Chapter 8

Understanding *Raga* Grammar: A Graph Theoretic Approach

“The syntax and the grammar of the language of music are not capricious; they are dictated by the texture and organization of the deep levels of the mind, so with mathematics.” ~ *H. E. Huntley*

8.1 Introduction:

In musicological research graph theory have been widely applied for discovering patterns in music as they provide a visual way to analyze a melodic sequence. The musical event considered for analysis is generally found to be constituted of a finite number of states. The transitions between the states usually occur at discrete points of time according to specified probabilities called transitional probabilities. The transition probabilities are then presented in terms of a (finite) matrix. However graph theoretic approach can also considered as an alternative for describing the Markov chain that can be observed from the transition of the finite states. A number of important properties of the Markov chain can then be deduced from the pictorial representation. In this chapter a graph theoretic approach have been employed to analyze the musical grammar of North Indian classical *ragas*. For our experiment we consider a musical composition belonging to *raga Bhupali* to explain the grammatical structure of the *raga* and its underlying features viz *Arohan-Avarohan* and *Catch Phrase (Pakad)*. The movement of the musical *notes* in the *alap* and a particular composition of *raga Bhupali* are modeled as a Markov chain. A comparison is then made between the weight matrices (t.p.m) of
the *alap* and the composition of *raga* Bhupali. This chapter is thus organised as follows. The materials and methods used in the chapter are discussed in Section 8.2. A brief review of literature on this chapter is presented in Section 1.4 of chapter 1. Section 8.3 contains the description of the structure of *raga* Bhupali along with the graphical representation of its essential features viz. Arohan-Avarohan and Catch Phrase (Pakad). Section 8.4 includes musical cycles of the *raga* obtained from Arohana-Avarohana and Pakad (AAP). In Section 8.5, the process of constructing musical seeds of various lengths has been demonstrated. Further the occurrence pattern of nodes corresponding to their positions in seeds of various lengths is presented in terms of digraphs. Section 8.6 and Section 8.7 includes Markov modeling of movement of *notes* occurring in *alap* and a particular composition of *raga* Bhupali. Finally a brief discussion on the results obtained is presented at the end of the chapter.

### 8.2 Materials and Methods:

Following definitions are used for the purpose of explaining digraphs of Music Theory.

**Musical Graph:** A musical graph is a disconnected digraph with only one dicomponent that describes the pattern of movements of *notes* (nodes) in a musical piece (complete or part). Symbolically a musical graph $M$ is an ordered quadruplet $(V(M), E(M), \psi(M), C(M))$ consisting a non-empty set $V(M) = \{i\}$ of vertices, a set $E(M) = \{e_{ij}\}$ of arcs which is disjoint from $V(M)$, an incidence function $\psi(M) = \{i,j\}$ that associates with each arc of $M$, an ordered pair of (not necessarily distinct) vertices $(i,j)$ of $M$. And $C(M)$ represents a complete musical graph corresponding to $M$. Symbolically $C(M) =$
\{i (e_{i1}, e_{i2},..., e_{ij},..., e_{ik}) \mid 1, 2,..., i, j..., k\} \text{ which represents the connections of the node } i \text{ with nodes } (1, 2,..., i, j,..., k) \text{ by the arcs } e_{i1}, e_{i2},..., e_{ij},..., e_{ik} \text{ respectively for all } i,j \text{ and } k.

**Musical Walk:** A *musical walk* from vertex $i$ to vertex $j$ is an alternating sequence of vertices and arcs, beginning with vertex $i$ and ending with vertex $j$ such that each arc is oriented from the vertex preceding it to the vertex following it. It contains no self loops and any tail node in a *musical walk* can have not more than two heads and vice versa.

Symbolically a *musical walk* $M$ is an ordered quadruplet viz. $(V(M), E(M), \psi(M), C(M))$ where $V(M) = \{i\}$ is a non-empty set of vertices. $E(M) = \{e_{ij}\}$ is a non-empty set of arcs which is disjoint from $V(M)$. $\psi(M) = \{i,j\}$ is an incidence function that associates with each arc of $M$, an ordered pair of vertices $(i,j)$ in two consecutive positions. And $C(M)$ represents a *complete musical graph* corresponding to $M$. Symbolically

$C(M) = \{i (e_{i+1}, e_{i+1}, i+1, i-1)\} \text{ which represents the connections of the node } i \text{ with nodes } (i+1,i-1) \text{ by the arcs } e_{i+1} \text{ and } e_{i-1} \text{ respectively for all } i.$

**Note:** A *musical walk* is a *musical graph* but the converse is not necessarily true.

**Multi-Musical Graph:** A *multi-musical graph* is a *musical graph* with $V(M) = \{i\}$, $\psi(M) = \{(i,j)\}$, $E(M) = \{e_{ij}\}$ and $C(M) = \{i (e_{ij}, e_{ji}, e_{ji}, e_{ji}, e_{ji},...), i, j, j, i, j,...\}$ for some $i,j$ in the graph.

**Musical Cycle:** A *musical cycle* of a particular length corresponding to a *note* (node) in a *musical graph* is defined to be a path through which it returns to itself for the first time in a musical event with minimum number of arcs. The number of arcs in a *musical cycle*
is the length of the cycle.

**Strength of a Node:** Strength $S(a)$ of a node $a$ is defined as the sum of number of arcs converging to it ($I_a$) and the number of arcs diverging out of it ($O_a$). Mathematically

$$S(a) = (I_a + O_a)$$ for $S(a) \in I$, where $I$ is a set of integers.

**Rank of the Strength:** Let $a_1, a_2, \ldots, a_k$ are $k$ notes (nodes) in a musical graph with strengths $S(a_1), S(a_2), \ldots, S(a_k)$ respectively. If the strengths are arranged in order of magnitude such that $S(a_1) < S(a_2) < \ldots < S(a_i) < \ldots < S(a_k)$, then $S(a_i)$ is said to be strength of rank $i$.

**Connectivity of a Musical Graph:** The connectivity of a musical graph $M$ may be defined as the total number of arcs in the graph divided by total number of nodes.

### 8.3 An Experiment with raga Bhupali:

Raga Bhupali is one of the sweetest evening melodies, derived from the Carnatic raga Mohanam. This raga belongs to Kalyan thaat and forbids the use of Madhyam(Ma) and Nishad(Ni). The raga belongs to Audava class and the set of notes used in this raga is given by $A = \{(S_l, R_l, G_l, P_l, D_l); (S_m, R_m, G_m, P_m, D_m); (S_h, R_h, G_h, P_h, D_h)\}$. Equivalently $A$ may be rewritten as

$$A=\{\{-1,-3,-5,-8,-10\}; \{1,3,5,8,10\}; \{13,15,17,20,22\}\}$$

The most important features for identification of any Indian classical raga are its

(i) Arohana (Ar)
(ii) Avarohana (Av) and

(iii) Pakad (Catch Phrase)

In case of raga Bhupali Ar, Av and Pakad are defined as

Arohana (Ar): S_m R_m G_m P_m, D_m, S_h

Avarohana(Av): S_h, D_m P_m, G_m, R_m, S_m

Pakad (Catch Phrase): G_m, R_m, S_m D_l, S_m R_m G_m, P_m G_m, D_m P_m G_m, R_m, S_m

(Note: Here the symbols “f”, “m” and “h” indicate lower, middle and higher octaves respectively. Thus S_m denotes the note Sa sung or played in the middle octave. The rest of the notes can also be similarly defined.)

The Ar and Av may be defined in terms of sequence \{X_n\}. We have already mentioned in chapter 2 and 3 that thaats as well as ragas are ordered sequence of notes formed by inclusion-exclusion theory. Let \{X_n, n=1,2,\ldots\} be a sequence of notes where \(X_n = i\) indicates the position of the \(i^{th}\) note at the time of \(n^{th}\) movement of notes during a performance. If \(X_{n+1} = j\) with \(j>i\), the movement is indexed by Ar. However if \(X_{n+1} = k\) for \(k<i\), the movement is denoted by Av. Graphically represented, the arohana and avarohana of most of the ragas will be found to be regularly ascending and descending straight lines or curves. However this does not hold true for vakra ragas. In vakra ragas, during the course of the arohana or avarohana or both, a prior note will be found to repeat itself. Raga Bhupali which is considered for analysis is a non-vakra raga.
Catch Phrase of a raga is a set of combinations of these ordered notes which are unique to a raga as formulated by musicologists. These short, independent and ordered phrases (combinations) are connected with the help of Ar and Av for the sake of performance of a musical episode, whenever necessary. This facilitates the creation of melody or style of music.

8.3.1 Arohana and Avarohana of raga Bhupali:

Fig 8.1: Digraph $M_1$ corresponding to Arohana and Avarohana of raga Bhupali:

Analysis of $M_1$:

Behavior of movements of the notes combined in Ar and Av may be presented by musical graph $M_1$ using state $i$ as a node for $i = S_m, R_m, G_m, P_m, D_m, S_h$. Here the symbols “$l$”, “$m$” and “$h$” indicate lower, middle and higher octaves respectively. Equivalently $i = 1, 3, 5, 8, 10, 13$. The Ar and Av of the raga do not contain self loops corresponding to the notes viz. 1, 3, 5, 8, 10, 13. However self loops are observed in songs of a raga. The above digraph $M_1$ may be represented by the quadruplet $(V(M_1), E(M_1), \psi(M_1), C(M_1))$ consisting of a non-empty set $V(M_1)$ of nodes, a set $E(M_1)$ of arcs, which is disjoint from $V(M_1)$, an incidence function $\psi(M_1)$ and the complete graph.
$C(M_i)$ where

\[ V(M_i)=\{1,3,5,8,10,13\} \] ... (8.1)

\[ E(M_i)=\{e_{1\ 3}, e_{3\ 5}, e_{5\ 8}, e_{8\ 10}, e_{10\ 13}, e_{13\ 10}, e_{10\ 8}, e_{8\ 5}, e_{5\ 3}, e_{3\ 1}\} \] ... (8.2)

where arc $e_{i\ j}$ represents the connection from $i^{th}$ node to $j^{th}$ node where $i,j = 1, 3, 5, 8, 10, 13$. The incidence function

\[ \psi (M_i) = \{(1,3),(3,5),(5,8),\ldots,(8,5),(5,3),(3,1)\} \] ... (8.3)

\[ C(M_i)= \{ \{1(e_{1\ 3})3\} \{3(e_{3\ 5}, e_{3\ 1})5,1\} \{5(e_{5\ 8}, e_{5\ 3})8,3\} \{8(e_{8\ 10}, e_{8\ 5})10,5\} \]

\{10(e_{10\ 13}, e_{10\ 8})13,8\} \{13(e_{13\ 10})10\} \} \] ... (8.4)

which presents a musical walk.

Since in this graph the arcs intersect at their ends they are planar and can be presented in a plane.
Adjacency matrix $A_1$ corresponding to $M_1$

<table>
<thead>
<tr>
<th>Notes</th>
<th>1</th>
<th>3</th>
<th>5</th>
<th>8</th>
<th>10</th>
<th>13</th>
<th>Outdegree (O)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
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<td>5</td>
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<td>1</td>
<td>0</td>
<td>1</td>
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<td>2</td>
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<td>8</td>
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<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>13</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

| In degree (I) | 1  | 2  | 2  | 2  | 2  | 1  |
| Polarity      | 0  | 0  | 0  | 0  | 0  | 0  |
| Strength=(I+O) | 2  | 4  | 4  | 4  | 4  | 2  |

Total strength = 20
8.3.2 *Catch phrase of raga Bhupali*:

Fig 8.2: Digraph $M_2$ corresponding to *catch phrase of raga Bhupali*:

![Diagaram](image)

Analysis of $M_2$

The catch phrase viz. $G_m$, $R_m$, $S_m D_l$, $S_m R_m$, $G_m$, $P_m G_m$, $D_m P_m G_m$, $R_m S_m$ which is an essential feature for *raga* identification may also be presented by a digraph using the rules of *arohana* and *avarohana*. This may be called “root to catch the catch phrase”.

The above digraph $M_2$ may be represented by the quadruplet $(V(M_2), E(M_2), \psi(M_2), C(M_2))$ consisting of a non-empty set $V(M_2)$ of nodes, a set $E(M_2)$ of arcs, which is disjoint from $V(M_2)$, an incidence function $\psi(M_2)$ and the *complete musical graph* $C(M_2)$ where

$$V(M_2) = \{-10, 1, 3, 5, 8, 10 \} \quad \ldots(8.5)$$

$$E(M_2) = \{e_{-10 \cdot 1}, e_{1 \cdot 3}, e_{3 \cdot 5}, e_{5 \cdot 8}, e_{5 \cdot 10}, e_{10 \cdot 8}, e_{8 \cdot 5}, e_{5 \cdot 3}, e_{3 \cdot 1}, e_{1 \cdot -10}\} \quad \ldots(8.6)$$

$$\psi(M_2) = \{(-10,1),(1,3),(3,5),(5,8),\ldots, (3,1)(1,-10)\} \quad \ldots(8.7)$$

$$C(M_2) = \{-10(e_{-10 \cdot 1}1) \{1(e_{1 \cdot 3}, e_{1 \cdot -10})3,-10\} \{3(e_{3 \cdot 5}, e_{3 \cdot 1})5,1\}$$
\{5(e_{5,8}, e_{5,10}, e_{5,3})8, 10, 3\} \{8(e_{8,5})5\} \{10(e_{10,8})8\} \quad \ldots \quad (8.8)

which represents a \textit{musical graph} but not a \textit{musical walk}.

\textbf{Adjacency matrix }A_2\textbf{ corresponding to }M_2

\begin{table}[h]
\centering
\begin{tabular}{|c|ccccccc|}
\hline
\textit{Notes} & -10 & 1 & 3 & 5 & 8 & 10 & \textit{Outdegree} \\
\hline
\textit{Outdegree} & & & & & & & \textit{(O)} \\
\hline
-10 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 2 \\
3 & 0 & 1 & 0 & 1 & 0 & 0 & 2 \\
5 & 0 & 0 & 1 & 0 & 1 & 1 & 3 \\
8 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
10 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
\hline
\textit{Indegree} & 1 & 2 & 2 & 2 & 2 & \textit{1} \\
\textit{Indegree} & & & & \textit{(I)} & & & \\
\hline
\textit{Polarity} & 0 & 0 & 0 & -1 & 1 & 0 & \textit{Total} \\
\textit{Polarity} & & & & & \textit{strength} = & & 20 \\
\hline
\textit{Strength} & 2 & 4 & 4 & 5 & 3 & 2 & \textit{=(I+O)} \\
\textit{Strength} & & & & & & & \\
\hline
\end{tabular}
\end{table}

\textbf{8.3.3 Properties of graphs }M_1\textbf{ and }M_2:\n
(1) Besides other differences, the matrices \(A_1\) and \(A_2\) also differs in polarity and strength with respect to the node 5. \(M_1\) and \(M_2\) are balanced digraphs.
(2) Since the diagonal elements of the matrices $A_1$ and $A_2$ are zero, there are no loops in either of the digraphs.

(3) A musical walk may be identified from digraph $M_1$ whereas digraph $M_2$ does not represent the same.

(4) From the digraph $M_1$ it is clear that there are five shortest musical cycles from nodes (1, 3, 5, 8, 10) with each cycle of length 2. Again from digraph $M_2$ there are three cycles of length 2, one from each node except for the node 5. There are 2 cycles, one with length 2 and other with length 3 from node 5.

8.4 Musical Cycles obtained by using Arohana, Avarohana and Pakad (AAP):

Let $j$ be a note(node) of a musical episode and $e_{jj}^{(n)}(k)$ be defined as the path where note (node) $j$ is visited by the musical episode for the first time in “$n$” steps provided it started with node $j$ initially. Here $n$ is defined as the minimum number of arcs in a particular path with $k$ nodes for all $k=1,2,3,\ldots$, $K$. $K$ is the total number of nodes corresponding to AAP. The longest path with ordered sequence of nodes in which the catch phrase of the raga is included is known as “root to catch the Catch Phrase”. Thus $\{e_{jj}^{(n)}(k)\}$ is a sequence of paths from node $j$ to $j$ which are automatically included in the “root to catch the catch phrase”. The all possible mutually independent paths taking $j = D_l$ may now be represented as follows:

\[(1) D_l S_m D_l; (2) D_l S_m R_m S_m D_l; (3) D_l S_m R_m G_m R_m S_m D_l; (4) D_l S_m R_m G_m P_m G_m R_m S_m D_l; (5) D_l S_m R_m G_m P_m G_m D_m P_m G_m R_m S_m D_l \ldots (8.9)\]

Hence the longest path i.e. path 5 is the root to catch the catch phrase.
Let a graph $M$ may now be defined by the quadruplet $(V(M), E(M), \psi(M), C(M))$ such that $V(M) = \{-10, 1, 3, 5, 8, 10\}$ and $E(M) = \{e_{i,j}\}$ where edge $e_{i,j}$ represents the connection from $i^{th}$ node to $j^{th}$ node for $i,j = -10, 1, 3, 5, 8, 10$. The incidence function is given by $\psi(M) = \{(i,j)\}$ where $(i,j)$ represents the pair of nodes namely $\{(-10, 1), (1,3), (3,5), (5,8), \ldots, (10,8), \ldots, (3,1), (1, -10)\}$. The above musical cycles (1 to 5) may then be represented in terms of $C(M)$ as follows:

**Cycle(1)** \(D_l S_m D_l\)

\[C(M) = \{-10 (e_{-10,1}) 1\} \{1 (e_{1,-10}) 10\}\]

**Cycle(2)** \(D_l S_m R_m S_m D_l\)

\[C(M) = \{-10 (e_{-10,1}) 1\} \{1 (e_{1,3}) 3\} \{3 (e_{3,5}) 1\} \{1 (e_{1,-10}) 10\}\]

**Cycle(3)** \(D_l S_m R_m G_m R_m S_m D_l\)

\[C(M) = \{-10 (e_{-10,1}) 1\} \{1 (e_{1,3}) 3\} \{3 (e_{3,5}) 5\} \{5 (e_{5,3}) 3\} \{3 (e_{3,1}) 1\} \{1 (e_{1,-10}) 10\}\]

**Cycle (4)** \(D_l S_m R_m G_m P_m G_m R_m S_m D_l\)

\[C(M) = \{-10 (e_{-10,1}) 1\} \{1 (e_{1,3}) 3\} \{3 (e_{3,5}) 5\} \{5 (e_{5,8}) 8\} \{8 (e_{8,5}) 5\} \{5 (e_{5,3}) 3\} \{3 (e_{3,1}) 1\} \{1 (e_{1,-10}) 10\}\]

**Cycle(5)** \(D_l S_m R_m G_m P_m G_m D_m P_m G_m R_m S_m D_l\)

\[C(M) = \{-10 (e_{-10,1}) 1\} \{1 (e_{1,3}) 3\} \{3 (e_{3,5}) 5\} \{5 (e_{5,8}) 8\} \{8 (e_{8,5}) 5\} \{5 (e_{5,10}) 10\} \{10 (e_{10,8}) 8\} \{8 (e_{8,5}) 5\} \{5 (e_{5,3}) 3\} \{3 (e_{3,1}) 1\} \{1 (e_{1,-10}) 10\}\]

where $D_l = -10$. 
These first passage paths from $D_l$ to $D_l$ are considered as musical cycles corresponding to AAP. The musical cycles (1-5) are then linearly expanded and are combined sequentially to construct an elongated path. The path thus obtained is the shortest elongated path that may be used for construction of chalans of the raga.

Algorithm for construction of shortest elongated path:

The shortest elongated path for first reach from first node to first node of a raga may be obtained in three steps.

**Step One:** Taking union of all the cycles we get

\[
\text{Cycle}(1) \cup \text{Cycle}(2) \cup \text{Cycle}(3) \cup \text{Cycle}(4) \cup \text{Cycle}(5)
\]

\[
= (D_l S_m D_l) \cup (D_l S_m R_m S_m D_l) \cup (D_l S_m R_m G_m R_m S_m D_l) \cup (D_l S_m R_m G_m P_m G_m R_m S_m D_l).
\]

**Step Two:** Taking the union of complete musical graphs for all the cycles we get a common complete musical graph viz.

\[
[{-10 (e_{-10}) 1} \{1 (e_{-10}) -10\}] \cup [{-10 (e_{-10}) 1} \{1 (e_{1,3}) 3\} \{3 (e_{3,1}) 1\} \{1 (e_{1,-10}) -10\}] \cup [{-10 (e_{-10}) 1} \{1 (e_{1,3}) 3\} \{3 (e_{3,5}) 5\} \{5 (e_{5,3}) 3\} \{3 (e_{3,1}) 1\} \{1 (e_{1,-10}) -10\}] \cup [{-10 (e_{-10}) 1} \{1 (e_{1,3}) 3\} \{3 (e_{3,5}) 5\} \{5 (e_{5,8}) 8\} \{8 (e_{8,5}) 5\} \{5 (e_{5,3}) 3\} \{3 (e_{3,1}) 1\} \{1 (e_{1,-10}) -10\}
\]


**Step Three:** Removing all the elements of type \( \{1 \, (e_{1-10}) -10\} \) and \( \{-10 \, (e_{-10 \, 1}) \, 1\} \) from the middle of the common *musical graph* we get another *complete musical graph* corresponding to the *shortest elongated* path. Equivalently the *complete musical graph* corresponding to the *shortest elongated* path may be obtained as follows:

\[
\{\{-10 \, (e_{-10 \, 1}) \, 1\}\, 1\} \{1\, (e_{1 \, 3}) \, 3\} \{3 \, (e_{3 \, 1}) \, 1\} \{1 \, (e_{1 \, 3}) \, 3\} \{3 \, (e_{3 \, 5}) \, 5\} \{5 \, (e_{5 \, 3}) \, 3\} \{3 \, (e_{3 \, 1}) \, 1\} \{1 \, (e_{1 \, 3}) \, 3\} \{3 \, (e_{3 \, 5}) \, 5\} \{5 \, (e_{5 \, 8}) \, 8\} \{8 \, (e_{8 \, 5}) \, 5\} \{5 \, (e_{5 \, 3}) \, 3\} \{3 \, (e_{3 \, 1}) \, 1\} \{1 \, (e_{1 \, 3}) \, 3\} \{3 \, (e_{3 \, 5}) \, 5\} \{5 \, (e_{5 \, 10}) \, 10\} \{10 \, (e_{10 \, 8}) \, 8\} \{8 \, (e_{8 \, 5}) \, 5\} \{5 \, (e_{5 \, 3}) \, 3\} \{3 \, (e_{3 \, 1}) \, 1\} \{1 \, (e_{1 \, 3}) \, -10\} \]
\]

…(8.10)

Thus we get a set of nodes corresponding to the *shortest elongated* path viz.

\[
\{D_l \, S_m \, R_m \, S_m \, R_m \, G_m \, R_m \, S_m \, R_m \, G_m \, P_m \, G_m \, R_m \, S_m \, R_m \, G_m \, P_m \, G_m \, D_m \, P_m \, G_m \, R_m \, S_m \, D_l\}
\]

...(8.11)

This path may now be used to construct *chalans of* *raga Bhupali*.

**8.5 Seeds and their presence in a musical composition:**

A musical *seed* of length \( m \) may be defined as a set of \( m \) *notes* with \((i,j)^{th}\) element \( d_{ij}\), where \( d_{ij}\) is the *note* \( j \) in \( i^{th}\) position in a composition for \( i = 1,2,\ldots, m\) and \( j \) belongs to the set \( A=\{j_1, j_2, \ldots, j_k\} \) of finite number \( k \) of *notes* used in a *raga* arranged in ascending order of *pitch*. Thus in a musical composition for any \( j \) defined above, the sets \( S_i=\{d_{ij}, \, d_{2j}, \ldots, d_{mj}\} \) which is followed by the set \( S_2 = \{d_{2j}, d_{3j}, \ldots, d_{m+j}\}\) and so on, may be used
to construct melody. Here \( j \) takes any value in \( A \) corresponding to its position \( i \). It is obvious that \( S_1 \cap S_2 = \{d_{2j}, \ldots, d_{nj}\} \). If the path length is \( n \) and \( i \) denotes the position of the note \( j \), \((r=1,2\ldots k)\) in a path then there are \( n-m+1 \) such sets that gives
\[
S_1 \cup S_2 \cup \ldots \cup S_{n-m+1} = \{d_{ij}, d_{2j}, \ldots, d_{n-m+j}, \ldots, d_{nj}\}.
\]

Thus seeds corresponding to different lengths constructed from shortest elongated path are given by

\[
\text{Length(1)}: \{D_l, S_m, R_m, S_m, R_m, G_m, R_m, S_m, R_m, G_m, P_m, G_m, R_m, S_m, P_m, G_m, D_m, P_m, G_m, R_m, S_m, D_l\} \\
\text{Length(2)}: \{D_l, S_m, R_m, S_m, R_m, G_m, P_m, G_m, R_m, S_m, R_m, G_m, P_m, G_m, R_m, S_m, D_m, P_m, G_m, R_m, S_m, D_l\} \\
\text{Length(3)}: \{D_l, S_m, R_m, S_m, R_m, G_m, P_m, G_m, R_m, S_m, R_m, G_m, P_m, G_m, R_m, S_m, D_m, P_m, G_m, D_m, P_m, G_m, R_m, S_m, D_l\}
\]

For demonstration purpose we may now write the adjacency matrix for seed of depth 2 as follows
### Adjacency matrix $A_s$ for seed of depth 2:

<table>
<thead>
<tr>
<th></th>
<th>$D_l$</th>
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<th>$R_m$</th>
<th>$G_m$</th>
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The obvious change in matrix $A_s$ is that the far left hand column of output matrix $A_s$ now contains 2 notes (prior states) as the seed rather than 1. Again it is observed that the Output adjacency matrix is always same with elements either 0 or 1 regardless of the depth. However order of the matrix turns out to be $(n-1)xk$ instead of $kxk$ where $n$ is length of shortest elongated path. Thus the matrix $A_s$ maintains the 'character' of the data used to construct it. Musically we might say it stores a sense of 'style'.

[http://explodingart.com/jmusic/jmtutorial/Markov1.html]
8.5.1 Occurrence pattern of the nodes corresponding to their positions in a musical event:

The shortest elongated path defined above may be used to construct seeds of various lengths. The musical seeds with length $m \geq 2$ can then be used for composing different melodies. The occurrence pattern of the nodes corresponding to their positions in seeds of various lengths may be presented in terms of a digraph.

The digraph of seeds with length 1, 2, 3 and 4 along with the position of the nodes in the shortest elongated path are given below for the purpose of demonstration and understanding.

**Transition pattern of seed of length 1**

$$i \rightarrow i+1 \rightarrow i+2 \rightarrow i+3 \rightarrow \ldots \rightarrow i+k-1 \ldots \rightarrow n$$

**Fig 8.3: Digraph of seed of length 1**

![Digraph of seed of length 1](image-url)
Transition pattern of seed of length 2

\[(i \rightarrow i+1) \rightarrow (i+1 \rightarrow i+2) \rightarrow (i+2 \rightarrow i+3) \rightarrow \ldots \rightarrow (i+k-1 \rightarrow i+k) \rightarrow \ldots \rightarrow (n-1 \rightarrow n)\]

Fig 8.4: Digraph of seed of length 2

Transition pattern of seed of length 3

\[(i \rightarrow i+1 \rightarrow i+2) \rightarrow (i+1 \rightarrow i+2 \rightarrow i+3) \rightarrow (i+2 \rightarrow i+3 \rightarrow i+4) \rightarrow \ldots \rightarrow (i+k-2 \rightarrow i+k-1 \rightarrow i+k) \rightarrow \ldots \rightarrow (n-2 \rightarrow n-1 \rightarrow n)\]

Fig 8.5: Digraph of seed of length 3
Transition pattern of seed of length 4

\((i \rightarrow i+1 \rightarrow i+2 \rightarrow i+3) \rightarrow (i+1 \rightarrow i+2 \rightarrow i+3 \rightarrow i+4) \rightarrow (i+2 \rightarrow i+3 \rightarrow i+4 \rightarrow i+5) \rightarrow (i+3 \rightarrow i+4 \rightarrow i+5 \rightarrow i+6) \rightarrow \ldots \rightarrow (i+k-3 \rightarrow i+k-2 \rightarrow i+k-1 \rightarrow i+k) \rightarrow \ldots \rightarrow (n-3 \rightarrow n-2 \rightarrow n-1 \rightarrow n)\)

**Fig 8.6: Digraph of seed of length 4**

It can be seen that the digraphs of the musical seeds corresponding to various lengths looks like ornaments.

### 8.6 Markov chain corresponding to movement of notes in Alap:

In this section a sample alap of raga Bhupali have been considered for analyzing the grammatical structure of the raga corresponding to the movement of notes in the alap of raga Bhupali. The alap is basically the opening section in the rendition of a typical North Indian classical raga where the various possibilities of exposing a raga are
explored. It gives valuable hints on aspects such as appropriate starting note, the typical phrases, the notes around which the raga may be elaborated and some other principle characteristics like arohana, avarohana and catch phrase. This justifies the reason for selecting the alap for analyzing the grammar of a raga.

For the purpose of analysis Markov chain have been employed to study the occurrence pattern of movement of notes in alap of raga Bhupali. Consider one step “note to note” transitions in a musical episode. Let $X_n = i$ be a random variable where $i$ is the note occurring in alap of raga Bhupali as an effect of $n^{th}$ transition for $i = P_l, D_l, S_m, R_m, G_m, P_m, D_m, S_h, R_h, G_h$ and $n = 1, 2, 3, \ldots$. Further it is assumed that

$$P\{X_{n+1} = j | X_1 = k, X_2 = l, \ldots, X_n = i\} = P\{X_{n+1} = j | X_n = i\} = p_{ij}$$

such that $\sum_i p_{ij} = 1$. Thus $\{X_n = i, n = 1, 2, 3, \ldots\}$ may be assumed to be a Markov chain with state space $S = \{D_l, S_m, R_m, G_m, P_m, D_m, S_h, R_h, G_h, P_h\}$ or equivalently $S = \{-10, 1, 3, 5, 8, 10, 13, 15, 17, 20\}$.

The parameters $p_{ij}$'s are estimated by the method of maximum likelihood.

$$\hat{p}_{ij} = \frac{n_{ij}}{n_i} \quad \text{with} \quad n_{ij} \text{ being number of times the system visited the state } j \text{ provided it entered state } i \text{ initially and } n_i \text{ is the total number of times the system is in state } i.$$

The transition probability matrix (weight matrix, in context of graph theory) corresponding to alap of Bhupali, its digraph, adjacency matrix and properties of digraph are discussed below.
Transition Probability Matrix (weight matrix) $W_1$ corresponding to sample *alap* of *raga Bhupali*:

$$W_1 =$$

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**Fig 8.7:** Digraph $M_3$ corresponding to sample *alap* of *raga Bhupali*:
Adjacency matrix \( A_3 \) corresponding to \( M_3 \)

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| Indegree (I) | 3 | 5 | 5 | 5 | 4 | 4 | 6 | 4 | 4 | 3 |
| Polarity     | 0 | -1 | 3 | -2 | 0 | 0 | -3 | 0 | 2 | 1 |
| Strength = (I+O) | 6 | 11 | 7 | 12 | 8 | 8 | 15 | 8 | 6 | 5 |
| Total Strength = | 86 |
8.6.1 Analysis of $M_3$:

The above digraph $M_3$ is a *musical graph* and may be represented by the quadruplet $((V(M_3), E(M_3), \psi(M_3), C(M_3))$ consisting of a non-empty set $V(M_3)$ of nodes, a set $E(M_3)$ of arcs, which is disjoint from $V(M_3)$, an incidence function $\psi(M_3)$ and a *complete musical graph* $C(M_3)$ where

$$V(M_3) = \{ D_l, S_m, R_m, G_m, P_m, D_m, S_h, R_h, G_h \}$$

or

$$V(M_3) = \{-10,1,3,5,8,10,13,15,17,20 \} \quad \text{...}(8.15)$$

$$E(M_3) = \{ (e_{-10} -10, e_{-10} 1), (e_{13} 1, e_{13} 3), (e_{15} 5, e_{15} 8), (e_{5} 5, e_{5} 8), (e_{10} 10, e_{10} 13, e_{10} 13, e_{10} 8), (e_{-10} 10, e_{10} 13, e_{10} 13, e_{10} 8), (e_{-10} 10, e_{10} 13, e_{10} 13, e_{10} 8), (e_{-10} 10, e_{10} 13, e_{10} 13, e_{10} 8), (e_{17} 20, e_{17} 15), (e_{20} 20, e_{20} 17) \} \quad \text{...}(8.16)$$

where edge $e_{ij}$ represents the connection from $i^{th}$ node to $j^{th}$ node where $i,j = -10,1,3,5,8,10,13,15,17,20$.

The incidence function

$$\psi(M_3) = \{ (-10,1), (13,1), (3,5), (5,8), ... , (13,10), (10,8), ... , (3,1), (1, -10) \}$$

\text{...}(8.17)
The complete musical graph of *alap* is

\[
C(M_3) = \left\{ \begin{align*}
(\{-10, -10, e_{-10} 1, e_{-10} 3\} \setminus \{-10, 1, 3\} \cup \{1(e_{1} 1, e_{1} 3, e_{1} 5, e_{1} 8, e_{1} 13, e_{1} -10)\}, 1, 3, 5, 8, 13, -10 \} \cup \{(3(e_{3} 5, e_{3} 1) \setminus 5, 1) \cup \{5(e_{5} 5, e_{5} 8, e_{5} 10, e_{5} 13, e_{5} 3, e_{5} 1, e_{5} -10)\}, 5, 8, 10, 13, 3, 1, 10 \} \cup \{8(e_{8} 10, e_{8} 13, e_{8} 5, e_{8} 3) \setminus 10, 13, 5, 3\} \cup \{10(e_{10} 13, e_{10} 15, e_{10} 17, e_{10} 8)\}, 13, 15, 17, 8\} \cup \{13(e_{13} 13, e_{13} 15, e_{13} 17, e_{13} 20, e_{13} 10, e_{13} 8, e_{13} 5, e_{13} 3, e_{13} 1)\}, 13, 15, 17, 20, 10, 8, 5, 3, 1\} \cup \{15(e_{15} 15, e_{15} 17, e_{15} 13, e_{15} 10)\}, 15, 17, 13, 10\} \cup \{17(e_{17} 20, e_{17} 15)\}, 20, 15\} \cup \{20(e_{20} 20, e_{20} 17)\}, 20, 17\} \right\}
\]

...(8.18)

### 8.6.2 Properties of *M₃*:

1. The non-empty set \(V(M_3)\) of nodes, may be written in terms of \(V(M_1)\) and \(V(M_2)\) such that

\[
V(M_3) = \{D_l, V(M_1), R_h, G_h, P_h\} = \{V(M_2), S_h, R_h, G_h, P_h\}.
\]

i.e. \(V(M_1)\), and \(V(M_2)\), are subsets of \(V(M_3)\).

Again

\[
V(M_1) \cup V(M_2) = \{D_l, S_m, R_m, G_m, P_m, D_m, S_h\} = \{-10, 1, 3, 5, 8, 10, 13\}
\]

this implies \(V(M_3) = \{V(M_1) \cup V(M_2), R_h, G_h, P_h\} = \{-10, 1, 3, 5, 8, 10, 13, 15, 17, 20\}\)

2. It can be shown that \(C(M_1)\) and \(C(M_2)\) are subsets of \(C(M_3)\) which shows that *ar* and *av* and root to catch the catch phrase are subsets of *alap*.

3. Graph \(M_3\) is not a simple graph but catch phrase and *arohan-avarohan* makes their presence visible in \(M_3\). Graph \(M_3\) is a balanced graph.
(4) The node $G_m$ has the highest polarity. There exist self loops for the nodes $D_l$, $S_m$, $G_m$, $S_h$, $R_h$, $P_h$ or (-10, 1, 5, 13, 15, 20) respectively.

(5) Since the chain is irreducible, the digraph $M_3$ has only one dicomponent.

**Note:** As it is mentioned already that in case of alap the first reach into a new octave can be a powerful event, we may reduce the size of the state space of alap by converting the ten state process into a three state process by writing $l = (D_l)$, $m = (S_m, R_m, G_m, P_m, D_m)$ and $h = (S_h, R_h, G_h, P_h)$. Hence $\{X_n = i, n = 1, 2, 3,...\}$ reduces to a Markov chain with state space $S = \{l, m, h\}$. The transition probability matrix corresponding to state space $S = \{l, m, h\}$ is presented in chapter 5 corresponding to a composition of raga Bhupali.

### 8.7 Comparison of alap and a composition of raga Bhupali:

A composition belonging to raga Bhupali has been considered and its corresponding weight matrix has also been derived. The weights are estimated by method of maximum likelihood. The digraph, adjacency matrix and weight matrix corresponding to the composition are presented below.
Transition Probability Matrix (weight matrix) $W_2$ corresponding to composition of

*raga Bhupali*:

$W_2 =$

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Fig 8.8: Digraph $M_4$ corresponding to a composition of *raga Bhupali*:
Adjacency Matrix corresponding to $M_4$

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8.7.1 Analysis of $M_4$:

The above digraph $M_4$ is a musical graph and may be represented by the quadruplet $(V(M_4), E(M_4), \psi(M_4), C(M_4))$ consisting of a non-empty set $V(M_4)$ of nodes, a set $E(M_4)$ of arcs, which is disjoint from $V(M_4)$, an incidence function $\psi(M_4)$ and a complete graph $C(M_4)$ where

$$V(M_4) = \{S_m, R_m, G_m, P_m, D_m, S_h, R_h, G_h\}$$

or

$$V(M_4) = \{1, 3, 5, 8, 10, 13, 15, 17\}$$

...(8.19)
\[ E(M_4) = \{ (e_{1\ 1}, e_{1\ 5}, e_{1\ 8}), (e_{3\ 5}, e_{3\ 1}), (e_{5\ 5}, e_{5\ 8}, e_{5\ 3}), (e_{8\ 8}, e_{8\ 10}, e_{8\ 13}, e_{8\ 5}), (e_{10\ 10}, e_{10\ 13}, e_{10\ 8}), (e_{13\ 13}, e_{13\ 15}, e_{13\ 10}, e_{13\ 8}), (e_{15\ 15}, e_{15\ 17}, e_{15\ 13}), (e_{17\ 15}) \} \]

where edge \( e_{ij} \) represents the connection from \( i^{th} \) node to \( j^{th} \) node where \( i, j = 1, 3, 5, 8, 10, 13, 15, 17 \). The incidence function

\[ \psi(M_4) = \{ (1, 3), (3, 5), (5, 8), (8, 10), \ldots, (13, 10), (10, 8), \ldots, (5, 3), (3, 1) \} \]

\[ \ldots \text{(8.20)} \]

The complete musical graph of alap is

\[ C(M_4) = \{ \{ 1(e_{1\ 1}, e_{1\ 5}, e_{1\ 8}), 1, 5, 8 \} \{ 3(e_{3\ 5}, e_{3\ 1}), 5, 1 \} \{ 5(e_{5\ 5}, e_{5\ 8}, e_{5\ 3}) \} \{ 8(e_{8\ 8}, e_{8\ 10}, e_{8\ 13}, e_{8\ 5}), 8, 10, 13, 5 \} \{ 10(e_{10\ 10}, e_{10\ 13}, e_{10\ 8}, e_{10\ 5}) \} \{ 13(e_{13\ 13}, e_{13\ 15}, e_{13\ 10}, e_{13\ 8}), 13, 15, 10, 8 \} \{ 15(e_{15\ 15}, e_{15\ 17}, e_{15\ 13}), 15, 17, 13 \} \{ 17(e_{17\ 15}) \} \} \]

\[ \ldots \text{(8.21)} \]

It may be seen that only a part of \( C(M_1) \) and \( C(M_2) \) is a sub-graph of \( C(M_4) \). This implies that arohan-avarohan and root to catch the catch phrase are not completely present in the musical graph of the composition.

8.7.2 Comparison of \( M_3 \) and \( M_4 \):

(1) In alap the highest strength is exhibited by the node 13 whereas in the composition the node 8 exhibits highest strength followed by node 5 (i.e. vadi note).
(2) The total strength of alap is greater than that of the composition.

(3) The whole catch phrase and Arohan-Avarohan is found to be a sub-graph of alap whereas only a part of catch phrase as well as arohan-avarohan is visible in the graph of the composition. This probably justifies the reason why grammar of a raga can be better explained by alap of the raga than a particular composition of the raga.

8.7.3 Distance between weights of alap and song of raga Bhupali:

The estimate of mean absolute difference of weights of alap and composition is given by \( W = \sum_i \sum_j |W_{ij} - W_{2j}| \) where \( n \) = total number of elements in the matrix \( |W_1 - W_2| \).

Here \( W \) serves as a crude measure of difference between alap and composition of raga Bhupali corresponding to their weights. The variance of the absolute difference of weights has also been obtained. Hence \( W= 0.0705 \) and variance =0.02.

Discussion:

Pattern presentation by graph theory definitely plays a significant role in the theory of music and it makes understanding of raga grammar a bit easier. The underlying features of ragas and the pattern of distribution of notes in the ragas as understood by traditional musicians with years of continuous practice and devotion can also be explained to some extent by the application of graph theory. However there remains much to be explored for proper understanding of the subject. But this is beyond the scope of this chapter.

Next chapter deals with allied ragas and graph theoretic presentation.