Appendix A

Generalized spin squeezing criterion and negativity of the covariance matrix

Korbicz et al. [80] have proposed generalized spin squeezing inequalities on multiqubit systems as a reflection of Peres-Horodecki criterion on the reduced two-qubit density operator of a symmetric $N$-qubit state which has been brought out in the form of a family of inequalities given by

$$\frac{4\langle \Delta J_n^2 \rangle}{N} < 1 - \frac{4\langle J_n^2 \rangle^2}{N^2},$$

where $J_n = \vec{J} \cdot \hat{n}$, with $\hat{n}$ denoting an arbitrary unit vector, $\langle \Delta J_n^2 \rangle = \langle J_n^2 \rangle - \langle J_n \rangle^2$ and $\vec{J} = \frac{1}{2} \Sigma_{\alpha=1}^N \vec{\sigma}_\alpha$ is the collective angular momentum of the multi-qubit system. These generalized spin squeezing inequalities are shown to be necessarily satisfied by every pairwise entangled symmetric $N$-qubit state.

We now show that the covariance matrix criterion discussed in Section 2.2 is equivalent to the generalized spin-squeezing inequalities. In order to do that, we express $\langle J_n \rangle$ and $\langle J_n^2 \rangle$ in terms of the two-qubit state variables $\vec{s}$ and $T$ (see Eq. (2.25) and Eq. (2.26)) of any random pair of qubits drawn from a symmetric $N$-qubit state. More
explicitly we have

\[
\langle J_n \rangle = \frac{1}{2} \sum_{\alpha=1}^{N} \sum_i \langle \sigma_{\alpha i} \rangle n_i = \frac{N}{2} \langle \vec{s} \cdot \vec{n} \rangle
\]

\[
\langle J_n^2 \rangle = \frac{N}{4} + \frac{1}{4} \sum_{i,j} \sum_{\alpha, \beta \neq \alpha} \langle \sigma_{\alpha i} \sigma_{\beta j} \rangle n_i n_j
\]

\[
= \frac{N}{4} + \frac{1}{2} \sum_{i,j} \sum_{\alpha=1, \beta > \alpha=1}^{N} \langle \sigma_{\alpha i} \sigma_{\beta j} \rangle n_i n_j
\]

\[
= \frac{N}{4} + \frac{1}{2} \sum_{i,j} \sum_{\alpha, \beta > \alpha} t_{ij} n_i n_j
\]

\[
= \frac{N}{4} + \frac{N(N-1)}{4} \sum_{i,j} t_{ij} n_i n_j
\]

i.e., \( \langle J_n^2 \rangle = \frac{N}{4} [1 + (N-1)n^TTn] \),

the inequalities given by (A.1) reduce to a simple form,

\[
\frac{N}{4} + \frac{N(N-1)}{4} (n^TTn) < \frac{N}{4} + \frac{N(N-1)}{4} (\vec{s} \cdot \vec{n})^2
\]

or

\[
n^T(T - \vec{s}\vec{s}^T)n < 0. \tag{A.3}
\]

In other words, bipartite reduction of an arbitrary symmetric multiqubit state, with \( T - \vec{s}\vec{s}^T < 0 \), satisfies generalized spin squeezing inequalities - implying that the system is pairwise entangled. This indeed offers an elegant connection on pairwise entanglement in symmetric \( N \)-qubit states resulting entirely due to its two qubit properties captured by the off-diagonal block of covariance matrix.
Appendix B

Counting the number of parameters

Let us consider a $N$-qubit system, density matrix of which is given by,

$$\hat{\rho} = \frac{1}{2^N} \sum_{\alpha_1,\alpha_2,\ldots,\alpha_N = 0,x,y,z} (\sigma_{\alpha_1} \sigma_{\alpha_2} \cdots \sigma_{\alpha_N}) T_{\alpha_1 \alpha_2 \cdots \alpha_N},$$  \hspace{1cm} (B.1)

where $T_{\alpha_1 \alpha_2 \cdots \alpha_N}$ denote the state parameters; $\sigma_0 = I$ denotes $2 \times 2$ identity matrix and $\sigma_x, \sigma_y, \sigma_z$ denote Pauli spin matrices.

Total number of independent parameters associated with a general $N$-qubit system is $2^{2N} - 1$, for a normalized density matrix. We may express the above state parameters as moments of $1^{\text{st}}, 2^{\text{nd}}, 3^{\text{rd}}$ order $\{T_{a_1 i_1 a_2 i_2 \cdots a_K i_K}\}$, $i_1, i_2, \ldots, i_K = x, y, z$, with $K = 1, 2, 3 \ldots N$; $a_1, \ldots, a_K$ denote qubit indices; here all the other $(N - K)$ indices are set equal to zero. A $K^{\text{th}}$ order moment has $3^K$ number of parameters. It may be noted that for a general multiqubit state, we have $N$ first order moments $T_{a_i}$ - (one each for the qubit index $a = 1, 2, \ldots, N$) - associated with all the $N$-qubits; $\frac{N(N-1)}{2}$ second order moments $T_{a_1 i_1, a_2 i_2}$; and extending this to $K$, we have $N C_K = \frac{N!}{(N-K)! K!}$ number of $K^{\text{th}}$ order moments.

For example for a two-qubit state, we have two sets of first order moments and one set of second order moments. And counting the number of parameters is clear now: $2^4 - 1 = 15$ for a unit trace Hermitian matrix $\rho$, which is expressed in terms of moment decomposition as $15 = 2 \times 3$ (first order moments) $+ 1 \times 3^2$ (second order moments).
Appendix B. Counting the number of parameters

For a general multiqubit state we have

\[ 2^{2N} - 1 = \sum_{r=1}^{N} N C_r \, 3^r, \]

as a counting rule.

Let us focus on symmetric multiqubit systems. By symmetric \( N \)-qubit states we mean those states, resulting when the addition of angular momenta of all the qubits give maximum value of \( J = \frac{N}{2} \) for total angular momentum. State space here is \( 2J+1 = N+1 \) dimensional; there are \( (N + 1)^2 - 1 = N^2 + 2N \) parameters associated with the state. To count this number, note that the state parameters \( T_{a_1 \, i_1 \, a_2 \, i_2 \ldots a_K \, i_K} \), are completely symmetric under interchange and we can drop the qubit index \( a_1, a_2, \ldots, a_K \) altogether for symmetric multiqubit systems. A symmetric tensor \( T_{i_1 \, i_2 \ldots i_K} \) with \( i_1, i_2, \ldots, i_K = x, y, z \) has \( \frac{(K+1)(K+2)}{2} \) independent components (For example \( K = 1 \) moment has 3 parameters for \( T_{i} \); \( K = 2 \) has 6 parameters for \( T_{ij} \) etc.). Moreover, there are constraints of the sort

\[ \sum_{i=x,y,z} J_i^2 = J(J+1) = \frac{N}{2} \left( \frac{N}{2} + 1 \right) \]

on these moments.

For example, take second order moments \( T_{ij} \), which are 6 in number (due to symmetry under interchange all the 9 components of \( T_{ij} \) are not independent). Further, they have one more constraint:

\[ \sum_{i=x,y,z} J_i^2 = J(J+1) = \frac{N}{2} \left( \frac{N}{2} + 1 \right) \]

(Note that this condition is not a part of symmetry under interchange of indices). So, we have only \( \frac{(K+1)(K+2)}{2} - 1 = 5 \) parameters when \( N = 2 \). For \( K = 3 \) we have (after taking care of symmetry under interchange of indices) \( \frac{(K+1)(K+2)}{2} = 10 \) parameters, which are further restricted by three more constraints coming from

\[ \langle (J_x^2 + J_y^2 + J_z^2) \rangle \]

So we have 10 - 3 = 7 independent components of third order moments. One more case will allow us to generalize this to \( K^{th} \) order moment: For fourth order symmetric tensor we have \( \frac{(K+1)(K+2)}{2} = 15 \) components; and there are 6 constraints

\[ \left( \langle J_x^2 + J_y^2 + J_z^2 \rangle \right)^2 \left( \langle J_m \, J_n \rangle \right) \]

So we have 10 - 3 = 7 independent components of third order moments. One more case will allow us to generalize this to \( K^{th} \) order moment: For fourth order symmetric tensor we have \( \frac{(K+1)(K+2)}{2} = 15 \) components; and there are 6 constraints

\[ \left( \langle J_x^2 + J_y^2 + J_z^2 \rangle \right)^2 \left( \langle J_m \, J_n \rangle \right) \]
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giving $15 - 6 = 9$ independent components. So we have $\frac{K(K-1)}{2}$ constraints on $\frac{(K+1)(K+2)}{2}$ components of symmetric tensors giving

\[
\frac{(K + 1)(K + 2)}{2} - \frac{K(K - 1)}{2} = (2K + 1)
\]

independent components for each $K^{th}$ order tensor. (The constraints imply that combinations of higher order moments result in lower order moments, which are already counted. To see this clearly, note that in the case of two-qubits we had $\text{Tr}(T) = 1$, where a combination of second order moments gives zeroth order moment). Total number of parameters for a symmetric multiqubit state can now be counted as follows: We have various symmetric tensors of rank $K$ (for which there are $(2K + 1)$ independent parameters, as shown above) with $K = 1, 2, \ldots, N$ denoting our state parameters and it is easy to see that

\[
\sum_{K=1}^{N} (2K + 1) = (N + 1)^2 - 1 = N^2 + 2N,
\]

thus matching the total number of parameters required to specify a symmetric $N$-qubit state.
Appendix C

Relationship between tensor parameters $t_{qq'}^{\kappa\kappa'}(N_1, N_2)$ and $t_{Q}^{K}(N)$

Let us evaluate the matrix elements of the density matrix given by Eq. (4.7) in the direct product basis $| j_1 m_1; j_2 m_2 \rangle$: (we are considering spins $j_1 = \frac{N_1}{2}$ and $j_2 = \frac{N_2}{2}$ in our case)

$$
\langle m'_1 m'_2 | \hat{\rho} | m_1 m_2 \rangle = \frac{1}{(2j_1 + 1)(2j_2 + 1)} \sum_{\kappa=0}^{N_1} \sum_{\kappa'=0}^{N_2} \sum_{\kappa=-\kappa}^{\kappa} \sum_{\kappa'=-\kappa'}^{\kappa'} \langle m'_1 | \hat{t}_{q q'}^{\kappa\kappa'} | m_1 \rangle \langle m'_2 | \hat{t}_{q q'}^{\kappa\kappa'} | m_2 \rangle t_{qq'}^{\kappa\kappa'}. 
$$

(C.1)

Using Hermiticity property Eq. (4.6) of the irreducible tensor operators:

$$
\hat{t}_{q}^{\kappa\dagger} = (-1)^{-q} \hat{t}_{-q}^{\kappa},
$$

$$
\hat{t}_{q'}^{\kappa\dagger} = (-1)^{-q'} \hat{t}_{-q'}^{\kappa'},
$$

and Eq. (4.2) for matrix elements:

$$
\langle j_1 m'_1 | \hat{t}_{q q'}^{\kappa\kappa'} | j_1 m_1 \rangle = C(j_1 \kappa j_1; m_1 - q m'_1) [\kappa],
$$

$$
\langle j_2 m'_2 | \hat{t}_{q q'}^{\kappa\kappa'} | j_2 m_2 \rangle = C(j_2 \kappa' j_2; m_2 - q' m'_2) [\kappa'],
$$

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Appendix C. Relationship between $t_{qq'}^{\kappa\kappa'}(N_1, N_2)$ and $t_Q^K(N)$

we get,

$$
\langle m'_1 m'_2 | \hat{\rho} | m_1 m_2 \rangle = \frac{1}{(2j_1 + 1)(2j_2 + 1)} \sum_{\kappa=0}^{N_1} \sum_{\kappa'=0}^{N_2} \sum_{q=-\kappa}^{\kappa} \sum_{q'=-\kappa'}^{\kappa'} (-1)^{-q-q'} C(j_1 \kappa j_1; m_1 - qm'_1) [\kappa] C(j_2 \kappa' j_2; m_2 - q'm'_2) [\kappa'] t_{qq'}^{\kappa\kappa'}.
$$

(C.2)

We have an alternative way of evaluating the matrix elements of Eq. (4.7), as follows:

$$
\langle m'_1 m'_2 | \hat{\rho} | m_1 m_2 \rangle = \sum_{J', M'} \langle m'_1 m'_2 | (j_1 j_2) J' M' \rangle \langle (j_1 j_2) J' M' | \hat{\rho} | (j_1 j_2) J M \rangle \langle (j_1 j_2) J M | m_1 m_2 \rangle
$$

(C.3)

where we have introduced complete set of coupled angular momentum states. Now, we use the representation given in Eq. (4.5) for density matrix of a symmetric system (for which we have $J = j_1 + j_2 = \frac{N_1}{2} + \frac{N_2}{2} = \frac{N}{2}$), so that the matrix element

$$
\langle (j_1 j_2) J' M' | \hat{\rho} | (j_1 j_2) J M \rangle = \langle (j_1 j_2) J M' | \hat{\rho} | (j_1 j_2) J M \rangle \delta_{J', J}
$$

and therefore, using the representation Eq. (4.5) for the density matrix, definition of CG coefficients and Eq. (4.6) to evaluate the matrix elements of irreducible tensor operators) we get,

$$
\langle m'_1 m'_2 | \hat{\rho} | m_1 m_2 \rangle = \frac{1}{(2J + 1)} \sum_{K, Q, M', M} C(j_1 j_2 J; m'_1 m'_2 M') t_Q^K (-1)^{-Q} C(J K J; M - Q M') [K] C(j_1 j_2 J; m_1 m_2 M).
$$

(C.4)

So, from Eq. (C.2) and Eq. (C.4), we obtain

$$
\frac{1}{(2j_1 + 1)(2j_2 + 1)} \sum_{\kappa=0}^{N_1} \sum_{\kappa'=0}^{N_2} \sum_{q=-\kappa}^{\kappa} \sum_{q'=-\kappa'}^{\kappa'} (-1)^{-q-q'} C(j_1 \kappa j_1; m_1 - qm'_1) [\kappa] \times C(j_2 \kappa' j_2; m_2 - q'm'_2) [\kappa'] t_{qq'}^{\kappa\kappa'}
$$

$$
= \frac{1}{(2J + 1)} \sum_{K, Q, M', M} (-1)^{-Q} C(j_1 j_2 J; m'_1 m'_2 M') C(J K J; M - Q M') \times C(j_1 j_2 J; m_1 m_2 M) [K] t_Q^K.
$$

(C.5)
Appendix C. Relationship between \( t_{qq'}^\kappa (N_1, N_2) \) and \( t_Q^K (N) \)

There are five CG coefficients in Eq. (C.5) and we may recall the definition of Wigner 9j symbol, which involves six CG coefficients as given below: (p.334 of Ref. [81]):

\[
\sum_{m_1, m_2} C(j_1, j_2; j_1; m_1, m_2) C(j_3, j_4; j_3, m_3, m_4) C(j_1, j_2; j_1, m_1, m_2) C(j_3, j_4; j_3, m_3, m_4) C(j_1, j_2; j_1, m_1, m_2) C(j_3, j_4; j_3, m_3, m_4) \times C(j_1, j_2; j_1, m_1, m_2) C(j_3, j_4; j_3, m_3, m_4) = \delta_{j_1, j_2} \delta_{m_1, m_2} \delta_{j_3, j_4} \delta_{j_1, j_2} \delta_{j_3, j_4}
\]

We now shift two of the CG coefficients from the LHS of Eq. (C.5), making use of their orthogonality properties: Let us first arrange the CG coefficients, using their symmetry properties as given below,

\[
C(j_1, j_2; j_1, m_1, m_2) = \frac{[j_1]}{[\kappa]} (-1)^{j_1 - m_1} C(j_1, j_2; j_1, m_1, m_2)
\]

\[
C(j_2, j_2; j_2, m_2, m_2) = \frac{[j_2]}{[\kappa]} (-1)^{j_2 - m_2} C(j_2, j_2; j_2, m_2, m_2)
\]

\[
C(J, K; M - QM') = \frac{[J]}{[K]} (-1)^{J - M} C(J, K; M - QM')
\]

Inserting this in Eq. (C.5), and shifting the factor \((-1)^{j_1 - m_1} (-1)^{j_2 - m_2}\) to the RHS we obtain

\[
\frac{1}{(2j_1 + 1)(2j_2 + 1)} \sum_{j_1, j_2} (-1)^{-q - q'} [j_1][j_2] C(j_1, j_2; j_1, m_1 + m_2; q) C(j_2, j_2; j_2, m_1 + m_2; q') \sum_{K, Q, M'} (-1)^{j_1 + j_2 - m_1 - m_2} (-1)^{-Q} C(j_1, j_2; j_1, m_1, M') \times (-1)^{J - M} C(J, K; M - QM') C(j_1, j_2; j_1, m_1, M) [K] t^{\kappa}_{qq'}
\]

Note that the \( C(j_1, j_2; j_1, m_1, M) \) in the RHS is non-zero (because \( C(j_1, j_2; j_1, m_1, M) = 0 \) if \( m_1 + m_2 \neq M \)) therefore, we may re express the factor \((-1)^{j_1 + j_2 - m_1 - m_2} \) in the RHS, so that it gets canceled with the one more factor which is there already:

\[
\frac{1}{[j_1][j_2]} \sum_{j_1, j_2} (-1)^{-q - q'} C(j_1, j_2; j_1, m_1 + m_2; q) C(j_2, j_2; j_2, m_1 + m_2; q') \sum_{K, Q, M'} (-1)^{-Q} C(j_1, j_2; j_1, m_1, M') C(j_1, j_2; j_1, m_1, M) [K] t^{\kappa}_{qq'}
\]

\[
= \frac{1}{[j_1][j_2]} \sum_{j_1, j_2} (-1)^{-Q} C(j_1, j_2; j_1, m_1, M') C(j_1, j_2; j_1, m_1, M) [K] t^{\kappa}_{qq'}
\]

\[
= \frac{1}{[j_1][j_2]} \sum_{j_1, j_2} (-1)^{-Q} C(j_1, j_2; j_1, m_1, M') C(j_1, j_2; j_1, m_1, M) [K] t^{\kappa}_{qq'}
\]

\[
= \frac{1}{[j_1][j_2]} \sum_{j_1, j_2} (-1)^{-Q} C(j_1, j_2; j_1, m_1, M') C(j_1, j_2; j_1, m_1, M) [K] t^{\kappa}_{qq'}
\]

\[
= \frac{1}{[j_1][j_2]} \sum_{j_1, j_2} (-1)^{-Q} C(j_1, j_2; j_1, m_1, M') C(j_1, j_2; j_1, m_1, M) [K] t^{\kappa}_{qq'}
\]

\[
= \frac{1}{[j_1][j_2]} \sum_{j_1, j_2} (-1)^{-Q} C(j_1, j_2; j_1, m_1, M') C(j_1, j_2; j_1, m_1, M) [K] t^{\kappa}_{qq'}
\]
Multiplying Eq. (C.9) by 
\[ C(j_1 j_1 \kappa_1; m_1 - m_1' q_1') C(j_2 j_2 \kappa_2'; m_2 - m_2' q_2') \]
and then summing over \( m_1, m_1' \), and \( m_2, m_2' \), we will be able to use the orthogonality property:

\[
\sum_{m_1, m_1'} C(j_1 j_1 \kappa_1; m_1 - m_1' q_1') C(j_1 j_1 \kappa_1'; m_1 - m_1' q_1') = \delta_{\kappa_1, \kappa_1'} \delta_{q_1, q_1'}
\]
\[
\sum_{m_2, m_2'} C(j_2 j_2 \kappa_1'; m_2 - m_2' q_2') C(j_2 j_2 \kappa_2'; m_2 - m_2' q_2') = \delta_{\kappa_1', \kappa_2'} \delta_{q_1', q_2'}
\]

we then obtain,

\[
\frac{1}{[j_1][j_2]} \sum_{\kappa, \kappa', q, q'} (-1)^{-q - q'} \delta_{\kappa, \kappa'} \delta_{q, q'} \delta_{\kappa_1, \kappa_1'} \delta_{\kappa_2, \kappa_2'} t_{\kappa_1 \kappa_1'; \kappa_2 \kappa_2'}^{q q'}
\]

\[
= \frac{1}{[J]^2} \sum_{M, M'} \sum_{m_1, m_2, K, Q} \sum_{\delta_1 \delta_2} \sum_{(1)^{\delta_1 + \delta_2}} C(j_1 j_2 J; m_1' m_2' M') C(J J K; M - Q M') C(j_1 j_2 J; m_1 m_2 M) \times C(j_1 j_1 \kappa_1'; m_1 - m_1' q_1') C(j_2 j_2 \kappa_2'; m_2 - m_2' q_2') [K] t^K_Q
\]

which is nothing but

\[
\frac{t_{\kappa_1' \kappa_2'; q_1' q_2'}}{[j_1][j_2]} = \frac{[j_1][j_2]}{[J]^2} \sum_{M, M'} \sum_{m_1, m_2, K, Q} \sum_{\delta_1 \delta_2} \sum_{(1)^{\delta_1 + \delta_2}} (-1)^{-Q} (-1)^{\delta_1 + \delta_2} [K] t^K_Q C(j_1 j_2 J; m_1' m_2' M') \times C(J J K; M - Q M') C(j_1 j_2 J; m_1 m_2 M) \times C(j_1 j_1 \kappa_1'; m_1 - m_1' q_1') C(j_2 j_2 \kappa_2'; m_2 - m_2' q_2').
\]

We now have 5 CG coefficients in the RHS of Eq. (C.11). Multiplying both sides by 
\[ C(\kappa_1' \kappa_2' K', q_1' q_2' Q'); \]
and taking a summation over \( q_1', q_2' \), and, further, identifying the following correspondence with Eq. (C.6),

\[
j_1 = j_1, \quad j_2 = j_2, \quad j_{12} = J, \quad m_1 = m_1, \quad m_2 = m_2, \quad m_{12} = M,
\]
\[
j_3 = j_1, \quad j_4 = j_2, \quad j_{34} = J, \quad m_3 = -m_1, \quad m_4 = -m_2, \quad m_{34} = -M,
\]
\[
j_{12} = J, \quad j_{34} = J, \quad m_{12} = M, \quad m_{34} = -M, \quad j = K, \quad m = Q,
\]
\[
j_{13} = \kappa_1, \quad j_{24} = \kappa_2, \quad m_{13} = q_1, \quad m_{24} = q_2, \quad j = K, \quad m = Q,
\]

(C.12)
Appendix C. Relationship between $t^{\kappa'_1 \kappa'_2}_{qq'}(N_1, N_2)$ and $t^K_Q(N)$

we obtain,

$$\sum_{q'_1, q'_2} t^{\kappa'_1 \kappa'_2}_{q'_1 q'_2} C(\kappa'_1 \kappa'_2 K'; q'_1 q'_2 Q') = [j_1] [j_2] [J] [\kappa'_1] [\kappa'_2] \sum_{K,Q} \left\{ \begin{array}{ccc} j_1 & j_2 & J \\ \kappa'_1 & \kappa'_2 & K \end{array} \right\} t^K_Q \delta_{\kappa, K'} \delta_{Q, Q'}$$

$$= [j_1] [j_2] [J] [\kappa'_1] [\kappa'_2] \left\{ \begin{array}{ccc} j_1 & j_2 & J \\ \kappa'_1 & \kappa'_2 & K' \end{array} \right\} t^{K'}_{Q'}.$$  \hspace{1cm} \text{(C.13)}

Now we obtain the equation relating $t^{\kappa'_1 \kappa'_2}_{qq'}$ and $t^K_Q$, by

(i) multiplying both sides of Eq. (C.13) by a CG coefficient $C(\kappa'_1 \kappa'_2 K'; q'_1 q'_2 Q')$ and summing over $K', Q'$,

(ii) further, using the orthogonality property of CG coefficients, viz.,

$$\sum_{K', Q'} C(\kappa'_1 \kappa'_2 K'; q'_1 q'_2 Q') C(\kappa'_1 \kappa'_2 K'; q'_1 q'_2 Q') = \delta_{q'_1, q'_1} \delta_{q'_2, q'_2}$$

we finally get our equation Eq. (4.16),

$$t^{\kappa'_1 \kappa'_2}_{qq'} = [j_1] [j_2] [J] [\kappa'_1] [\kappa'_2] \sum_{K', Q'} C(\kappa'_1 \kappa'_2 K'; q'_1 q'_2 Q') = [j_1] [j_2] [J] [\kappa'_1] [\kappa'_2] \left\{ \begin{array}{ccc} j_1 & j_2 & J \\ \kappa'_1 & \kappa'_2 & K' \end{array} \right\} t^{K'}_{Q'}.$$  \hspace{1cm} \text{(C.14)}

or

$$t^{\kappa' \kappa'}_{qq'} = [j_1] [j_2] [J] [\kappa] [\kappa'] \sum_{K,Q} C(\kappa \kappa' K; qq' Q) = [j_1] [j_2] [J] [\kappa] [\kappa'] \left\{ \begin{array}{ccc} j_1 & j_2 & J \\ \kappa & \kappa' & K \end{array} \right\} t^K_Q.$$  \hspace{1cm} \text{(C.15)}

\textbf{Inverse relation of Eq. (4.16)}

An inverse relation connecting $t^K_Q$ with $t^{\kappa'_1 \kappa'_2}_{qq'}$ is obtained by

(i) multiplying both sides of Eq. (C.13) with $t^K_Q$ with

$$\left\{ \begin{array}{ccc} j_1 & j_2 & J' \\ \kappa'_1 & \kappa'_2 & K' \end{array} \right\};$$  \hspace{1cm} \text{summing over $\kappa'_1, \kappa'_2,$}

and
(ii) using the orthonormality of $9j$ symbols (p.335 of Ref. [81]) viz.,

$$
\sum_{\kappa_1', \kappa_2'} [\kappa_1']^2 [\kappa_2']^2 \begin{pmatrix}
\kappa_1 & \kappa_2 & J \\
\kappa_1' & \kappa_2' & J'
\end{pmatrix}
= \frac{1}{[j_1][j_2][J]} \delta_{J, J'}
$$

We obtain,

$$
\sum_{\kappa_1', \kappa_2', q', q_2} [\kappa_1'] [\kappa_2'] \nu_{q_1 q_2} C(\kappa_1, \kappa_2 K'; q_1, q_2 Q') \begin{pmatrix}
\kappa_1 & \kappa_2 & J \\
\kappa_1' & \kappa_2' & J'
\end{pmatrix}
= \frac{[j_1][j_2]}{[J]^3} t_{Q'}^{K'}
$$

re arranging the terms we get

$$
t_{Q'}^{K'} = \sum_{\kappa_1', \kappa_2', q', q_2} [\kappa_1'] [\kappa_2'] \frac{[J]^3}{[j_1][j_2]} \nu_{q_1 q_2} C(\kappa_1, \kappa_2 K'; q_1, q_2 Q') \begin{pmatrix}
\kappa_1 & \kappa_2 & J \\
\kappa_1' & \kappa_2' & J'
\end{pmatrix}
$$

or

$$
t_{Q}^{K} = \sum_{\kappa, \kappa', q, q'} [\kappa] [\kappa'] \frac{[J]^3}{[j_1][j_2]} \nu_{q q'} C(\kappa, \kappa' K'; q q' Q') \begin{pmatrix}
\kappa & \kappa' & J \\
\kappa & \kappa' & J'
\end{pmatrix}
$$

which is nothing but the inverse relation of Eq. (4.16).
Appendix D

Derivation of equation (4.18)

Consider the relationship between collective state parameters $t_{qq'}^{\kappa'}(N_1, N_2)$ and $t_Q^K(N)$ which is given by Eq. (4.16):

$$t_{qq'}^{\kappa'} = [j_1][j_2][J][\kappa][\kappa'] \sum_{K,Q} \left\{ \begin{array}{ccc} j_1 & j_2 & J \\ j_1 & j_2 & J \\ \kappa & \kappa' & K \end{array} \right\} t_Q^K C(\kappa\kappa'K;qq'Q).$$

To evaluate $t_{q0}^{\kappa 0}$ substitute $\kappa' = 0$ and $q' = 0$ in above equation, we get,

$$t_{q0}^{\kappa 0} = [j_1][j_2][J][\kappa] \sum_{K,Q} \left\{ \begin{array}{ccc} j_1 & j_2 & J \\ j_1 & j_2 & J \\ \kappa & 0 & K \end{array} \right\} t_Q^K C(\kappa 0K;q0Q)\delta_{\kappa \kappa} \delta_{Q q}. \quad (D.1)$$

After simplification we get,

$$t_{q0}^{\kappa 0} = [j_1][j_2][J][\kappa] \left\{ \begin{array}{ccc} j_1 & j_2 & J \\ j_1 & j_2 & J \\ \kappa & 0 & \kappa \end{array} \right\} t_{q}^{\kappa}. \quad (D.2)$$

Evaluating $9j$ symbol in above equation using following set of relations:
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(i) Applying symmetry property of $9j$ symbol which is given in Eq. (4), Eq. (5) and Eq. (6) in Page. 342 of Ref. [81] in the above equation:

\[
\begin{pmatrix}
j_1 & j_2 & J \\
j_1 & j_2 & J \\
\kappa & 0 & \kappa \\
\end{pmatrix} = (-1)^{2j_1+2j_2+2J+2\kappa} \begin{pmatrix}
j_1 & J & j_2 \\
j_1 & J & j_2 \\
\kappa & \kappa & 0 \\
\end{pmatrix} = \begin{pmatrix}
j_1 & J & j_2 \\
j_1 & J & j_2 \\
\kappa & \kappa & 0 \\
\end{pmatrix}.
\]

(ii) Now, applying Eq. (2) in Page. 357 of Ref. [81] on $9j$ symbol i.e.,

\[
\begin{pmatrix}
a & b & c \\
d & e & c \\
g & g & 0 \\
\end{pmatrix} = \frac{W(bcgd; ae)}{[e][g]},
\]

we get

\[
\begin{pmatrix}
j_1 & J & j_2 \\
j_1 & J & j_2 \\
\kappa & \kappa & 0 \\
\end{pmatrix} = \frac{W(Jj_2\kappa j_1; j_1 J)}{[j_2][\kappa]}.
\]

But, from Eq. (11) in Page. 291 of Ref. [81] we have

\[
W(bcgd; ae) = (-1)^{b+c+g+d} \begin{pmatrix}
b & c & a \\
d & g & e \\
\end{pmatrix}
\]

therefore,

\[
\frac{W(Jj_2\kappa j_1; j_1 J)}{[j_2][\kappa]} = (-1)^{\kappa} \begin{pmatrix}
J & j_2 & j_1 \\
j_1 & \kappa & J \\
\end{pmatrix}
\]

or using symmetry property Eq. (2) in Page. 298 of Ref. [81] we get

\[
\frac{(-1)^{\kappa}}{[j_2][\kappa]} \begin{pmatrix}
J & j_2 & j_1 \\
j_1 & \kappa & J \\
\end{pmatrix} = \frac{(-1)^{\kappa}}{[j_2][\kappa]} \begin{pmatrix}
j_2 & j_1 & J \\
\kappa & \kappa & J \\
\end{pmatrix}.
\]

(iii) From Eq. (3) in Page. 300 of Ref. [81] to $6j$ symbol we have

\[
\begin{pmatrix}
a & b & a+b \\
d & e & f \\
\end{pmatrix} = (-1)^{a+b+d+e}
\]

\[
\sqrt{\frac{(2a+2b+1)(a+b+d+e)![(a+b-d-e)!a+b-d-e)!(-a+f)!(-b+d+f)!}{(2a+2b+1)(-a+b+d+e)(a+b+f)!a+b+f+1)(b+d-f)!b+d+f+1)!}}
\]
for which $6j$ symbol obtained in the above step will become

$$\left\langle \begin{array}{c} j_2 \ j_1 \\ \kappa \ j_1 \end{array} \right\rangle = \frac{1}{[j_2][\kappa]} \left[ \frac{(2j_1)!(2j_2)!(2J + \kappa + 1)!(2J - \kappa)!(j_1 - j_2 + J)!}{(2J + 1)!^2(j_2 - j_1 + J)!(2j_1 - \kappa)!(2j_1 + \kappa + 1)!} \right]^{\frac{1}{2}}. \quad (D.3)$$

Therefore we have

$$\left\langle \begin{array}{c} j_1 \ j_2 \ J \\ \kappa \ 0 \ \kappa \end{array} \right\rangle = \frac{1}{[j_2][\kappa]} \left[ \frac{(2j_1)!(2j_2)!(2J + \kappa + 1)!(2J - \kappa)!(j_1 - j_2 + J)!}{(2J + 1)!^2(j_2 - j_1 + J)!(2j_1 - \kappa)!(2j_1 + \kappa + 1)!} \right]^{\frac{1}{2}}. \quad (D.3)$$

Substitute Eq. (D.3) in Eq. (D.2) and after simplification, with $j_1 = \frac{N_1}{2}$, $j_2 = \frac{N_2}{2}$, $J = \frac{N}{2}$, we get

$$t^{\kappa 0}_{q_0}(N_1, N_2) = \mathcal{P}_\kappa(N_1, N_2) \ t^\kappa_q(N), \quad (D.4)$$

where

$$\mathcal{P}_\kappa(N_1, N_2) = \frac{1}{(N)!} \sqrt{\frac{(N_1 + 1)N_1!(N_2 + \kappa + 1)!(\frac{N_1}{2} - \frac{N_2}{2} + \frac{N}{2} + \kappa)!(N - \kappa)!}{(N + 1)(\frac{N_2}{2} - \frac{N_1}{2} + \frac{N}{2})!(N + \kappa + 1)!(N_1 - \kappa)!}}$$

or, since $N_2 = N - N_1$,

$$\mathcal{P}_\kappa(N_1, N_2) = \frac{N_1!}{(N)!} \sqrt{\frac{(N_1 + 1)(N + \kappa + 1)!(N - \kappa)!}{(N + 1)!(N + \kappa + 1)!(N_1 - \kappa)!}};$$

$\kappa = 0, 1, 2, \ldots N_1$, which is the expression for $t^{\kappa 0}_{q_0}$ given in Eq. (4.18).