CHAPTER - VI

AN INVERSION FORMULA FOR FOX-H-TRANSFORM

1. In a very recent paper Verma has introduced Fox-H-Transform as

\[ \phi(u) = \int_0^\infty H(u \xi) f(\xi) d\xi \]  

where the kernel \( H(x) \) is defined by (1.1.6).

We shall, however, take the kernel in the form (1.1.7) and put (6.1.1) as

\[ \phi(u) = \lambda u \int_0^\infty H_{\rho, \nu}^{m, n} \left[ \frac{\xi (a, b)}{\eta (c, d)} \right] f(\xi) d\xi \] 

According to Braakma

\[ H_{\rho, \nu}^{m, n} \left[ \frac{\xi (a, b)}{\eta (c, d)} \right] = O(|x|^\sigma) \text{ for large } x, \]

where \( \sum \epsilon_i - \sum f_i < 0 \) and \( \sum c_i - \sum d_i + \sum \epsilon_i - \sum f_i > 0 \)

\[ \sigma_i = \max R \left( \frac{a_i - 1}{e_i} \right), \quad (i = 1, 2, \ldots, n) \]

and

\[ H_{\rho, \nu}^{m, n} \left[ \frac{\xi (a, b)}{\eta (c, d)} \right] = O(|x|^\gamma) \text{ for small } x, \]

where \( \sum \epsilon_i - \sum f_i \leq 0 \) and \( \gamma = \min R \left( \frac{\epsilon_i}{f_i} \right), (i = 1, 2, \ldots, n) \)

1. Verma, C.B.L. (1966) - - - - - (261)
2. Braakma, B.L.J. (1963) - - - - - (61)
The object of the present chapter is to invert the integral (6.1.2) and to obtain a solution which may serve as a key formula from which a number of inversion formulae for various transform pairs may be deduced as particular cases, by specializing the parameters involved.

Conditions for validity of each transform-pair may, however, be decided individually.

Besides the inversion formula we give here a general relation involving this transform and deduce two results of Sharma therefrom.

2. THE MAIN RESULT:

Theorem 1:
If \( f(x) \) and \( \phi(x) \) belong to \( L(\alpha, \infty) \)
where \( \phi(u) \) is given by (6.1.2), then.

\[
(6.2.1) \quad f(x) = \frac{c}{d} \int_{a}^{\infty} \frac{\phi(u)}{u} \left[ \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \left( \frac{(1 - a_{n,l} - e_{n,l}, \beta_{n,l}, \gamma_{n,l})}{(1 - a_{n,l} - e_{n,l}, \beta_{n,l}, \gamma_{n,l})} \right)^{\alpha} \right] \right] \left[ \sum_{l=1}^{\infty} \left( \frac{(1 - a_{n,l} - e_{n,l}, \beta_{n,l}, \gamma_{n,l})}{(1 - a_{n,l} - e_{n,l}, \beta_{n,l}, \gamma_{n,l})} \right)^{\beta} \right] \left[ \sum_{l=1}^{\infty} \left( \frac{(1 - a_{n,l} - e_{n,l}, \beta_{n,l}, \gamma_{n,l})}{(1 - a_{n,l} - e_{n,l}, \beta_{n,l}, \gamma_{n,l})} \right)^{\gamma} \right] \right] du
\]

provided

(i) \( 0 \leq u \leq \beta, \ 1 \leq \alpha \leq \gamma, \ \sum_{i=1}^{r} e_{i} - \sum_{i=1}^{2} f_{i} < 0 \),

(ii) \( f(x) \) is, continuous at \( x = u, \ u > 0, \ s = k + it, \ 0 < t < \infty \).

\*\*\*\*

1. Sharma, K.C. (1964) b, -- -- -- --(219)
(iii) \( a + \sigma_2 + 1 > 0, \ b + \sigma_1 + 1 < 0, \) where

\[
 f(x) = \begin{cases} \frac{1}{(1 + x)^{a}} & \text{for small } x \\ \frac{1}{(1 + x)^{b}} & \text{for large } x \end{cases}
\]

and \( \sigma_1, \sigma_2 \) are defined as in (6.1.3), (6.1.4),

(iv) \( 1 + \sigma_1 < \lambda < 1 + \sigma_2 \),

(v) The H-function and the integral in (6.2.1) exist.

**Proof.** To prove the theorems of this Chapter we make use of (2.3.1) and (2.3.2).

Multiplying both sides of (6.1.2) by \( u^{-s} \) and integrating with respect to \( u \) between \( 0 \) and \( \infty \), we get with the help of (2.3.2)

\[
 \int_{0}^{\infty} f(x) x^{s-1} dx = \frac{1}{\lambda} \prod_{j=1}^{m} \left\{ 1 - \lambda_j + f_j (s-1) \right\} \int_{0}^{\infty} \frac{\phi(u) du}{u^{s+1}}
\]

Therefore using Inversion Theorem for Mellin Transforms and interpreting with the help of (1.1.7) we get (6.2.1) under the conditions stated.

**SPECIAL CASES.**

I. Putting

\[
 \lambda = \frac{1}{\sqrt{2}}, \ \omega = 1, \ \eta = 0 = \beta, \ \gamma = 2, \ \zeta = \frac{1}{2}, \ \lambda_1 = \frac{1}{4} + \frac{\nu}{2}, \ \lambda_2 = \frac{1}{4} - \frac{\nu}{2}, \ \beta = \frac{1}{2}, \ \gamma_2 = \frac{1}{2} = \gamma_3, \]

we get
\[
\frac{\phi(u)}{u} = \frac{1}{\sqrt{\pi}} \int_0^\infty H_{\mu, 2}^{1,0} \left[ \begin{array}{c}
\frac{ux}{2} \\
\left( \frac{1}{4} + \frac{\nu}{2}, \frac{1}{2} \right), \left( \frac{1}{4} - \frac{\nu}{2}, \frac{1}{2} \right) \end{array} \right] f(x) \, dx
\]

\[
= \int_0^\infty (\chi u)^{\frac{1}{2}} J_\nu (u x) f(x) \, dx
\]

and

\[
f(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{\phi(u)}{u} \cdot H_{\mu, 2}^{1,0} \left[ \begin{array}{c}
\frac{ux}{2} \\
\left( \frac{1}{4} + \frac{\nu}{2}, \frac{1}{2} \right), \left( \frac{1}{4} - \frac{\nu}{2}, \frac{1}{2} \right) \end{array} \right] \, du
\]

\[
= \int_0^\infty (u x)^{\frac{1}{2}} J_\nu (u x) \frac{\phi(u)}{u} \, du
\]

provided \( a + \frac{3}{2} + R(\nu) > 0 \), \( b + 1 < 0 \), \( \int_0^\infty \phi(u) u^{s-1} \, du \)

and \( \int_0^\infty x^{s-1} f(x) \, dx \) converge absolutely, \( f(x) \)

is continuous at \( x = u, u > 0, s = k + t, 0 < t < \infty, \)

\( 1 < k < \frac{3}{2} + R(\nu) \).

This agrees with the inversion formulae of

the Hankel-Fourier sine-and Fourier cosine -

Transforms.

II. Putting \( \lambda = \frac{1}{\sqrt{2}}, \ n = 2, \ m = 0, \ \beta = 1, \ \gamma = 3, \ c = \frac{1}{2} \)

\( a_1 = \frac{-3}{4} - \frac{1}{4}, \ a_2 = \frac{1}{2}, \ b_1 = -\frac{3}{2} + \frac{1}{4}, \ b_2 = -\frac{1}{2} + \frac{3}{4}, \)

\( b_3 = -\frac{1}{4}, \ f_1 = f_2 = f_3 = \frac{1}{2}, \)

we get

1. Ditkin, V. A. &
Prudnikov, A. P. pp. 71-72 (1965)---------(84)
\[ \frac{\phi(u)}{u} = \frac{1}{\sqrt{\pi}} \int_0^\infty H^{2,2}_1 \left( \frac{u}{x} \right) H_{2,3} \left( \frac{x}{u} \right) f(x) dx \]

and

\[ f(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{\phi(u)}{u} H^1_{2,3} \left( \frac{u}{x} \right) f(x) dx \]

provided \( a + \frac{3}{2} + R(\nu) > 0 \), \( b+1 < 0 \), \( \int_0^\infty \phi(u) u^{-s-1} du \)

and \( \int_0^\infty f(x) x^{s-1} dx \) converge absolutely, \( f(x) \) is continuous at \( x = u, u > 0 \), \( s = k + i \tau, c > \tau \leq \infty \), and \( 1 < k < \frac{3}{2} + R(\nu) \).

Similarly, we can show that if \( \frac{\phi(u)}{u} \) is the H-Transform of \( f(x) \), then \( f(x) \) is the Y-Transform of \( \frac{\phi(u)}{u} \). This agrees with the inversion formulae of Y- and H-Transform pairs.

III. Putting \( \lambda = \frac{1}{2\sqrt{\pi}}, c = \frac{1}{2} \), \( f_1 = f_1, n = \alpha = \beta, \nu = \omega = 2 \)

\[ \lambda_1 = \frac{1}{4} + \frac{\nu}{2}, \lambda_2 = \frac{1}{4} - \frac{\nu}{2} \]

we obtain

\[ \frac{\phi(u)}{u} = \frac{1}{2\sqrt{\pi}} \int_0^\infty H^{2,2}_1 \left( \frac{u}{x} \right) f(x) dx \]

and

\[ f(x) = \frac{1}{2\sqrt{\pi}} \int_0^\infty (ux)^\frac{1}{2} K_{\nu}(ux) f(x) dx \]
so that
\[
f(x) = \int_{0}^{\infty} \frac{\phi(u)}{u} H_{\nu,2} \left[ \left( \frac{\nu x}{2} \right) \left( \frac{\nu x}{2} + \frac{1}{2} \right) \right] \, du
\]
\[
= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\phi(u)}{u} \left( \frac{1}{\Gamma(\nu)} \right) \, du \int_{-i\infty}^{i\infty} \frac{(ux)^{-s} \, ds}{\Gamma(\frac{3}{4} - \frac{\nu x}{2} - \frac{s}{2}) \Gamma(\frac{3}{4} + \frac{\nu x}{2} - \frac{s}{2})}
\]
\[
= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{(ux)^{-s} \, ds}{\Gamma(\frac{3}{4} + \frac{\nu x}{2} - \frac{s}{2}) \Gamma(\frac{3}{4} - \frac{\nu x}{2} - \frac{s}{2})}
\]
provided \( a + \frac{3}{2} + R(\nu) > 0 \), \( \int_{0}^{\infty} \phi(u) \, u^{-s-1} \, du \) and \( \int_{0}^{\infty} x^{s-1} f(x) \, dx \) converge absolutely, \( f(x) \) is continuous at \( x = u, \ u > 0 \), \( s = k + it, \ 0 < t < \infty \), \( k < \frac{3}{2} + R(\nu) \).

This is Sharma's result.

IV. Putting \( \lambda = \frac{1}{\sqrt{2}} \), \( m = 1 \), \( u = c = 0 \), \( q = 2 \), \( c = \frac{1}{2} = f_1 \),
\[
f_2 = \frac{c}{x}, \ b_1 = \frac{1}{2} + \frac{1}{q}, \ b_2 = \frac{\nu x}{2} - \nu + \frac{c}{q},
\]
we get
\[
\frac{\phi(u)}{u} = \frac{1}{\sqrt{2}} \int_{0}^{\infty} \frac{b}{b_0} \left( \frac{\nu x}{2} \left( \frac{\nu x}{2} + \frac{1}{2} \right), \left( \frac{\nu x}{2} - \nu + \frac{c}{q} \right) \right) f(x) \, dx
\]
\[
= \frac{1}{2\nu} \int_{0}^{\infty} \left( \frac{\nu x}{2} + \frac{1}{2} \right) \int_{\nu x}^{\infty} \left( \frac{x^2 u^2}{q} \right) f(x) \, dx
\]

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\[ f(x) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} \frac{\Phi(u)}{u} H_{0,1} \left[ \left( \frac{u}{2} + \frac{1}{2} \right)^{\frac{1}{2}} \left( \frac{1}{2} - \frac{u}{2} \right)^{\frac{1}{2}} \right] du \]

\[ = \frac{2 + \nu - \frac{1}{2} (1 + \nu)}{\sqrt{\mu}} \int_0^{\infty} \left( \frac{u}{\mu} \right)^{\frac{1}{2}} \left( \frac{1}{u} \right)^{\frac{1}{2}} \frac{1}{\sqrt{\mu}} \frac{\Phi(u)}{u} du \]

which agrees with the result given by Agrawal\(^1\).

V. We note that \( e^{-px} = H_{0,1} \left[ px \mid (0,1) \right] \).

Specializing the parameters in (6.1.2) we see that

\[ \Phi(p) = \int_0^{\infty} \left[ \frac{1}{p} \left( 1 + \frac{1}{p} \right) \right] f(x) dx = \int_0^{\infty} e^{-px} f(x) dx \]

is the classical Laplace Transform. The corresponding Inversion Formula becomes

\[ f(x) = \int_0^{\infty} \frac{\Phi(p)}{p} H_{0,1} \left[ \left( \frac{1}{p} \right) \left( 1 + \frac{1}{p} \right) \right] dp \]

\[ = \int_0^{\infty} \frac{\Phi(p)}{p} \cdot \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{(px)^{-s}}{(1-s)} ds dp \]

\[ = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\chi}{\left( 1 + \frac{1}{s} \right)} \int_0^{\infty} \Phi(p) p^{-s-1} dp ds \]

1. Agrawal, R.P. (1963) -- -- -- -- (4)
Therefore from Copson

\[ f(x) = \frac{1}{2\pi i} \int_{H_{-i\infty}}^{H_{+i\infty}} \Phi(s) \chi^s \frac{1}{s} dt ds = \frac{1}{2\pi i} \int_{H_{-i\infty}}^{H_{+i\infty}} e^{-t} \frac{1}{t} dt ds \]

Finally using inversion theorem for Mellin transform we get

\[ f(x) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{e^{-t}}{t} \phi(t/x) dt \]

which is the classical inversion formula.

3. Theorem 1 can be put in a slightly different form namely:

\[ f(x) = \frac{1}{\lambda} \int_{0}^{\infty} \frac{\Phi(u)}{u^{\frac{1}{2}}} \left[ \frac{x}{\lambda u} \right] \frac{\left\{ (1-s_{m}, s_{n}), s_{m}, s_{n} \right\}}{\left\{ (1-s_{m}, s_{n}), s_{m}, s_{n} \right\}} \]

1. Copson, E.T. (p.231) (1951) - - - (32)
under relevant conditions

NOTE:

From the proof of Theorem 1 and the Mellin transform of

\[ \int_{c}^{\infty} \left[ c \mathbf{X} \right]\{a_p, e_p\} \{b_q, f_q\} \, dx \]

given in (2.3.2) it can be easily seen that if

\[ \phi(u) = \lambda u \int_{c}^{\infty} H_{c, c} \left[ c \left( u x \right)^{\nu} \left\{ a_p, e_p \right\} \left\{ b_q, f_q \right\} \right] f(x) \, dx \]

then

\[ f(x) = \frac{1}{\lambda} e^{\frac{1}{\lambda} \int_{c}^{\infty} \phi(u) H_{c, c} \left[ c \left( u x \right)^{\nu} \left\{ a_p, e_p \right\} \left\{ b_q, f_q \right\} \right] du} \]

Similarly the result (6.3.1) can be modified.

4. THEOREM 2: If

(i) \( a + \sigma_i > 0 \), \( b + \sigma_i > 0 \),

(ii) \( a' + \sigma_i > 0 \), \( b' + \sigma_i > 0 \),

(iii) conditions (2.3.1) hold,

then

\[ \frac{A t}{\lambda} \int_{0}^{\infty} \frac{\phi(u)}{u} H_{y, y} \left[ \frac{a u t + \lambda_5}{\sigma_i} \left\{ a_w, e_w \right\} \left\{ b_w, f_w \right\} \right] \, du \]

(6.4.1)

\[ = \int_{0}^{\infty} f(x) \left[ c x \left\{ a_w, e_w \right\} \left\{ b_w, f_w \right\} \left\{ a_{w+}, e_{w+} \right\} \right] \, dx \]
provided the $K$-function and the integral on the right of (6.4.1) exist and

$$\phi(u) = \begin{cases} O(|u|^\epsilon) & \text{for small } u \\ O(|u|^\epsilon') & \text{for large } u \end{cases}$$

$$\sigma_i' = \max R \left( \frac{u_i^\epsilon - 1}{c_i} \right), \quad i = 1, 2, \ldots, \beta$$

$$\sigma_i'' = \min R \left( \frac{\lambda_{i, \iota}}{d_{i, \iota}} \right), \quad i = 1, 2, \ldots, \alpha$$

This may be easily obtained by multiplying both sides of (6.1.2) by $\int_{\gamma, \beta} A_{ut} \left[ \frac{\Gamma(\mu, c_r)}{\Gamma(\lambda_m, d_s)} \right]$ integrating with respect to $u$ between 0 and $\infty$, and applying (2.3.1)

**Particular Cases:**

**Case I.** Putting $\lambda = \frac{1}{2\sqrt{\pi}}, \quad c = \frac{1}{2}, \quad \mu = 2 = q, \quad \eta = c = \rho, \quad f = \frac{1}{2} = f$:

$$\beta, \lambda_1 = \frac{1}{4} + \frac{\nu}{2}, \lambda_2 = \frac{1}{4} - \frac{\nu}{2}, \lambda_3 = \frac{1}{4} - \frac{\nu}{2}$$

$$\lambda_1 = \frac{3}{4} - \frac{\nu}{2}, \lambda_2 = \frac{3}{4} - \frac{\nu}{2}, \lambda_3 = \frac{1}{4} + \frac{\nu}{2}, \lambda_3 = \frac{1}{4} - \frac{\nu}{2}$$

$$d_1 = d_2 = d_3 = \frac{1}{2} = c_i$$, we get from (6.1.2) and (6.4.1) respectively.

$$\frac{\phi(u)}{u} = \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} (ux)^{\frac{1}{2}} K_{\nu} (ux) f(x) \, dx$$
and 
\[ \sqrt{2} \pi \int_0^\infty \frac{\phi(\nu)}{\nu^2} (ut)^{1/2} H_{-\nu}^1 (ut) \, d\nu \]

\[ = \frac{1}{t} \int_0^\infty f(x) H_{1/2}^1 \left( x \left| \frac{\nu}{t} \right| \frac{\nu}{t} \right) \, dx \]

Putting 
\[ \psi(\nu, t) = \sqrt{\frac{\pi}{t}} \int_0^\infty \frac{\phi(\nu)}{\nu^2} (ut)^{1/2} H_{-\nu}^1 (ut) \, d\nu \]

and simplifying we see that 
\[ t^{-3/2} \psi(\nu, t) = \int_0^\infty x^{\nu-1} (x^2 + t^2)^{-1/2} \, dx \]

Applying Mellin’s Inversion Theorem we obtain under appropriate conditions 
\[ f(x) = \frac{x^2 + t^2}{2\pi i} \int_{\mathbb{R} - i\infty}^{\mathbb{R} + i\infty} x^{-\nu - 1/2} t^{-3/2} \psi(\nu, t) \, d\nu \]

which is given by Sharma.1

Case II. Putting 
\[ \lambda = \frac{1}{2j\pi}, \ c = \frac{1}{2} = f_1 = f_2 = A, \ \nu = z = 2, \]

\[ n = 0 = p, \ \lambda_1 = \frac{1}{4} + \frac{\nu}{2}, \ \lambda_2 = \frac{1}{4} - \frac{\nu}{2}, \ \alpha = 1, \ \beta = 0 = \gamma \]

\[ \delta = 2, \ \lambda_1 = \frac{1}{4} + \mu + \frac{\nu}{2}, \ \lambda_2 = \frac{1}{4} - \mu + \frac{\nu}{2}, \ \delta_1 = \frac{1}{2} = \delta_2 \]

we get
\[ \frac{\Phi(u)}{u} = \sqrt{\frac{2}{\pi}} \int_0^\infty (ux)^{1/2} K_{\nu}(ux) f(x) \, dx \]
and
\[ t^{M+1/2} \int_0^\infty (ut)^{1/2} \int_0^\infty (ut)^{\mu-1} \phi(u) \, du \]
\[ = 2 \int_0^\infty \left( \frac{x}{t} \right)^{1/2} \left( \int_0^\infty (t+\mu)f(x) \, t^{3/2} \right) \frac{\mu-1}{(x^2+t^2)^{M+1}} \, dx \]
or
\[ \left( \frac{2t}{t^2} \right)^{-\mu-1/2} \pi^{1/2} \psi(y) = \int_0^\infty \left( \frac{x^2}{t^2} \right)^{1/2} \left( \int_0^\infty (t+\mu)f(x) \, \frac{x}{(x^2+t^2)^{M+1}} \right) \, dx \]
where
\[ \psi(y) = \int_0^\infty (ut)^{1/2} \int_0^\infty (ut)^{\mu-1} \phi(u) \, du \]

Inverting this integral with the help of
Inversion Theorem for Mellin Transforms we get
under relevant conditions.

\[ \int_0^{(\mu+1)} f(x) = \pi^{-1/2} \left( \frac{2}{t} \right)^{-\mu-1/2} \left( t^2 + x^2 \right)^{1/2} \int_{-\infty}^{(\mu+1)} \psi(y) \, dy \]
which is known.  

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PART - II