CHAPTER - II

TWO THEOREMS ON $\mathbb{H}$-FUNCTION

1. The object of this Chapter is to obtain two theorems on $\mathbb{H}$-Functions. Theorem I is an "Addition Theorem" on $\mathbb{H}$-Function. The result on G-Function by Edelstein follows as an immediate consequence of this theorem. In Theorem II, we evaluate an integral involving product of two $\mathbb{H}$-Functions. Since $\mathbb{H}$-Function is more general than even Meijer's G-Function, the result obtained here becomes a master or key formula from which a large number of relations can be deduced for functions appearing in applied Mathematics and Mathematical Physics.

We shall make use of the following special cases of $\mathbb{H}$-Function in this Chapter:

\begin{align}
(2.1.1) \quad & H_{\ell, z}^{m, n} \left[ \kappa \left| \frac{1}{2} \right| \left( \frac{a_1, a_2, \ldots, a_p}{b_1, b_2, \ldots, b_q} \right) \right] = \frac{1}{\ell} \mathbb{C}_{\ell}^{m, n} \left( \kappa \left| \frac{1}{2} \right| \frac{a_1, a_2, \ldots, a_p}{b_1, b_2, \ldots, b_q} \right) \\
(2.1.2) \quad & H_{\ell, z}^{2, 0} \left[ \kappa \left| \frac{\ell}{2} + \frac{1}{4} + \frac{1}{2}, \frac{1}{2} \right| \left( \frac{\ell}{2} + \frac{1}{4} - \frac{1}{2}, \frac{1}{2} \right) \right] \\
& = \kappa \left( \frac{\ell + \frac{1}{2}}{2} \right) K_{\ell, \frac{1}{2}}(2 \kappa) \\
\end{align}

\begin{align}
(2.1.3) \quad H_{\mu, \nu}^{(\ell-\ell+1, 1)} (x) &= 
(2.1.4) \quad H_{0, \nu}^{(\ell+1, 1/2)} (x) = \frac{1}{2} \int_{-\nu}^{\nu} \gamma(x)
(2.1.5) \quad H_{0, 1}^{(\ell+1, 1/2)} (x) = e^{-x}
(2.1.6) \quad H_{0, 2}^{(\ell+1, 1/2)} (x) = 2 \pi^{-1/2} \cos 2x
(2.1.7) \quad H_{0, 2}^{(\ell+1, 1/2)} (x) = 2 \pi^{-1/2} \sin 2x
\end{align}

2. **Theorem I**

\begin{align}
(2.2.1) \quad H_{\mu, \nu}^{(\ell+1, 1/2)} \left[ \gamma \left\{ (x, y) \right\} \right] &= H_{\mu, \nu}^{(\ell+1, 1/2)} (x) + \\
&\quad + \sum_{k=1}^{\infty} \left( \frac{\gamma}{k} \right)^{1/2} \frac{1}{\sqrt{k}} H_{\mu, \nu}^{(\ell+1, 1/2)} \left[ \gamma \left\{ (0, 1) \right\} \right]
\end{align}

Provided

\begin{align}
\gamma &= \sum_{j=1}^{m} f_j - \sum_{j=m+1}^{n} f_j + \sum_{j=1}^{n} \varepsilon_j - \sum_{j=n+1}^{p} \varepsilon_j > 0,
\arg \left( \frac{\gamma}{\pi} \right) < \frac{1}{2} \gamma \pi, \quad \arg \left( \frac{\gamma}{\pi} \right) < \frac{1}{2} \gamma \pi, \quad \left| \frac{\gamma}{\pi} \right| < 1.
\end{align}

And the series on the right is convergent.

**Proof:** we have

\begin{align}
H_{\mu, \nu}^{(\ell+1, 1/2)} \left[ \gamma \left\{ (x, y) \right\} \right] &= \\
= \frac{1}{2\pi i} \int_{L} \cdots \frac{1}{2
\int_{L} \frac{1}{\prod_{j=m+1}^{n} (1-\gamma_j + \varepsilon_j) \prod_{j=m+1}^{p} (\varepsilon_j - \varepsilon_j)} (x+y) ds
\[ (33) \]
\[
\frac{1}{2\pi i} \int_{L} \prod_{j=1}^{m} \frac{(L_j - \xi_j^2)}{L_j} \prod_{j=1}^{n} \frac{(1 - \alpha_j^2 + \eta_j^2)}{\eta_j} \frac{s}{\alpha_j^2 \eta_j} \left[ 1 + \sum_{k=1}^{\infty} \left( \frac{y}{\alpha_k^2} \right) \left( \frac{-1}{\xi_k^2} \right) \frac{s}{\alpha_k^2 \xi_k} \right] ds.
\]

Provided \[ \left| \frac{y}{\alpha} \right| < 1 \]

Now
\[ (-1)^{\xi} (-s)_2 = \frac{\sqrt{(1+s)}}{\sqrt{(1+s-x^2)}} \]

So we get
\[
H_{p,q}^{m,n} \left[ \left( \alpha \right)^{\frac{1}{2}} \left( \eta \right)^{\frac{1}{2}} \right] \left\{ \{(\alpha_p, \eta_p)\} \right\} \left\{ \{(\alpha_q, \eta_q)\} \right\}
\]

\[ + \frac{1}{2\pi i} \sum_{k=1}^{\infty} \left( \frac{y}{\alpha_k} \right)^{\xi} \left( \frac{1}{\xi_k} \right)^{\frac{1}{2}} \int_{L} \prod_{j=1}^{m} \frac{(L_j - \xi_j^2)}{L_j} \prod_{j=1}^{n} \frac{(1 - \alpha_j^2 + \eta_j^2)}{\eta_j} \frac{s}{\alpha_j^2 \eta_j} \left[ 1 + \sum_{k=1}^{\infty} \left( \frac{y}{\alpha_k^2} \right) \left( \frac{-1}{\xi_k^2} \right) \frac{s}{\alpha_k^2 \xi_k} \right] ds \]

\[ = H_{p,q}^{m,n} \left[ \left( \alpha \right)^{\frac{1}{2}} \left( \eta \right)^{\frac{1}{2}} \right] \left\{ \{(\alpha_p, \eta_p)\} \right\} \left\{ \{(\alpha_q, \eta_q)\} \right\} + \sum_{k=1}^{\infty} \left( \frac{y}{\alpha_k} \right)^{\xi} \left( \frac{1}{\xi_k} \right)^{\frac{1}{2}} H_{p+1,q+1}^{m,n+1} \left[ \left( \alpha \right)^{\frac{1}{2}} \left( \eta \right)^{\frac{1}{2}} \right] \left\{ \{(\alpha_p, \eta_p)\} \right\} \left\{ \{(\alpha_q, \eta_q)\} \right\} \]

It is easy to see that the term by term integration is valid as the integrand does not involve the argument \[ \frac{y}{\alpha} \] and conditions (A) justify the existence of every H-function in the series.

Moreover the series \[ \sum_{k=1}^{\infty} \left( (-1)^{\xi} (-s)_2 \left( \frac{y}{\alpha} \right)^{\xi} \left( \frac{1}{\xi_k} \right)^{\frac{1}{2}} \right) \] is uniformly convergent as long as \[ \left| \frac{y}{\alpha} \right| < 1 \].

1. Bateman manuscript project, (p. 68 art. 2.1.6)
Putting \( e_1 = e_2 = \cdots = e_\ell = 1 = f_1 = f_2 = \cdots = f_\ell \)
we obtain Edelstein's result which has been used by him in molecular quantum mechanics.

3. **Theorem II**

If \( \lambda < 1 \)

\[
\begin{align*}
&H^m \left[ \alpha x^\lambda \left\{ \left( \begin{array}{c} a_1 \\ \vdots \\ a_n \\ \frac{e_n}{e_j} \end{array} \right) \right\}, \left\{ \left( \begin{array}{c} f_1 \\ \vdots \\ f_\ell \end{array} \right) \right\} \right] \in L(0, \infty) \\
&\text{and } M \left\{ H_{\alpha, t} \left[ \beta x^\lambda \left\{ \left( \begin{array}{c} c_1 \\ \vdots \\ c_\ell \end{array} \right) \right\}, \left\{ \left( \begin{array}{c} d_1 \\ \vdots \\ d_n \end{array} \right) \right\} \right] \right\} \in L(-\infty, \infty) \end{align*}
\]

where \( s = \alpha + i \gamma \) and \( M \{ f(x); s \} \) denotes the Mellin Transform of \( f(x) \), then

\[
\int_0^\infty H^m \left[ \alpha x^\lambda \left\{ \left( \begin{array}{c} a_1 \\ \vdots \\ a_n \\ \frac{e_n}{e_j} \end{array} \right) \right\}, \left\{ \left( \begin{array}{c} f_1 \\ \vdots \\ f_\ell \end{array} \right) \right\} \right] H_{\alpha, t} \left[ \beta x^\lambda \left\{ \left( \begin{array}{c} c_1 \\ \vdots \\ c_\ell \end{array} \right) \right\}, \left\{ \left( \begin{array}{c} d_1 \\ \vdots \\ d_n \end{array} \right) \right\} \right] dx
\]

\[
\int_0^\infty H^m \left[ \alpha x^\lambda \left\{ \left( \begin{array}{c} a_1 \\ \vdots \\ a_n \\ \frac{e_n}{e_j} \end{array} \right) \right\}, \left\{ \left( \begin{array}{c} f_1 \\ \vdots \\ f_\ell \end{array} \right) \right\} \right] H_{\alpha, t} \left[ \beta x^\lambda \left\{ \left( \begin{array}{c} c_1 \\ \vdots \\ c_\ell \end{array} \right) \right\}, \left\{ \left( \begin{array}{c} d_1 \\ \vdots \\ d_n \end{array} \right) \right\} \right] dx
\]

\[
= \frac{1}{\sigma^\lambda \beta^\lambda} \cdot \int_{b+t, \bar{z}+\bar{t}} \left[ \alpha x^\lambda \left\{ \left( \begin{array}{c} a_1 \\ \vdots \\ a_n \\ \frac{e_n}{e_j} \end{array} \right), \left\{ \left( \begin{array}{c} c_1 \\ \vdots \\ c_\ell \end{array} \right) \right\} \right\} \right] \left[ \beta x^\lambda \left\{ \left( \begin{array}{c} a_1 \\ \vdots \\ a_n \\ \frac{e_n}{e_j} \end{array} \right), \left\{ \left( \begin{array}{c} c_1 \\ \vdots \\ c_\ell \end{array} \right) \right\} \right\} \right] dx
\]

provided that \( \sigma > 0, \lambda > 0 \) and the conditions 
(1), (ii) along with one of the conditions in (iii) and (iv) given below are satisfied.

(1) \(- \sigma \min_j R \left( \frac{e_j}{f_j} \right) < R(s) < \frac{\sigma}{\bar{z}^\ell} - \sigma \max_{1 \leq j \leq n} R \left( \frac{e_j}{e_j} \right) \)

(2) \(- \lambda \min_j R \left( \frac{d_j}{b_j} \right) < R(s) < \frac{\lambda}{\bar{z}^\ell} - \lambda \max_{1 \leq j \leq \ell} R \left( \frac{d_j}{b_j} \right) \)

1. Edelstein, L.A. (1964) - - - - - - (88)
(35)

\[ \lambda \min R \left( \frac{d_j}{u_j} \right) + \sigma \min_{1 \leq i \leq m} R \left( \frac{b_i}{f_i} \right) + 1 > 0 \]

\[ \sigma \left[ \frac{1 - \max_{e_j} R(e_j)}{e_j} \right] + \lambda \left[ \frac{1 - \max_{c_i} R(c_i)}{c_i} \right] > 1 \]

\[ \gamma > 0, \quad |\text{ang}(\alpha)| < \frac{1}{2} \gamma \pi \]

or

\[ \gamma > 0, \quad |\text{ang}(\alpha)| \leq \frac{1}{2} \gamma \pi \quad \text{and} \quad R(\delta+1) < 0, \quad R(\delta+1') < 0, \]

\[ R(\delta+1-\frac{1}{\lambda} \sum_{J} \xi_{j} + \frac{1}{\lambda} \sum_{J} u_{j}) < 0, \]

(iv)

\[ \gamma' > 0, \quad |\text{ang}(\beta)| < \frac{1}{2} \gamma' \pi \]

or

\[ \gamma' > 0, \quad |\text{ang}(\beta)| \leq \frac{1}{2} \gamma' \pi \quad \text{and} \quad R(\delta+1) < 0, \quad R(\delta+1') < 0, \]

\[ R(\delta+1-\frac{1}{\lambda} \sum_{J} \xi_{j} + \frac{1}{\lambda} \sum_{J} u_{j}) < 0, \]

where

\[ \gamma = \sum_{j=1}^{n} e_j - \sum_{j=n+1}^{p} e_j + \sum_{j=1}^{m} f_j - \sum_{j=m+1}^{q} f_j \]

\[ \gamma' = \sum_{j=1}^{q} f_j - \sum_{j=1}^{p} f_j + \sum_{j=1}^{t} u_j - \sum_{j=1}^{t} u_j \]

(v)

\[ \delta = \frac{1}{2} (\gamma - \delta) + \sum_{j=1}^{p} \xi_j - \sum_{j=1}^{m} c_j \]

\[ \delta' = \frac{1}{2} (\gamma' - \delta') + \sum_{j=1}^{q} f_j - \sum_{j=1}^{t} c_j \]

Proof: If \( F(s) = M \{ f(x) ; s \} \) and \( G(s) = M \{ g(x) ; s \} \) denote the Mellin Transforms of \( f(x) \) and \( g(x) \) respectively, then by Faltung Theorem

1. Sneddon, I. N. (1951) - (p. 48) - (227)
\[(36)\]
\[\int_0^\infty f(x) g(x) \, dx = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) G(1-s) \, ds\]

development provided that \(f(x)\) belongs to \(L(0, \infty)\) and \(G(1-s)\) belongs to \(L(-\infty, 0)\), \(s = a + i \tau\).

Now, from the definition of the Mellin transform and (1.1.7), we easily see that

\[M \{ \mathcal{H}_{\alpha, \beta}^\mu \left[ \frac{x^\mu}{\alpha \beta} \right] \} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{(-1)^{\alpha+j} \pi^{-\frac{1}{2}n}}{\prod_{j=1}^n \Gamma(1-\alpha_j) \Gamma(1-\beta_j) \prod_{j=d+1}^e \Gamma(1-\alpha_j - \beta_j) \Gamma(1-\beta_j)} x^{-s} \, ds\]

provided

\[(\alpha)\] \(-\infty < \min_{1 \leq j \leq d} R(\frac{\beta_j}{\alpha_j}) < \max_{1 \leq j \leq e} R(\frac{\alpha_j}{\beta_j}) < \infty\)

and

\[(\beta)\]

\[\theta > 0, \quad |\arg(\mu)| \leq \frac{1}{2}\theta\pi, \quad \text{or} \quad \theta > 0, \quad |\arg(\mu)| \leq \frac{1}{2}\theta\pi, \quad \text{and} \quad R(\mu + 1) < 0.\]

where

\[\theta = \sum_{j=1}^d \eta_j - \sum_{j=d+1}^e \eta_j + \sum_{j=1}^d \beta_j - \sum_{j=d+1}^e \beta_j\]

\[\phi = \frac{1}{2} (u - v) + \sum_{j=1}^d \beta_j - \sum_{j=1}^d \alpha_j\]
Using (2.3.2) to find the Mellin Transforms of
\[ H_{\mu, \nu}^{m, n} \left[ \frac{\sigma}{\gamma} \left| \frac{(a_p, b_p)}{c_p, d_p} \right| \right] \text{ and } H_{\alpha, \beta}^{k, l} \left[ \frac{\lambda}{\mu} \left| \frac{(c_1, d_1)}{e_1, f_1} \right| \right] \]
we apply Theorem (B) above and interpret the right-hand side by means of (1.1.7) to get the desired result under conditions stated.

Special Case:—
Putting \( \lambda = \sigma = 1 \) and \( \epsilon_1 = \epsilon_2 = \ldots = \epsilon_p = 1 \)
\( f_1 = f_2 = \ldots = f_p = 1 = k_1 = k_2 = \ldots = k_t = u_1 = u_2 = \ldots = u_t \)
then in view of (2.1.1) and properties of G-function, we have,
\[ \int_0^\infty G_{m, n}^{m, n} \left( \frac{\alpha}{\gamma} \left| \frac{a_1, \ldots, a_p}{b_1, \ldots, b_t} \right| \right) \quad G_{\alpha, \beta}^{k, l} \left( \frac{\mu}{\lambda} \left| \frac{c_1, \ldots, c_p}{d_1, \ldots, d_t} \right| \right) \quad d\alpha \]
\[ = \frac{1}{x} G_{q, k}^{n+k, l+m} \left( \frac{\mu}{\lambda} \left| \frac{-b_1, \ldots, -b_m, c_1, \ldots, c_q, -b_m, \ldots, -b_q}{-a_1, \ldots, -a_n, d_1, \ldots, d_t, -a_n, \ldots, -a_p} \right| \right) \]
under relevant conditions. This is a known result.

If further we put
(a) k=2=t, \( \ell=0=r \), \( \lambda = 1 \), \( \beta = \frac{u}{2} \), \( u_1 = \frac{1}{2} = u_2 \)
\( a_1 = \ell_1 + \frac{1}{4} = \frac{u}{2} \), \( a_2 = \ell_2 + \frac{1}{4} = \frac{u}{2} \)
and
(b) k=2=t, \( \ell=0 \), \( r=1 \), \( \lambda = 1 \), \( \beta = u \), \( k_1 = 1 = u_1 = u_2 \)
\( c_1 = k_1 - k_{11} \), \( a_1 = \ell - k_{11} + k_{11} \), \( a_2 = \ell - k_{11} - k_{11} \)

1. Bateman Manuscript Project, p.209(8),(9) (1953) a = -- -(38)
2. Bateman Manuscript Project, p.422 -- (1954) b = -- -(42)
in (2.3.1), then by virtue of (2.1.2) and (2.1.3)
we get the results obtained by Gupta viz,

\[
\int_{0}^{\infty} \mathcal{K}_{\lambda}(u, \lambda) \mathcal{H}_{p, q} \left( \frac{\kappa}{\lambda} \right)^{\frac{m+n}{2}} \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m+n+1}{2}\right)} \left\{ (ap, ep) \right\} \left\{ (ap, ep) \right\} \, \lambda \, d\lambda
\]

and

\[
\int_{0}^{\infty} \mathcal{K}_{\lambda}(u, \lambda) \mathcal{H}_{p, q} \left( \frac{\kappa}{\lambda} \right)^{\frac{m+n-1}{2}} \frac{\Gamma\left(\frac{m+n-1}{2}\right)}{\Gamma\left(\frac{m+n}{2}\right)} \left\{ (ap, ep) \right\} \left\{ (ap, ep) \right\} \, \lambda \, d\lambda
\]

respectively with necessary conditions stated therein.

It is interesting to observe that in view of the relations (2.1.4), (2.1.5), (2.1.6) and (2.1.7), the results on Hankel, Laplace and the Fourier Transforms become particular cases of our result.

---