \) and \( Y \) are independently distributed, we consider two cases (i) \( \lambda \) is unknown (ii) \( \lambda \) is known.

In both cases of Case-I we have taken \( \alpha = 1.5, \lambda = 2.0 \) and \( \kappa = 2.0, 2.5, 3.0, 4.0 \) respectively. In case – (i) of Case – II value of parameters are \( \alpha = 2.0, \lambda = 1.0 \) and \( \beta = 2.5, 4.0, 5.0, 10.0 \) and in case-(ii) of Case – II, \( \alpha = 2.0, \lambda = 3.0 \) and \( \beta = 2.5, 4.0, 5.0, 10.0 \) respectively. In all the above cases we consider the following small sample sizes; \( (m,n) = (15,15), (20,20), (25,25), (15,20), (20,15), (15,25), (25,15), (30,50), (50,30) \). All the results are based on 1000 Replications.

In Case-I first we consider the case when the common scale parameter \( \alpha \) is unknown. From the sample, we compute the estimate of \( \alpha \) by solving the non-linear equation (3.28) using iterative algorithm. We have used the initial estimate to be 0.9 and the iterative process stops when the difference between the two consecutive iterates are less than \( 10^{-4} \). Once we estimate \( \alpha \), we estimate \( \lambda \) and \( \kappa \) using (3.29) and (3.30) respectively. Finally we obtain the MLE of \( R \) using (3.26). We report the Average Biases and Mean Square Errors (MSE’S) over 1000 replications in Table 3.5. We do same procedure for all other cases. Table 3.7 represents Average Biases and Mean Square Error (MSE’S) when \( \lambda \) is unknown.

Some of the points are quite clear from this experiment. Even for small sample sizes, the performances of the MLEs are quite satisfactory in terms of biases and MSE’S. It is observed that when \( m=n \) and \( m, n \) increases the MSE’S decreases. It verifies the consistency property of the MLE of \( R \). For fixed \( n \), as \( m \) increases, the
MSE’s decreases. Similarly, for fixed m, as n increases the MSE’s decreases as
Expected.

Now let us consider the case when the common scale parameter is known. In
this case we estimate \( \lambda \) and \( \kappa \) using (3.29) and (3.30) respectively. We obtain the
MLE of R using (3.26). We report the Average Biases and Mean Square Errors (MSE’S)
in Table 3.6.

In this case as expected for all the methods when m=n and m; n increase then
the average biases and MSE’s decrease. For fixed m as n increases the MSE’s
decreases and also for fixed n as m decreases the MSE’s decreases. Similarly we do
same procedure for all other cases. Table 3.8 represents Average Biases and Mean
Square Error (MSE’S) when \( \lambda \) is known.

3.7.3 Bootstrap Confidence Intervals for R

We propose to use two confidence intervals based on the parametric bootstraps
methods;

(i) Percentile bootstrap method (we call it from now on as Boot-p) based on
the idea of Efron (1982).

(ii) Bootstrap – t method (we refer it as Boot – t from now on) based on the
idea of Hall (1988).

We illustrate briefly how to estimate confidence intervals of R using both methods.
Here we discuss both the bootstraps method for case-1.

**Boot-p Methods:**

**Step-1:** From the sample \( \{x_1, x_2, \ldots, x_n\} \) and \( \{y_1, y_2, \ldots, y_m\} \), compute \( \hat{\xi}, \hat{\xi}^* \) and \( \hat{\xi}^* \).

**Step-2:** Using \( \hat{\xi} \) and \( \hat{\xi}^* \) generate a bootstrap sample \( \{x_1^*, x_2^*, \ldots, x_n^*\} \) and similarly
using \( \hat{\xi}^* \) and \( \hat{\xi}^* \) generate a bootstrap sample \( \{y_1^*, y_2^*, \ldots, y_m^*\} \). Based on
\( \{x_1^*, x_2^*, \ldots, x_n^*\} \) and \( \{y_1^*, y_2^*, \ldots, y_m^*\} \), compute the bootstrap estimate of R using
(3.26), say \( R^* \).
**Step-3:** Repeat step-2 NBOOT times.

**Step-4.** Let $G(x) = P(R^* \leq x)$, be the cumulative distribution function of $\tilde{R}$. Define $\hat{R}_{\text{Boot} - \rho(x)} = G^{-1}(x)$ for a given $x$. The approximate $100(1-\gamma)$% confidence interval of $R$ is given by

$$\left( \hat{R}_{\text{Boot} - \rho\left(\frac{\gamma}{2}\right)}, \hat{R}_{\text{Boot} - \rho\left(1-\frac{\gamma}{2}\right)} \right)$$

**Bootstrap-t Confidence Interval**

**Step-1:** From the sample $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_m\}$, compute $\hat{\mu}$, $\hat{\sigma}$ and $\hat{\sigma}^2$.

**Step-2:** Using $\hat{\mu}$ and $\hat{\sigma}^2$ generate a bootstrap sample $\{x_1^*, x_2^*, \ldots, x_n^*\}$ and similarly using $\hat{\sigma}$ and $\hat{\sigma}^2$ generate a bootstrap sample $\{y_1^*, y_2^*, \ldots, y_m^*\}$. Based on $\{x_1^*, x_2^*, \ldots, x_n^*\}$ and $\{y_1^*, y_2^*, y_m^*\}$, compute the bootstrap estimate of $R$ using (3.26), says $R^*$ and the following statistic:

$$T^* = \frac{\sqrt{m} (\hat{R}^* - \hat{R})}{\sqrt{V(R^*)}}$$

Compute $V(R^*)$ using Fisher Information Matrix.

**Step-3:** Repeat step-2 NBOOT times.

**Step-4.** From the NBOOT $T^*$ values obtained, determine the upper and lower bound of the $100(1-\gamma)$% confidence interval of $R$ as follows: Let $H(x) = P(T^* \leq x)$ be the cumulative distribution function of $T^*$. For a given $x$, define

$$\hat{R}_{\text{Boot} - t} = \hat{R} + m \frac{1}{2} \sqrt{V(\hat{R})} H^{-1}(x)$$

Here also, $V(\hat{R})$ can be computed using Fisher Information Matrix. The approximate $100(1-\gamma)$% confidence interval of $R$ is given by

$$\left( \hat{R}_{\text{Boot} - t - \rho\left(\frac{\gamma}{2}\right)}, \hat{R}_{\text{Boot} - t - \rho\left(1-\frac{\gamma}{2}\right)} \right)$$
### Table-3.5 Average Biases and MSE's ($\alpha$ - unknown)

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### Table-3.6. Average Biases and MSE’s ($\alpha$ - known)

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### Table-3.7. Average Biases and MSE’s (\(\lambda\) - unknown)

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### Table-3.8. Average Biases and MSE’s (\(\lambda\) - known)

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In each cell, first row represents the Average Biases and second row represents Mean Square Error.

### 3.8 Conclusion

In this chapter, we have discussed inferences on Marshall-Looking Generalized Exponential distribution. In the introductory section graphs of probability density function, reliability function, hazard function and reverse hazard function have drawn for different values of $\alpha$ and $\lambda$=1 to study about that distribution. It has been seen that exponential distribution is a particular case of MOGE. In section 3.2 different estimation procedures like MLE, Bootstrap, LSE, PCE and Bayes of parameters $\alpha$ and $\lambda$ are considered. Section 3.3 covers Interval estimation methods. In this section we have discuss Likelihood based confidence interval and Bootstrap confidence interval estimates of the parameters $\alpha$ and $\lambda$. Section 3.4 deals with Testing of hypothesis. Simulation study is carried out in section 3.5

Numerical examples are considered in section 3.6. Estimation of $R = P(Y < X)$ are considered in section 3.8.