CHAPTER - I

INTRODUCTION

"The object of the present dissertation is to investigate certain properties of H-Function of Fox, to utilize them to generalize various Integral Transforms and to obtain some results on Self-reciprocal Functions with a view to enlarge the field of application of Special Functions appearing in Applied Mathematics and Mathematical Physics."

The H-Function embraces almost all the functions encountered in the solution of problems in Applied Mathematics, Physics and Technology; and it is the most general character of the H-Function that has motivated us to carry out investigations about it. In this connection it is significant to note that many of the well-known results of various Special Functions may be easily derived from those of H-Function by giving suitable values to the parameters involved."

(1.1) HISTORICAL INTRODUCTION

Till the beginning of the eighteenth century there was no distinction between Pure and Applied Mathematics and the great Mathematicians of the time were amazingly dextrous in the most difficult art of collaboration between theoretical investigations."
and Physical requirements. On account of "Colossal out-put of both Pure and Applied Mathematics" due to Bernoullis, Euler, D'Alembert, Clairaut, Laplace, Bessel, Legendre and Monge "it became humanly impossible by the middle of the nineteenth century for a man to attain the first rank as a Scientist and as a Mathematician". Thus the situation in this period led to the study of Mathematics as an independent branch of Science.

The investigations about the oscillations of heavy chains, vibrations of a stretched membrane, elliptic motion, dynamical astronomy, in the early eighteenth century led to the discovery of Bessel coefficients (later extended to Bessel Functions) and similarly other Special Functions were born in Science and Technology. Thus most of the Special Functions appeared either as solutions of differential equations of Applied Mathematics and Mathematical Physics or as limiting processes on infinite series required in Physical situations.

When the Special Functions fulfilled the immediate Scientific purposes of applications in the physical problems, "they were exploited by numerous analysts whose interests were purely Mathematical." 

We outline a brief history of the development of $H$-functions, self-reciprocal functions, fractional integrals, and integral transforms.

(A1) **Gauss' Hypergeometric Function.**

In 1812 Gauss$^1$ obtained the differential equation (Gauss' equation or hypergeometric equation)

$$3(1-3) \frac{d^2u}{d\zeta^2} + [c - (a+b+1)3] \frac{du}{d\zeta} - abu = 0$$

satisfied by

$$F(a, b; c; \zeta)$$

\[ (1, 1, 1) = 1 + \frac{a}{1, c} \zeta + \frac{a(a+1)b(b+1)}{1, 2, c(c+1)} \zeta^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{1, 2, 3, c(c+1)(c+2)} \zeta^3 \]

and proved that the series on the right is absolutely convergent if $|\zeta| < 1$ and divergent if $|\zeta| > 1$.

When $|\zeta| = 1$, the series is absolutely convergent if $R(a+b-c) < 0$. It is assumed that $c$ is not a negative integer. He further pointed out that for special values of its letters this series represented nearly all functions then known. Kummer$^2$ obtained twenty-four solutions of this differential equation and the contributions made by Goursat$^3$ are also noteworthy.

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1. Gauss, C.F. (1866) - - - - - - - - - (107)
2. Kummer, E.E. (1836) - - - - - - - - - (138)
3. Goursat, E. (1881) - - - - - - - - - (109)
(A9) GENERALIZED HYPERGEOMETRIC FUNCTION.

The Gauss' hypergeometric series has been generalized in various ways by Haine, Appell, and others. One way of generalization is by the introduction of more parameters like a, b, c. In the compact notation invented by Pochhammer and modified by Barnes we write the generalized hypergeometric series as

\[ \sum_{n=0}^{\infty} \frac{[a_1]_n [a_2]_n \cdots [a_k]_n}{[b_1]_n [b_2]_n \cdots [b_q]_n} \frac{z^n}{n!} \]

where \( [\alpha]_n = \alpha(\alpha+1)(\alpha+2) \cdots (\alpha+n-1) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \)

\( [\alpha]_0 = 1 \), and no b is a non-positive integer.

The earliest reference of the generalized hypergeometric series is found in 1828 when Clausen introduced it in the case \( p=3, q=2 \) and proved

\[ \left( \sum_{n=0}^{\infty} \frac{2a \cdot a+b \cdot 2b}{a+b+\frac{1}{2} \cdot 2a+2b} \frac{z^n}{n!} \right)^2 = \sum_{n=0}^{\infty} \frac{2a \cdot a+b \cdot 2b}{a+b+\frac{1}{2} \cdot 2a+2b} \frac{z^n}{n!} \]

Pochhammer discussed the cases \( q=3, q=4, (p \leq q) \), \( p=q+1 \) and found multiple integral representations of \( p^q \).

1. Haine, E. (1878) (128)
2. Appell, Paul (1926) (9)
5. Clausen, Thomas (1828) (31)
(8)

\[ p^F_q \] converges under the following conditions:

(i) \( p < q+1 \), for all \( a_n, a_{n+1}, \ldots, b_n, b_{n+1}, \ldots \) real or complex, or

(ii) \( p=q+1 \) and \( |z| < 1 \), for all \( a_n, a_{n+1}, \ldots, b_n, b_{n+1}, \ldots \)

or \( z=k \), and \( R \left[ \sum_{i}^{q} b_i - \sum_{i}^{q+1} a_i \right] > 0 \),

or \( z=-1 \), and \( R \left[ \sum_{i}^{q} b_i - \sum_{i}^{q+1} a_i \right] > -1 \).

\[ p^F_q \] diverges for all \( z \neq 0 \) if \( p > q+1 \).

The generalized hypergeometric series have been studied extensively. Amongst numerous authors who made notable contributions may be mentioned Bailey, Barnes, Bateman, Burchnall, Chaundy, Dougall, Erdelyi, Fasenmyer, Fox, Goldstein, Hardy, Luke, Coleman, Rainville, Rice, Saalschütz, Srivastava, Whipple and Wimp. The confluent hypergeometric functions of Whittaker\(^2,3\) have been used with advantage in various Integral Transforms. \(^4,7\)

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1. References to relevant literature are given in (13)-(23), (30), (34)-(36), (63), (64), (77), (78), (87), (92), (93), (95), (100), (102), (108), (124), (142), (185), (186), (191), (202), (229), (266)-(269), (276).

2. Whittaker, E.T. \(1904\) \( \quad \) \(1904\) \( \quad \) \(270\)

3. Whittaker, E.T. & Watson, G.N. \(1946\) \( \quad \) \(1946\) \( \quad \) \(271\)

4. Mainra, V.P. \(1961\) \( \quad \) \(1961\) \( \quad \) \(148\)

5. Meijer, G.S. \(1941\) \( \quad \) \(1941\) \( \quad \) \(166\)

6. Rathie, P.N. \(1965\) \( \quad \) \(1965\) \( \quad \) \(127\)

7. Varma, R.S. \(1951\) \( \quad \) \(1951\) \( \quad \) \(241\)
The discovery of Gauss' hypergeometric function and its generalizations played the most fundamental role in the advance of the theory of Special Functions. In an attempt to give a meaning to $p^q$ when $p > q + 1$, Meijer in 1936, and MacRobert in 1937-38 respectively introduced G and E-Functions. Originally E-function was defined in terms of a multiple integral, but afterwards E-function was expressed as:

$$E \left( \begin{array}{c} b; a_1, a_2, \ldots, a_p; z \\ b_1, b_2, \ldots, b_q; z \end{array} \right) = \frac{1}{2\pi i} \int_L \frac{\left( \prod_{k=1}^{b} \Gamma \left( b + \frac{z}{b} \right) \right)^{b}}{\prod_{t=1}^{q} \Gamma \left( b_t - \frac{z}{b} \right)} dz$$

where $z = \gamma + i \eta$, $L$ is taken upwards along the $\eta$-axis (when $p < q + 1$, the contour is bent to the left at both ends) with loops, if necessary, to ensure that the pole at the origin lies to the left and the poles at $a_1, a_2, \ldots, a_p$ to the right of the contour. Zero and negative integral values of the $a$'s and $b$'s are excluded and $a$'s must not differ by integral values.

1. Meijer, C.S. (1936) (164)
2. MacRobert, T.M. (1937-38) (143)
3. Bateman Manuscript Project P.204 (1953) (38)
Meijer originally defined the G-function as a sum of the series of generalized hypergeometric functions but later in 1941 he replaced the definition by one in terms of Mellin-Barnes type integral as:

\[ G_{p,q}^{m,n}(x \left| a_1, \ldots, a_p \right. b_1, \ldots, b_q) = \frac{1}{2\pi i} \int \frac{i \prod_{j=1}^{m} \Gamma((\alpha_j - \gamma_j) \prod_{j=1}^{n} \Gamma((1 - \alpha_j + \gamma_j) x \overset{\gamma_j}}{\prod_{j=m+1}^{q} \Gamma(1 - \alpha_j + \gamma_j) \prod_{j=n+1}^{p} \Gamma(\alpha_j - \gamma_j)} \]  

where \( x \neq 0 \), an empty product is interpreted as unity \( 1 \leq m \leq q, 0 \leq n \leq p \), and the parameters are such that no pole of \( \prod_{j=1}^{m} (\alpha_j - \gamma_j) \), \( j = 1, 2, \ldots, m \) coincides with any pole of \( \prod_{j=1}^{n} (1 - \alpha_j + \gamma_j) \), \( j = 1, 2, \ldots, n \). \( L \) is a contour similar to that in (1.1.3) and parameters satisfy certain conditions.

It may be mentioned here that the E-Function of MacRobert was found to be particular case of G-Function, the relation being

\[ G_{p+1,q+1}^{p,1}(x \left| \begin{array}{c} 1, \beta_1, \ldots, \beta_q \\ \alpha_1, \ldots, \alpha_p \end{array} \right) = E \left[ \begin{array}{c} \alpha_2 : \alpha_3 : \cdots : \alpha_p : x \beta_1 : \cdots : \beta_q \end{array} \right] \]  

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1. Meijer, C.S. (1941) - - - - - - - (166)
2. Bateman Manuscript (1953) a, - - - - - - - (38)
   Project p. 207
Integrals of the type (1.1.3) were first introduced by Pincherle in 1888 and their theory was developed by Mellin in 1910. These integrals were also used by Barnes for a complete integration of the hypergeometric differential equation.

The integral

\[
\int \frac{1}{2\pi i} \int_{\gamma - i \infty}^{\gamma + i \infty} \prod_{j=1}^{m} \frac{1}{(a_j + A_j \cdot s)} \prod_{j=1}^{n} \frac{1}{(d_j - B_j \cdot s)} \frac{1}{\prod_{j=1}^{p} (c_j + C_j \cdot s)} \frac{1}{\prod_{j=1}^{q} (d_j - D_j \cdot s)} ds
\]

is a typical Mellin-Barnes integral, \( \gamma \) is real, all the \( A_j, B_j, C_j, D_j \), are positive and the path of integration is a straight line parallel to the imaginary axis with indentation, if necessary, to avoid the poles of the integrand. The integral was discussed in detail by Dixon and Ferrar in 1936.

(A4) H-FUNCTION OF FOX

In 1961 Fox introduced an H-Function which is more general than even Meijer's G-function. He obtained the Symmetrical Fourier kernel as

\[
H(x) = \frac{1}{2\pi i} \int \frac{\prod_{m=1}^{q} \Gamma(b_m + c_m \cdot y) \prod_{n=1}^{p} \Gamma(a_n - e_n \cdot y) \cdot y^{-2} dy}{\prod_{m=1}^{q} \Gamma(b_m + c_m - e_m \cdot y) \prod_{n=1}^{p} \Gamma(a_n - e_n + e_n \cdot y)}
\]

1. Pincherle, S. (1888) - - - - - - - (177)
2. Mellin, H.J. (1910) - - - - - - - (167)
3. Barnes, E.W. (1908) - - - - - - - (32)
4. Dixon, A.L. & Ferrar, W.L. (1936) - - - - - - - (85)
5. Fox, Charles (1961) - - - - - - - (105)
where \( c_n > 0, \ n = 1, 2, \ldots, q \) \( e_n > 0, \ n = 1, 2, \ldots, p \); all the poles of the integrand are simple and the contour \( L \) is a straight line parallel to the imaginary axis in \( \gamma = \arg \gamma \) plane and the poles of \( \sqrt{(b_m + c_m \gamma)} \) lie on the left while those of \( \sqrt{(a_n - e_n \gamma)} \) on the right of \( L \) and

\[
2 \left( \sum_{m=1}^{q} c_m - \sum_{n=1}^{p} e_n \right) > 0.
\]

Braaksm\(\text{a}^{1}\) studied the asymptotic expansions and analytic continuations of this Function in 1963.

We shall, however, define the H-Function as\(^{2}\):

\[
\mathcal{H}_{\beta, \gamma}^{m,n} \left( \frac{(a_1, A_1, \ldots, a_p, A_p)}{(b_1, B_1, \ldots, b_q, B_q)} \right) \equiv \mathcal{H}_{\beta, \gamma}^{m,n} \left( \frac{(a_1, A_1)}{(b_1, B_1)} \right)
\]

(1.1.7)

\[
= \frac{1}{2\pi i} \int_{L} \prod_{j=1}^{m} \frac{\prod_{j=1}^{n} ((1 - a_j + A_j s) \prod_{j=1}^{\rho} \sqrt{(a_j - A_j s)})}{\prod_{j=1}^{\rho} \sqrt{(1 - b_j - B_j s) \prod_{j=1}^{\rho} \sqrt{(a_j - A_j s)}}} \ dx ds
\]

where \( s = \sigma + i \tau \), \( x \) is not equal to zero, empty product is taken as unity, \( p, q, m, n \) are integers satisfying \( 1 \leq m \leq q, 0 \leq n \leq p \), \( A_j, (j=1,2,\ldots,p), B_j, (j=1,2,\ldots,q) \) are positive numbers and \( a_j, (j=1,2,\ldots,p) \)

1. Braaksm\(\text{a}, \ B. \ L. \ J. \) (1963) -- -- -- -- -- -- -- (61)
2. Gupta, K. C. (1965) -- -- -- -- -- -- -- (114)
$b_j, (j=1,2,\ldots, q)$ are complex numbers such that no pole of \( \sqrt{(b_i^h - B_h \cdot s)}, (h=1,2,\ldots, m) \) coincides with any pole of \( \sqrt{(1 - a_i^h + A_i^h \cdot s)}, (i=1,2,\ldots, n) \), that is,

\[
(\alpha) \quad a_i^h (b_i^h + \mu) \neq (a_i^h - \lambda - 1) B_i^h
\]

\[
[ \mu, \lambda = 0,1,2,\ldots; h=1,2,\ldots, m; i=1,2,\ldots, n ]
\]

The contour $L$ runs from $\sigma - i \infty$ to $\sigma + i \infty$ such that the points

\[
S = \frac{b_i^h}{B_i^h} \quad (h=1,2,\ldots, m; \mu = 0,1,2,\ldots)
\]

which are the poles of \( \sqrt{(b_i^h - B_i^h \cdot s)}, (h=1,2,\ldots, m) \) lie on the right and the points

\[
S = \frac{a_i^h - \lambda - 1}{A_i^h} \quad (i=1,2,\ldots, m; \lambda = 0,1,2,\ldots)
\]

which are the poles of \( \sqrt{(1 - a_i^h + A_i^h \cdot s)}, (i=1,2,\ldots, n) \) lie on the left of $L$. Such a contour exists on account of the above relation $\alpha$. The parameters are so chosen that the integral exists. The $H$-functions studied in the sequel carry with them the above assumptions. Here \( \{ (a_\beta, A_\beta) \} \) stands for the sequence $(a_1, A_1), (a_2, A_2), \ldots, (a_\beta, A_\beta)$. We shall also use \( \{ (a_n, \beta, A_n, \beta) \} \) to denote the sequence $(a_n, A_n), (a_{n+1}, A_{n+1}), \ldots, (a_\beta, A_\beta)$ unless the alternative notation is expressly introduced.
(11)

(B) SELF-RECIPIROCAL FUNCTIONS.

A function \( f(x) \) satisfying the integral equation,

\[
f(x) = \int_0^\infty f(y) k(xy) dy
\]

is said to be self-reciprocal for the kernel \( k(x) \).

If \( k(x) = x^{\frac{1}{2}} J_{2\nu}(x) \), \( f(x) \) is self-reciprocal for the Hankel transform of order \( \nu \) and is called \( R_{2\nu} \).

Towards the beginning of the nineteenth century, stray examples of Self-reciprocal functions were found in the works of various Mathematicians who were not even conscious of their existence.\(^1\)

As early as 1811, Laplace\(^2\), for instance, gave the formulae

\[
\int_0^\infty \frac{\cos x}{\sqrt{x}} dy = \frac{\sqrt{\pi}}{2} = \int_0^\infty \frac{\sin x}{\sqrt{x}} dy
\]

which were given by Pringsheim in the form

\[
\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\cos xy}{\sqrt{y}} dy = \frac{1}{\sqrt{x}} = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin xy}{\sqrt{y}} dy
\]

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1. Mehrotra, B.M. (1934-35) -- -- -- -- -- (161)
2. Laplace, P.S. (1898) -- -- -- -- -- (139)
3. Pringsheim, A (1910) -- -- -- -- -- (183)
Again Laplace\(^1\) gave the formula
\[
\int_0^\infty e^{-ax} \cos bx \, dx = \frac{\sqrt{\pi}}{2a} e^{-\frac{x^2}{4a^2}}
\]
and similar examples may be traced in the works of other authors.

The subject of Self-reciprocal Functions in the present form was given by Hardy and Titchmarsh\(^2\) who investigated in detail the general classes of Self-reciprocal Functions and framed rules of transformation of one class into another.\(^3\) The treatment is based on the rigorous principles of analysis. In this connection the contributions made by Bailey, Hardy and Titchmarsh,\(^4\) Mehrotra\(^5\) are note-worthy along with the work continued by different writers.\(^7\) All these Mathematicians considered the Functions Self-reciprocal for the \(J_{\nu}\) - or Hankel transforms. In 1931 Watson\(^8\) introduced a generalized kernel \(\omega_{\mu, \nu}\) and the development towards

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1. Laplace, P.S. (1898) - - - (139)
2. Hardy, G.H. & Titchmarsh, E.C. (1930) - - - (126)
3. Titchmarsh, E.C. (1948) - - - (234)
4. Bailey, W.N. (1930) a, b, (1932) - - (15) to (17)
5. Hardy, G.H. & Titchmarsh, E.C. (1933) - - - (127)
7. Some of the references are given in (44), (169), (170), (171), (243), (246), (260).
8. Watson, G.N. (1931) - - - (265).
the generalization of the kernel started.

As said earlier, Fox, in 1961, gave a class of H-functions as the most general Symmetrical Fourier kernel. Since all symmetrical Fourier kernels can be associated with Self-reciprocal Functions and vice-versa, it is natural to investigate the classes of functions Self-reciprocal for the Symmetrical Fourier kernel given by Fox.

(C) FRACTIONAL INTEGRALS.

The integral
\[ I_\alpha f = \frac{1}{(\alpha)} \int_0^z f(t)(z-t)^{\alpha-1} \, dt \]
is called Riemann-Liouville (fractional) integral of order \( \alpha \), and
\[ K_\alpha f = \frac{1}{(\alpha)} \int_0^\infty f(t)(t-z)^{\alpha-1} \, dt \]
is known as Weyl (fractional) integral of order \( \alpha \).

\( \alpha \) and \( z \) are complex numbers, the paths of integration in \( I_\alpha f \) and \( K_\alpha f \) are \( t = z \mu, 0 \leq \mu < 1 \) and \( t = z + u, u > 1 \), or \( t = z + u, u > 0 \) respectively.

Hardy & Littlewood discussed these integrals.

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1. Work in this direction was carried out by R.P. Agarwal (2) K.P. Hatanagar (45)-(48), V.P. Mainra (149), Roop Narain (133)-(201), B. Singh (223), S. P. Singh (226), V.K. Varma (242)-(248)
Kober and Erdelyi gave some extensions of these operators. Kober, for instance, gave the operators:

\[ I_{\gamma, \alpha}^+ f = \frac{z^{-\gamma - \alpha}}{\Gamma(\alpha)} \int_0^z (z - t)^{\alpha - 1} t^{-\gamma} f(t) \, dt \]

\[ K_{\gamma, \alpha}^- f = \frac{z^{-\gamma}}{\Gamma(\alpha)} \int_{\rho(i\infty)}^\infty (t - z)^{\alpha - 1} t^{-\gamma} f(t) \, dt \]

\[ I_{\gamma, \alpha}^- f = \frac{z^{-\gamma - \alpha}}{\Gamma(\alpha)} \int_{\rho(i\infty)}^\infty (t - z)^{\alpha - 1} t^{-\gamma} f(t) \, dt \]

\[ K_{\gamma, \alpha}^+ f = \frac{z^{-\gamma}}{\Gamma(\alpha)} \int_0^z (z - t)^{\alpha - 1} t^{-\gamma - \alpha} f(t) \, dt \]

\( \gamma \) being a complex parameter.

The functions \( \frac{z^{-\gamma + \alpha}}{\Gamma(\alpha)} I_{\gamma, \alpha}^+ f \) and \( \frac{z^{-\gamma}}{\Gamma(\alpha)} K_{\gamma, \alpha}^+ f \) are the 'Riemann Liouville integrals of order \( \alpha \)' of \( z f(z) \) and \( z^{-\gamma - \alpha} f(z) \) respectively. The functions \( \frac{z^{-\gamma + \alpha}}{\Gamma(\alpha)} I_{\gamma, \alpha}^- f(z) \) and \( \frac{z^{-\gamma}}{\Gamma(\alpha)} K_{\gamma, \alpha}^- f(z) \) are similarly the 'Weyl integrals of order \( \alpha \)' of \( z f(z) \) and \( z^{-\gamma - \alpha} f(z) \).

Kober also discussed the connection of these operators and Mellin transforms. Connection of Fractional Integrals and Hankel transforms was given by Erdelyi & Kober and Erdelyi.

1. Kober, Hermann (1940) - - - - - - (135)
2. Erdelyi, Arthur (1940) - - - - - - (93)
3. Erdelyi, A & Kober, H. (1940) - - - - - - (93)
Earlier Fractional Integrals occurred in the
solution of definite integrals of Linear Differential
Equations. Ince used these under the title of
Euler Transforms. Zygmund used these operators
in Fourier Series and Doetsch illustrated their
utility in Laplace Transforms. Erdelyi used the
fractional integration by parts in the discussion
of some properties of hypergeometric functions.
Widder discussed the Abel's integral equation.

\[ J = \int_0^+ f \]

Riesz developed the theory of Fractional Integrals
of several variables and used these integrals in
the solution of partial differential equations.
Baker and Copson also made use of these operators.
Erdelyi in 1950-51 and Buschman in 1955 further
generalized these operators. Still further
generalizations due to Saxena and Parashar extend

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1. Ince, E.L. (1927) 2. Zygmund, Antoni (1936)
5. Widder, D.V. (1941) 6. Riesz, Marcel (1943)
11. Parashar, B.P. (1968)
the field of application of these integrals.

In recent years there has been an extensive use of these operators in the formal solution of Integral Equations, Dual Integral Equations, Chain Transform theory and Self-reciprocal Functions. Reference may be made to the works of Fox, Erdelyi and Sneddon, RoopNarain Saxena, Singh and others.

(D) INTEGRAL TRANSFORMS.

The Laplace Transform has passed through very interesting phases of development. Originating in the form of "Operational Calculus" finding various uses in physical and Technical problems it formed the basis of "Symbolic Calculus". Now it is conventional tool used in the solution of Differential Equations, Boundary Value problems and a variety of other fields of physical and technical nature.

Laplace Transform has been generalized by various authors and the contributions in this regard by Meijer and Varma are note-worthy.

1. Fox, Charles (1958), (1965) - - - - (104), (106)
2. Erdelyi, A. & Sneddon, I.N. (1962) - - - - - - (99)
3. RoopNarain (1966-67) - - - - - - (200)
4. Saxena, R.K. (1967) - - - - - - (207)
5. Singh, B. (1966) - - - - - - (223)
6. Meijer, C.S. (1940)&(1941) - - - - (165), (166)
7. Varma, R.S. (1947)&(1961) - - - - (240), (241)
Besides Laplace Transform the work on other Integral Transforms is also available. Stieltjes Transforms which are iterated Laplace Transforms have been studied by Widder and Titchmarsh. Closely related to Laplace Transform are the Fourier Transforms in which the kernels are \( \sin px \), \( \cos px \), and \( e^{-ipx} \), and also the Mellin Transforms where the kernel is \( x^{s-1} \).

The Integral Transform involving \( J_{\nu} (px) \) in the Kernel is called Hankel Transform. Similarly \( Y_{\nu} \)-Transform and \( H_{\nu} \)-Transform have respectively \( Y_{\nu} (px) \), the Bessel function of the second kind and \( H_{\nu} (px) \), the Struve function in the kernel.

Titchmarsh, Campbell & Foster, Sneddon and Wiener have written standard Texts on the theory and application of various transforms, Tricomi has also studied Hankel Transform at length. Fox gave a generalization of Fourier Bessel Integral Transform.

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1. Widder, D.V. (1941)
2. Titchmarsh, E.C. (1948)
4. Sneddon, I.N. (1951)
5. Wiener, N. (1933)
6. Tricomi, F. (1935)
7. Fox, C. (1929)
Recently finite Legendre Transformation

\[ f_n = \int p_n(x) F(x) dx \]

has been studied by Churchill, Tranter, Ta Li, Buschman & Demaskos and Buschman.

The operational representations of certain important functions of Mathematical Physics have given impetus to the image concept of symbolic calculus and the idea has been extended to include functions like

\[ I_f(p) = \int_a^b K(p, x)f(x) dx \]

where \( K(p, x) \), \( f(x) \) are such that the integral is convergent. \( I_f(p) \) is known as the Integral representation of \( f(x) \) by the kernel \( K(p, x) \).

It is interesting to observe that all the Integral transforms are of the form \((\beta)\) which may, therefore, be called most general Integral Transform.

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1. Churchill, R.V. (1954) - - - - (80)
2. Tranter, C.J. (1950) - - - - (235)
3. Li, Ta (1960) - - - - (141)
4. Buschman, R.G. & Demaskos, N.J. (1960) - - - - (489)
SPECIAL FUNCTIONS AND APPLICATION OF INTEGRAL TRANSFORMS.

Originally the functions of Mathematical Physics appeared as expressions in series and integrals and most of these expressions were very complicated to handle. "Over a thousand Special Functions have been deemed of sufficient interest to merit more or less detailed investigation since the beginning of the eighteenth century". There was no theory which could satisfactorily give the general methods of derivation of the properties common to all these functions until Cauchy (1825) began to evolve a systematic theory of Functions of a Complex Variable. The theories of Cauchy and Riemann are regarded as natural outgrowth of Lagrange's introduction in 1773 of the potential in Newtonian gravitation.

The idea of operational representation of a function was also successfully applied by Van der Pol & Nissen to study the properties of Special Functions.

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1. Bell, E.T. (p.469) (1945) - - - - - - (43)
2. Pol, Vander & Nissen, K.F. (1932) - - - - - - (181)
Most of the work on classical special functions is available in the monumental compilations of Bateman Manuscript Project and the references given there at the end of each Chapter. Hobson, Lebedev, MacRobert, Rainville, Sansone, Sneddon, Szegö, Truesdell, Watson and Whittaker & Watson have written very interesting texts on Special Functions.

The work on New Special Functions is still in the form of Research Papers. Reference may be made to the Generalized Associated Legendre Functions introduced by Kuipers & Meulenbeld and the like. The research in this direction is going on so rapidly that it is difficult to enumerate all new functions.

1. Bateman Manuscript Project (1953)a, b, (1954)a, b, (1955) - - - - - - -(38)-(42)
2. Hobson, E.W. (1931) - - - - - - -(129)
3. Lebedev, N.N. (1965) - - - - - - -(140)
4. MacRobert, T.M. (1962), (1967) - - - -(146), (147)
5. Rainville, E.D. (1963) - - - - - - -(186)
6. Sansone, G. (1959) - - - - - - -(203)
7. Sneddon, I.N. (1966) - - - - - - -(228)
8. Szegö, G. (1939) - - - - - - -(233)
9. Truesdell, C.A. (1948) - - - - - - -(237)
10. Watson, G.N. (1922) - - - - - - -(264)
11. Whittaker, E.T. & Watson, G.N. (1946) - - - - - - -(271)
12. Kuipers, L. & Meulenbeld, B. (1957) - - - - - - -(137)
13. Kuipers, L. (1958) - - - - - - -(136)
14. Meulenbeld, B. (1959) - - - - - - -(163)
The applications of Special Functions in aerodynamic and fluid mechanics, electromagnetic theory, thermodynamics, geophysics, quantum mechanics, nuclear physics and a variety of other fields have increased the utility of such functions. New applications of classical Special Functions are found among others in the works of Ashour, Bhonsle, Cholewinski & Haimo, Haimo, Mehta, Popes, Varma. Hence the importance of the study of Special Functions hardly needs any emphasis.

Integral Transforms and their Inversion formulae have played a prominent role in the solution of boundary value problems of heat conduction, vibrations of elastic strings and membranes. The field of applications of the Integral Transforms is being extended to new problems of space travel. In problems of modern technology polynomial approximations of Integral Transforms and asymptotic expansions are required for computation of certain results.

<table>
<thead>
<tr>
<th>Author</th>
<th>Year(s)</th>
<th>Notes</th>
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<tbody>
<tr>
<td>Ashour, A. A.</td>
<td>(1950), (1965)a, b</td>
<td>(10) - (12)</td>
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<tr>
<td>Bhonsle, B.R.</td>
<td>(1957), ... (1967)</td>
<td>(53) - (59)</td>
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<td>Cholewinski, F.M.</td>
<td>(1963)</td>
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<td>Mehta, D.K.</td>
<td>(1967)</td>
<td>(163)</td>
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<tr>
<td>Popes, G. I. a</td>
<td>(1964)</td>
<td>(182)</td>
</tr>
<tr>
<td>Varma, R. C.</td>
<td>(1967)</td>
<td>(239)</td>
</tr>
<tr>
<td>Wimp, Jett</td>
<td>(1961)</td>
<td>(274)</td>
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(1.3) REVIEW OF PREVIOUS RESULTS

The known results on classical special functions are so numerous that they cannot be compiled into a single volume. Even the results on \( F, G \)-and \( H \)-Functions are difficult to be recorded here for want of space. Reference may be made to the works of Bhise, Edelstein, Fields & Luke, Gupta, Gupta & Jain, MacRobert, Meijer, Ragab, Rathie, Roy, Narain, Saxena, Sharma, Verma, Wimp, Wimp & Luke and others. These workers have obtained numerous integrals, Finite and Infinite Series expansions, Symmetrical and Unsymmetrical Fourier kernels and Chain transforms. The work continues unabated on account of the importance of these functions in various fields.

One striking feature about the present-day work is its trend towards generalization. With \( H \)-function in the kernel Verma introduced a transform which gives most of the classical transforms on specialization of parameters. Agarwal has extended Meijer's \( G \)-function to two variables and Sharma has

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1. \( (49)-(52),(88),(101),(114),(115),(116),(117),(144)-(146),(166),(184),(189),(195)-(199),(201),(204),(205),(214)-(219),(262)-(260),(275),(276) \)
2. \( (6)-(8),(24)-(28),(110)-(112),(118),(119),(132),(133),(134),(172),(174)-(176),(203),(210),(211),(213),(220),(221),(232),(249),(262),(263) \)
3. Verma, C.B.L. (1956) \( - - - - - - (261) \)
4. Agarwal, R.P. (1955) \( - - - - - - (5) \)
5. Sharma, B.L. (1965),(1967) \( - - - (212),(214) \)
introduced another function of two variables by means of Mellin-Barnes type double integrals.

In 1934-35 Shastri\(^1\) obtained certain results on simultaneous operational calculus and subsequently Bose, Chakrabarty made certain investigations about the Laplace transform in two variables. The work is now available in the form of a text by Ditkin and Prudnikov. Rathie generalized this transform.

As described earlier Symmetrical Fourier kernels have been generalized in various ways, and each generalization has given rise to new classes of Self-reciprocal Functions. Self-reciprocal functions into complex variables were studied by Gray, Erdélyi, Reed, Agarwal, Bhatnagar, Singh, Gupta. Single and double Euler Transforms of the generalized hypergeometric functions were given by Rainville and Abdul-Halim and Al-Salam. Generalizations\(^15\) of Double Euler

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1. Shastri, N.A. (1934-35) -- (222)
2. Bose, S.K. (1962) -- (60)
5. Rathie, F.N. (1965) -- (187)
6. Gray, A.C. (1931) -- (113)
7. Erdélyi, A. (1936-37) -- (91)
8. Reed, I.S. (1944) -- (190)
15. The relevant papers are Singh, R.P. (224); Jain, R.N. (131); Srivastava H.M. & Singh, J.P. (231); Srivastava H.M. & Joshi C.M., (230);
Transforms were introduced by Singh, Jain, Srivastava & Singhal and Srivastava & Joshi.

(1.4) THE PRESENT COURSE OF STUDY.

Integral Transforms are extensively used in solving Differential Equations, Integral Equations, Boundary value problems and other fields of technology. They are also used in evaluating certain infinite Integrals. The practical applications of these transforms depend upon available tables of Integral Transforms. Some of the recent generalizations of these transforms require the extension of these tables. Therefore an effort towards the evaluation of very general integrals, investigation of inversion formulas of general transforms and other results of general character is justified.

Most of the classical special functions are particular cases of G-function. The appearance of H-function seems to have met the long-felt need of the scientists working in Applied Mathematics and Theoretical Physics since the H-function yields almost all Functions in Common use.

Apart from these considerations, the H-function has aroused interests which are purely Mathematical because "the urge to consolidate premises, to unify
concepts, to penetrate the variety and particularity of the manifest world to the undifferentiated unity that lies beyond is not only the leaven of Science, but is the loftiest passion of the human intellect."

So the work is justified from theoretical point of view also.

The thesis has been divided into three parts. The first part, consisting of five Chapters deals with H-function in one variable. The next four Chapters form the second part and contain interesting investigations in two variables about the H-function while the last three chapters which constitute the third part give some miscellaneous results, e.g. on Inverse functions, generalized heat equation and generating functions.

Our study of the H-functions begins from Chapter II with the evaluation of a very important useful and general integral involving the product of two H-functions. This contains a huge variety of results as particular cases, and can be used in various ways. This fact is illustrated in subsequent chapters. Besides this, Chapter II includes an "Addition Theorem" for H-functions, the immediate particular case of which is the corresponding theorem on G-functions by Edelstein.

Chapter III deals with evaluation of certain infinite integrals involving H-functions, modified Bessel functions of the third kind, and Appell’s function $F_4$. On specialization of parameters many results follow besides those of Saxena, Sharma and Rathie.

Another illustration of the use of the integral given in Chapter II is provided by the contents of Chapter IV. We have derived New Self-reciprocal functions for different Symmetrical Fourier kernels with the help of this integral and results of Verma. The results seem to be interesting.

In Chapter V we have given functions which are self-reciprocal for a very general Symmetrical Fourier Kernel and belong to the class $A(w, \alpha)$ defined by Hardy and Titchmarsh. Functions $f(x)$ of $A(w, \alpha)$ Self-reciprocal for the kernel:

\[
\lambda \int_{2p, 2p} \left[ \lambda^2 x \right]^{-\frac{1}{2}} \left\{ (1 - a_x, e_x) \right\} \left\{ (a_x - e_x, e_x) \right\} \left\{ (e_x, c_x) \right\} \left\{ (1 - c_x, c_x) \right\}
\]

3. Rathie, P.N. (1965)
5. Hardy, G.H. & Titchmarsh, E.C. (1930)
are denoted by
\[ R \left\{ \left( 2a_p - 1 - e_p \right) \right\} \]
\[ R \left\{ \left( 2a_p - 1 + e_p \right) \right\} \]
and contain as particular cases the most generalized
self-reciprocal functions.
\[ R \left( z_1, \ldots, z_{2p} \right) \]
\[ R_{2u,...,2w,(\lambda_1,\ldots,\lambda_n)} \]
for the kernel \[ \sum_{2u,...,2w,\lambda_1,\ldots,\lambda_n} (x) \] developed by
1. Varma. Other generalizations are particular cases
of this. The transformations from one class of
self-reciprocal functions to another class have
been effected by Fractional Integrals.

Chapter VI deals with an Inversion formula for
2. Fox-H-Transform introduced by Verma. This formula
is so general that the relevant formulae of Hankel,
K-, Y-, H-Transforms follow as particular cases. This
also gives inversion formulae of Laplace and generalized
Laplace Transforms besides a huge variety of new
results which can be obtained from it. This Chapter
also contains a result of a general character which

yields the investigations of recent research workers.

In Chapter VII we have given a generalization of Laplace Transform in two variables and a general transform in two variables with \( H \)-functions in the Kernel. The Inversion formulae have also been given. On Specialization of parameters we can obtain various transforms in two variables along with their inversion formulae.

We study generalized Double Euler Transforms of generalized hypergeometric functions in Chapter VIII. The usefulness of these operators is illustrated by means of mechanical proofs of certain known generating functions and results on generalized hypergeometric functions.

Very general classes of self-reciprocal functions in two complex variables for a very general Symmetrical Fourier kernel are given in Chapter IX. Almost all the known results can be derived as particular cases.

We have extended the \( H \)-function to two variables in Chapter X. The definition yields the functions of Agrawal and Sharma for particular

1. Agrawal, R.P. (1965) - - - - (5)
2. Sharma, B.L. (1965) - - - - (212)
values of the parameters. Besides this an integral involving product of three H-functions has been evaluated in terms of H-function of two variables.

Realizing the utility of "inverse" functions in Quantum Mechanics for the determination of two-body potential from experimental phase shifts we have investigated the inverse functions of H-functions in Chapter XI. Results of Mavromatis & Schilcher on "inverse" functions follow as particular cases.

As a diversion from H-function we have given a conventional solution of generalized heat-equation of Haimo in Chapter XII. This contains a variety of results as particular cases.

In Chapter XIII we have introduced a very general class of polynomials by means of Rodrigue's Formula and found two generating functions. The polynomials include all the classical polynomials along with those introduced by Chatterjea.

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The present work is an humble attempt to obtain some of the most general results hitherto uninvestigated. We have purposely avoided 'deducing' many examples, since, as is naturally in case of general results, a huge number of them can be derived by giving suitable values to the parameters. This has been done to keep the work small in size.