CHAPTER - XIII

GENERATING FUNCTIONS OF A GENERALIZED POLYNOMIAL

1. A polynomial is generally defined by means of:

   (i) a generating function, or
   (ii) Rodrigue's Formula, or
   (iii) an explicit expression like hypergeometric function etc.

   Proceeding from any one definition the other two can be formulated with the help of certain manipulations. The object of this Chapter is to define a very general polynomial by means of Rodrigue's Formula and deduce therefrom its explicit expression and two generating functions.

The polynomial is defined as

\[
F^{(c)}_{\nu, \lambda, \mu} \left[ \begin{array}{c} \nu; \alpha, \beta; \gamma, \delta; \gamma \beta \end{array} \right] = \frac{1}{c^n} (x - \lambda)^{-\alpha} (x + \mu)^{-\beta} e^{\nu x^2} \int \left[ (x - \lambda)^{\lambda + \alpha} (x + \mu)^{\lambda + \beta} e^{-\beta x^2} \right]
\]

(13.1.1)
where $D \equiv \frac{d}{dx}$; $h, k, n$ are non-negative integers and $\alpha, \beta, \xi, \lambda, \mu, \nu, \xi$, are real numbers.

It is evident from the nature of the parameters that almost all the classical orthogonal polynomials can be derived from (13.1.1)

2. **Explicit Formula:**

Using the formula

$$
\frac{1}{ln} \frac{d^n y}{dx^n} = \sum \frac{1}{l^2} \kappa^n_l f(u)
$$

where

$$y = f(u), \quad u = \phi(x)$$

$$\kappa^n_l \equiv \text{Coeff. of } h^n \text{ in } [\phi(x+h) - \phi(x)]^r$$

and the summation extends to all positive integral values of $r$ from $r = 1$ to $r = n$, we see that

$$
(13.2.1) \quad \frac{1}{lt} \frac{d^t (e^{\lambda x^t})}{dx^t} = \frac{e^{\lambda x^t}}{x^t} \sum_{i=1}^{t} \frac{\beta^i x^i}{i!} \sum_{j=0}^{i} (-1)^j \binom{i}{j} \binom{t}{j} x^j
$$

Also we know that

$$
(13.2.2) \quad \frac{d^n (u v w)}{dx^n} = \sum \frac{d^n u}{dx^n} \frac{d^s v}{dx^s} \frac{d^t w}{dx^t}
$$

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1. Edwards, J. (pp. 104, 67) (1950) ------(90)
The summation being extended to all positive integral values of \( r, s, t \) inclusive of zero, which satisfy \( r + s + t = n \).

Solving

\[
D^n \left[ \left( x - \lambda \right)^{\alpha} \left( \mu + m \right)^{\beta} \right]
\]

with the help of (13.2.1) and (13.2.2) we get the explicit form of the polynomial (13.1.1) as

\[
\binom{c}{n} \sum_{q, s, t} \left[ \binom{\alpha + k}{q} \binom{\beta + q}{s} \left( x - \lambda \right)^{\alpha - q} \left( \mu + m \right)^{\beta - s} x^{r + s + t} \right.
\]

where the first summation extends to all positive integral values of \( q, s, t \) inclusive of zero, which satisfy \( q + s + t = n \).

Putting \( \alpha = a, \lambda = 0 = \mu = \beta = \lambda, \ c = 1 \), the polynomial reduces to the one given by Chatterjea\(^1\) in the form

\[
\binom{c}{n} \left[ x ; a, \lambda, \beta \right].
\]

So that (13.2.3) gives,

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(13.2.4)

\[
F^{(i)}_{n, a, b} (x; a, b; \alpha, \beta; \gamma, \delta; \mu, \nu) = x^{-a} e^{-x} \sum_{i=0}^{\infty} \frac{\beta^{i}}{i!} \sum_{j=0}^{i} \frac{\gamma^{j}}{j!} (-1)^{i} \binom{i}{j} x^{\alpha+i} \gamma^{j} \delta^{i} \nu^{\mu+i} = F_{n}^{(i)} (x; a, \alpha, b, \delta, \nu),
\]

(Using Vandermonde's Theorem)

3. Generating Function for \( F^{(i)}_{n, a, b} (x; a, b; \alpha, \beta; \gamma, \delta; \mu, \nu) \).

We have by Lagrange's Formula

\[
f(z) = f(x) + \sum_{n=1}^{\infty} \frac{t^{n}}{n!} D^{n-1} \left( \phi(x) \right) f(x)\]  \(1\)

where \( z = x + t \cdot \phi(x) \); \( \phi(x) \), \( f(x) \) are derivable at \( z = x \) and \( \phi(x) \neq 0 \).

Differentiating (13.3.1) with regard to \( t \) and substituting \( F(z) \) for \( \phi(x) \) we get

\[
F(z) = F(x) + \sum_{n=0}^{\infty} \frac{t^{n}}{n!} D^{n} \left[ \{ \phi(x) \}^{n}, F(x) \right]
\]

1. Whittaker E.T. & Watson G.N. (p.133)
Now let
\[ z = x + \frac{t}{c} (z - \lambda) (z + \mu) \hat{h} \]
\[ \phi(x) = (x - \lambda) (x + \mu)^\beta \hat{c} \]
and
\[ f(x) = (x - \lambda) (x + \mu)^\beta e^{-\beta x^2} \]

then (13.3.2) gives after a little simplification
\[ \sum_{n=0}^{\infty} \frac{t^n}{n!} f^{(n)}_{x \lambda \beta \lambda x \mu \beta \lambda} \]
\[ = \left( \frac{\beta - \lambda}{x - \lambda} \right)^\alpha \left( \frac{\beta + \mu}{x + \mu} \right)^\beta \frac{e^{-\beta x^2}}{1 - \frac{t}{c} (3 - \lambda) (3 + \mu) \frac{\hat{h}}{\hat{c}}} \left( \frac{(\hat{h} + \lambda) \hat{h} + \mu - \lambda}{\hat{h} + \mu - \lambda} \right) \]

with \( z = x + \frac{t}{c} (z - \lambda) (z + \mu) \hat{h} \).

as the required generating function.

**Particular Cases:**

On specialization of parameters we obtain the generating functions of almost all the classical orthogonal polynomials. Here we record only two cases.

I. Putting \( \lambda = \mu = 1, \hat{h} = \hat{c} = 1, c = 2, \beta = 0 \)
we get from Rodrigue's formula for Jacobi Polynomials

\[ \text{(13.1.1)} \]

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1. Rainville, E.D. (pp. 257(6) 271(6) (1963)----(126)
\[
F_{n,\lambda, \phi}^{-1} \left[ x; \alpha, \beta; \lambda, \phi \right] = \left( \frac{\lambda \phi}{x - \lambda \phi} \right)^{\frac{1}{\phi}} \Gamma \left( \frac{\lambda \phi}{x - \lambda \phi} \right)^{\lambda} \Gamma \left( \frac{\lambda \phi}{x - \lambda \phi} \right)^{-\beta} \]

so that (13.3.3) gives

\[
\sum_{n=0}^{\infty} t^n \left( \frac{\beta - 1}{x - \lambda \phi} \right)^{\lambda} \left( \frac{\lambda \phi}{x - \lambda \phi} \right)^{\lambda} \left( 1 - \frac{\lambda \phi}{x - \lambda \phi} \right)^{-\beta}
\]

with 
\[
\beta = x + \frac{t}{\lambda \phi} (x - 1)(x + 1)
\]

Whence,
\[
\beta = \frac{1 - \rho}{\rho}, \quad \rho = \sqrt{-2xt + t^2}, \quad \text{and hence}
\]

(13.3.4)
\[
\sum_{n=0}^{\infty} t^n \left( \frac{\beta - 1}{x - \lambda \phi} \right)^{\lambda} \left( \frac{\lambda \phi}{x - \lambda \phi} \right)^{\lambda} \left( 1 - \frac{\lambda \phi}{x - \lambda \phi} \right)^{-\beta} (1 + \rho)^{-\beta} (1 - \rho)^{-\beta}
\]

which is well-known

II. Putting \( \alpha = \alpha, \beta = \beta = \lambda = \mu = h, \quad c = 1 \)

we get from (13.1.1) and (13.2.4)

\[
F_{n,\lambda, \phi}^{(1)} \left[ x; a, \beta; \lambda, h, \phi \right] = F_{n,\lambda, \phi}^{(1)} \left[ x; a, h, \beta \right]
\]

and therefore (13.3.3) gives

(13.3.5)
\[
\sum_{n=0}^{\infty} \frac{t^n}{\lambda \phi^n} \left( \frac{\beta - 1}{x - \lambda \phi} \right)^{\lambda} \left( \frac{\lambda \phi}{x - \lambda \phi} \right)^{\lambda} \left( 1 - \frac{\lambda \phi}{x - \lambda \phi} \right)^{-\beta} \left( 1 - th \frac{x - \lambda \phi}{\lambda \phi} \right)^{-\beta}
\]

which is known

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1. Rainville, E.D. (p.271(6)) (1963) --- (186)

2. Chatterjee, S.K. (p.342) (1966) b---(76)
Generating function for \( F_{n, \lambda, \mu}^{(c)} \left[ \alpha; x - \lambda n, \beta - \lambda n; k, \lambda, \mu, \Delta \right] \)

On putting \( \phi(\beta) = 1 \) in (13.3.1) we obtain
\[
\sum_{n=0}^{\infty} \frac{t^n}{n!} \mathcal{D}^n f(x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \left[ (x - \lambda)^k (x + \mu)^\lambda \right]^n \mathcal{D}^n f(x) = f \left[ (x + t(x - \lambda)(x + \mu))^\lambda \right]
\]

Changing \( t \) to \( (x - \lambda)^k (x + \mu)^\lambda \) we get
\[
\sum_{n=0}^{\infty} \frac{t^n}{n!} \left[ (x - \lambda)^k (x + \mu)^\lambda \right]^n \mathcal{D}^n f(x) = f \left[ (x + t(x - \lambda)(x + \mu))^\lambda \right]
\]

If we substitute \( f(y) = (y - \lambda)^\lambda (y + \mu)^\mu e^{-\frac{y^2}{2c}} \)

it is easy to obtain another generating function in the form:
\[
\sum_{n=0}^{\infty} \frac{t^n}{n!} \mathcal{F}_{n, \lambda, \mu}^{(c)} \left[ \alpha; x - \lambda n, \beta - \lambda n; k, \lambda, \mu, \Delta \right] = \left[ 1 + \frac{t}{c} (x - \lambda)(x + \mu) \right]^\lambda \left[ 1 + \frac{t}{c} (x - \lambda)(x + \mu) \right]^\mu \frac{\lambda - 1}{\lambda} (x - \lambda)^k (x + \mu)^\mu \frac{\lambda}{\mu} (x - \lambda)^k (x + \mu)^\mu \frac{\lambda}{\mu}
\]

Putting \( h = k = 1 = \lambda = \mu, c = 2, p = 0 \) we get a known generating function for Jacobi Polynomials in the form
\[
\sum_{n=0}^{\infty} \frac{t^n}{n!} P_n^{(\alpha-n, \beta-n)}(x) = \left[ 1 + \frac{t}{c} (x + 1) \right]^\lambda \left[ 1 + \frac{t}{c} (x - 1) \right]^\mu
\]

Specializing the parameters we get similarly the generating function of \( F_n^{(b)} \left[ \alpha; x - \lambda n, k, \mu \right] \) given by Chatterjea.