CHAPTER-III

AXIALLY SYMMETRIC COSMOLOGICAL MODELS IN SOME SCALAR TENSOR THEORIES OF GRAVITATION*

*Work in this chapter resulted in the following publications:


3.1 INTRODUCTION:

In recent years, the researchers have shown their keen interest in the alternative theories of gravitation. One of the most popular theories among them is a scalar tensor theory of gravitation, proposed by Brans and Dicke (1961).

Several aspects of strange quark matter attached to string cloud have been extensively investigated by many authors. Dey et al. (1998) have obtained new sets of EOS for strange matter based on a model of inter quark potential which has the following features: (a) asymptotic freedom, (b) confinement at zero baryon density and deconfinement at high baryon density, (c) chiral symmetry restoration and (d) gives stable uncharged $\beta$-stable matter. Adhav et al. (2008, 2009) have discussed string cloud and domain walls with quark matter in $n$-dimensional Kaluza-Klein cosmological model in general relativity and strange quark matter attached to string cloud in Bianchi type-III space time in general relativity. Khadekar et al. (2009) have confined their work to the quark matters attached to the topological defects in general relativity. Khadekar and Rupali Wanjari (2012) have discussed geometry of quark and strange quark matter in higher dimensional general relativity.
Rao and Neelima (2013) have discussed axially symmetric space time with strange quark matter attached to the string cloud in self creation theory and general relativity and established that the additional condition, special law of variation of Hubble parameter proposed by Berman (1983), taken by Katore and Shaikh (2012) in general relativity is superfluous. Rao and Sireesha (2012c) have studied Bianchi type-II, VIII and IX cosmological models with strange quark matter attached to string cloud in Brans-Dicke theory of gravitation.

Barber (1982) proposed two self-creation cosmologies by modifying the Brans-Dicke (1961) theory and general relativity. These modified theories create the universe out of self contained gravitational and matter fields. Brans (1987) has pointed out that Barber’s first theory is not only in disagreement with experiment, but is actually inconsistent. Barber’s second theory is a modification of general relativity to a variable G - theory. In this theory the scalar field does not directly gravitate, but simply divides the matter tensor, acting as a reciprocal gravitational constant. It is postulated that the scalar field couples to the trace of the energy momentum
tensor. A comparison with Einstein’s equations shows that the Barber’s theory goes over to General relativity in every respect in case of coupling constant $\eta \to 0$, $\phi = \text{constant} = G^{-1}$. With a small value of $\eta$ there would have been a violent period of matter creation in the earliest stages of the Big-Bang. However, such a view is probably unable to explain all the observed helium abundance and other mechanisms. Barber (1982) and Soleng (1987) have discussed the FRW models while Reddy and Venkateswarlu (1989) have studied the Bianchi type-VI$_0$ cosmological model in Barber’s second theory of gravitation. Wang (2004, 2005, 2006), Bali and Dave (2002), Bali and Pradhan (2007) and Tripathy et al. (2010) have studied the Bianchi type cosmological model in the presence of cosmic strings and bulk viscosity. Rao and Sireesha (2012d) have studied Bianchi type-II, VIII and IX string cosmological models with bulk viscosity in Brans-Dicke theory of gravitation.

In this chapter, we have discussed axially symmetric perfect fluid as well as strange quark matter attached to string cloud cosmological models in a scalar tensor theory of gravitation proposed by Brans and Dicke (1961) and string cosmological model with bulk viscosity in self creation theory of gravitation.
This chapter is organized as follows: In section 3.2, the field equations and their solutions are obtained for axially symmetric metric in the presence of strange quark matter attached to string cloud and in section 3.3, the field equations and their solutions are obtained for axially symmetric metric in the presence of perfect fluid distribution in Brans-Dicke theory of gravitation. In section 3.4, the field equations and their solutions are obtained for axially symmetric metric in the presence of string cosmological model with bulk viscosity in self creation theory of gravitation. In section 3.5, some important features of the models with conclusions are discussed.

### 3.2 METRIC AND FIELD EQUATIONS:

We consider the axially symmetric metric in the form

\[
ds^2 = -dt^2 + A^2 (dx^2 + f^2(\chi)d\phi^2) + B^2 dz^2 \tag{3.2.1}\]

where A, B are functions of ‘t’ only and f is a function of the coordinate ‘\chi’ only.
The non-vanishing components of the Einstein’s tensor are given as,

\[ G^1_1 = G^2_2 = \frac{\dddot{A}}{A} + \frac{\ddot{A} \dot{B}}{AB} + \frac{\dddot{B}}{B} \]  \hspace{1cm} (3.2.2)

\[ G^3_3 = \frac{2\dddot{A}}{A} + \left(\frac{\dddot{A}}{A}\right)^2 - \frac{1}{A^2} \left(\frac{f''}{f}\right) \]  \hspace{1cm} (3.2.3)

\[ G^4_4 = \left(\frac{\dddot{A}}{A}\right)^2 + \frac{2\dddot{A} \ddot{B}}{AB} - \frac{1}{A^2} \left(\frac{f''}{f}\right) \]  \hspace{1cm} (3.2.4)

Here the over head dot denotes differentiation with respect to ‘t’ and the over head dash denotes differentiation with respect to ‘χ’.

Brans - Dicke field equations for combined scalar and tensor field are given by

\[ G_{ij} = -8\pi\phi^{-1}T_{ij} - \omega\phi^{-2}\left(\phi_{i},\phi_{j} - \frac{1}{2} g_{ij}\phi_{k}\phi_{,k}\right) \]

\[ -\phi^{-1}(\phi_{;j} - g_{ij}\phi_{,k}) \]  \hspace{1cm} (3.2.6)

and \( \phi_{,k}^k = 8\pi(3 + 2\omega)^{-1}T \)  \hspace{1cm} (3.2.7)
where $G_{ij} = R_{ij} - \frac{1}{2} R g_{ij}$ is an Einstein tensor, $R$ is the scalar curvature, $\omega$ and $n$ are constants, $T_{ij}$ is the stress energy tensor of the matter and comma and semicolon denote partial and covariant differentiation respectively.

Also, we have energy – conservation equation

$$T;_{ij} = 0.$$ \hspace{1cm} (3.2.8)

This is a consequence of the field equations (3.2.6) and (3.2.7).

The energy momentum tensor for string cloud is given by

$$T_{ij} = \rho u_i u_j - \rho_s x_i x_j$$ \hspace{1cm} (3.2.9)

where $\rho$ is the rest energy density for the cloud of strings with particles attached to them and $\rho_s$ is the string tension density and they are related by

$$\rho = \rho_p + \rho_s$$ \hspace{1cm} (3.2.10)

where $\rho_p$ is the particle energy density. We know that string is free to vibrate. The different vibrational models of the string represent the different types of particles because these different
models are seen as different masses or spins. Therefore, here we will take quarks instead of particles in the string cloud. Hence we consider quark matter energy density instead of particle energy density in the string cloud.

In this case from (3.2.10), we get

\[ \rho = \rho_q + \rho_s + B_c \]  

(3.2.11)

From (3.2.10) & (3.2.11), we have energy momentum tensor for strange quark matter attached to the string cloud (Yavuz et al. 2005) as

\[ T_{ij} = (\rho_q + \rho_s + B_c)u_i u_j - \rho_s x_i x_j \]  

(3.2.12)

where \( u_i \) is the four velocity of the particles and \( x_i \) is the unit space like vector representing the direction of string.

We have \( u_i \) and \( x_i \) with

\[ u_i u^i = -x_i x^i = 1 \text{ and } u_i x^i = 0 \]  

(3.2.13)

We have taken the direction of string along z-axis.
Then the components of energy momentum tensor are

\[ T_1^1 = T_2^2 = 0, T_3^3 = \rho_s, T_4^4 = \rho \]  \hspace{1cm} (3.2.14)

where \( \rho \) and \( \rho_s \) are functions of ‘t’ only.

The field equations (3.2.6) & (3.2.7) for the metric (3.2.1), with the help of equations (3.2.9) to (3.2.14), can be written as

\[
\frac{\ddot{A}}{A} + \frac{\dot{A} \dot{B}}{AB} + \frac{\ddot{B}}{B} + \frac{1}{2} \frac{\omega \dot{\phi}^2}{\phi^2} + \frac{\ddot{\phi}}{\phi} + \frac{\dot{\phi}}{\phi} \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right) = 0 
\]  \hspace{1cm} (3.2.15)

\[
\frac{2\ddot{A}}{A} + \left( \frac{\dot{A}}{A} \right)^2 - \frac{1}{A^2} \left( \frac{f''}{f} \right) + \frac{1}{2} \frac{\omega \dot{\phi}^2}{\phi^2} + \frac{\ddot{\phi}}{\phi} + \frac{\dot{\phi}}{\phi} \left( \frac{2\dot{A}}{A} \right) = 8\pi \phi^{-1} \rho_s 
\]  \hspace{1cm} (3.2.16)

\[
\left( \frac{\dot{A}}{A} \right)^2 + \frac{2\dot{A} \dot{B}}{AB} - \frac{1}{A^2} \left( \frac{f''}{f} \right) - \frac{1}{2} \frac{\omega \dot{\phi}^2}{\phi^2} + \frac{\ddot{\phi}}{\phi} + \frac{\dot{\phi}}{\phi} \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right) = 8\pi \phi^{-1} \rho 
\]  \hspace{1cm} (3.2.17)

\[
\ddot{\phi} + \frac{\dot{\phi}}{2} \left( 2\frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right) = 8\pi (3 + 2\omega)^{-1} (\rho + \rho_s) 
\]  \hspace{1cm} (3.2.18)

\[
\dot{\rho} + (\rho + \rho_s) \left( 2\frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right) = 0 
\]  \hspace{1cm} (3.2.19)
Here the overhead dot denotes differentiation with respect to ‘t’ and the overhead dash denotes differentiation with respect to ‘\( \chi \)’.

**SOLUTIONS OF THE FIELD EQUATIONS:**

From (3.2.16) & (3.2.17), we can observe that it is possible to separate the terms of \( f(\chi) \) to one side and the terms of \( A(t), B(t), \rho(t) & \rho_\chi(t) \) to another side. Hence we can take each part is equal to a constant. So,

\[
\frac{f''}{f} = k^2, \quad k^2 \text{ is a constant.} \tag{3.2.20}
\]

If \( k=0 \), then \( f(\chi) = c_1 \chi + c_2, \quad \chi > 0 \)

where \( c_1 \) and \( c_2 \) are integrating constants.

Without loss of generality, by taking \( c_1 = 1 \) and \( c_2 = 0 \),

we get \( f(\chi) = \chi \).
Now the field equations (3.2.15) to (3.2.19) will reduce to

\[
\frac{\ddot{A}}{A} + \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}}{B} + \frac{1}{2} \omega \dot{\phi}^2 + \frac{\ddot{\phi}}{\phi} + \ddot{\phi} \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right) = 0
\]  \hspace{1cm} (3.2.21)

\[
\frac{2\dot{A}}{A} + \left( \frac{\dot{A}}{A} \right)^2 + \frac{1}{2} \omega \dot{\phi}^2 + \frac{\ddot{\phi}}{\phi} + \ddot{\phi} \left( \frac{2\dot{A}}{A} \right) = 8\pi \phi^{-1} \rho_s
\]  \hspace{1cm} (3.2.22)

\[
\left( \frac{\dot{A}}{A} \right)^2 + \frac{2\dot{A}\dot{B}}{AB} - \frac{1}{2} \omega \dot{\phi}^2 + \ddot{\phi} \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right) = 8\pi \phi^{-1} \rho
\]  \hspace{1cm} (3.2.23)

\[
\ddot{\phi} + \ddot{\phi} \left( 2 \frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right) = 8\pi (3 + 2\omega)^{-1} (\rho + \rho_s)
\]  \hspace{1cm} (3.2.24)

\[
\dot{\rho} + (\rho + \rho_s) \left( 2 \frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right) = 0
\]  \hspace{1cm} (3.2.25)

among the above five field equations (3.2.21) to (3.2.25), the first four equations are independent involving five unknowns \(A, B, \rho, \rho_s\) and \(\phi\). Hence, in order to get a deterministic solution we take the following linear relationship between the metric potentials \(A\) and \(B\), i.e., \(A = B^n\), where \(n\) is an arbitrary constant.
Using this relation, the field equations (3.2.21) to (3.2.24),
can be written as

\begin{align*}
(n+1) \frac{\ddot{B}}{B} + n^2 \frac{\dot{B}^2}{B^2} + \frac{\omega \ddot{\phi}^2}{2 \phi^2} + \frac{\ddot{\phi}}{\phi} + (n+1) \frac{\dot{B} \ddot{\phi}}{B \phi} &= 0 \\
2n \frac{\ddot{B}}{B} + (3n^2 - 2n) \frac{\dot{B}^2}{B^2} + \frac{\omega \ddot{\phi}^2}{2 \phi^2} + \frac{\ddot{\phi}}{\phi} + 2n \frac{\dot{B} \ddot{\phi}}{B \phi} &= 8\pi \phi^{-1} \rho_s \\
(n^2 + 2n) \frac{\ddot{B}^2}{B^2} - \frac{\omega \ddot{\phi}^2}{2 \phi^2} + (n+1) \frac{\dot{B} \ddot{\phi}}{B \phi} &= 8\pi \phi^{-1} \rho \\
\ddot{\phi} + (2n+1) \phi \frac{\dot{B}}{B} &= 8\pi (3 + 2\omega)^{-1} (\rho + \rho_s) 
\end{align*} 

(3.2.26)-(3.2.29)

From (3.2.26) with \( A = B^n \), we get

\begin{align*}
B &= [k_3(k_1t + k_2)]^{1/k_3} \\
A &= [k_3(k_1t + k_2)]^{n/k_3}, \text{ where } k_3 = \frac{n^2 + n + 1}{n + 1}, n \neq -1
\end{align*} 

(3.2.30)-(3.2.31)

and

\( \phi = (k_1t + k_2)^r \) 

(3.2.32)

and \( k_1 \neq 0 \) & \( k_2 \) are arbitrary constants.
The metric (3.2.1) can now be written as

\[
ds^2 = dt^2 - [k_t(k_t + k_2)]^{2n/k_3} [d\chi^2 + \chi^2 d\phi^2] - [k_t(k_t + k_2)]^{2/k_3} dz^2
\]  

(3.2.33)

From (3.2.28), we get the string energy density

\[
\rho = \frac{k_t^2}{8\pi k_3^2} \left[ n(n+2) + (n+1)rk_3 - \frac{\omega}{2} r^2k_3^2 \right] (k_t + k_2)^{r-2}
\]  

(3.2.34)

From (3.2.27), we get the string tension density

\[
\rho_s = \frac{k_t^2}{8\pi k_3^2} \left[ 3n^2 + (r-1)k_3(2n+rk_3) + \frac{\omega}{2} r^2k_3^2 \right] (k_t + k_2)^{r-2}
\]  

(3.2.35)

The string particle density is given by

\[
\rho_p = \rho - \rho_s = \frac{k_t^2}{8\pi k_3^2} \left[ (rk_3[(n+1) - (r-1)k_3]) - 2n[(n-1) + (r-1)k_3] \right] (k_t + k_2)^{r-2}
\]  

(3.2.36)

The quark energy density is given by

\[
\rho_q = \rho - B_C = \frac{k_t^2}{8\pi k_3^2} \left[ n(n+2) + (n+1)rk_3 - \frac{\omega}{2} r^2k_3^2 \right] \times (k_t + k_2)^{r-2} - B_C
\]  

(3.2.37)
The quark pressure is given by

\[
p_q = \frac{\rho q}{3} = \frac{k_1^2}{24\pi k_3} \left[ n(n+2) + (n+1)rk_3 - \frac{\omega}{2} r^2 k_3^2 \right]
\times (k_1 t + k_2) r^{-2} - \frac{B_c}{3}
\]

(3.2.38)

This solution satisfies the field equations (3.2.26) to (3.2.29) provided the arbitrary constants are related by

\[
(r - 2)k_3[n(n+2) + (n+1)rk_3 - \frac{\omega}{2} r^2 k_3^2]
+ (2n+1)(3 + 2\omega)k_3[r(r-1)k_3 + (2n+1)r] = 0
\]

(3.2.39)

**Axially symmetric cosmological model with strange quark matter attached to string cloud in Brans-Dicke theory of gravitation:**

If \( r = \frac{-2n}{(\omega + 2)(n^2 + n + 1)} \), the metric (3.2.33) together with (3.2.32) and (3.2.34) to (3.2.38) represents axially symmetric cosmological model with strange quark matter attached to string cloud in Brans-Dicke theory of gravitation.
where \( \omega \) and \( n \) are related by

\[
\omega^2(n^5 + 9n^4 + 24n^3 + 25n^2 + 11n + 2) + \omega(4n^5 + 35n^4 + 84n^3 + 82n^2 + 35n + 6) + (4n^5 + 34n^4 + 76n^3 + 70n^2 + 28n + 4) = 0 \quad (3.2.40)
\]

Axially symmetric cosmological model with strange quark matter attached to string cloud in General Relativity:

If \( r = 0 \), from (3.2.32), we get

\( \phi = \text{constant} \)

From (3.2.34), we get the string energy density

\[
\rho = n(n + 2)(n + 1)^2 \frac{k_1^2}{8\pi(n^2 + n + 1)^2} (k_1 t + k_2)^{-2} \quad (3.2.41)
\]

From (3.2.35), we get the string tension density

\[
\rho_s = n(n^2 - 1)(n + 2) \frac{k_1^2}{8\pi(n^2 + n + 1)^2} (k_1 t + k_2)^{-2} \quad (3.2.42)
\]

The string particle density is given by

\[
\rho_p = \rho - \rho_s = \frac{n(n + 2)(n + 1)k_1^2}{4\pi(n^2 + n + 1)^2} (k_1 t + k_2)^{-2} \quad (3.2.43)
\]
The quark energy density is given by

\[ \rho_q = \rho - B_C = \frac{n(n + 2)(n + 1)^2 k_1^2}{8 \pi (n^2 + n + 1)^2} (k_1 t + k_2)^2 - B_C \]  \hspace{1cm} (3.2.44)

The quark pressure is given by

\[ p_q = \frac{\rho q}{3} = \frac{n(n + 2)(n + 1)^2 k_1^2}{24 \pi (n^2 + n + 1)^2} (k_1 t + k_2)^2 - \frac{B_C}{3} \]  \hspace{1cm} (3.2.45)

Thus (3.2.33) together with (3.2.41) to (3.2.45) represents axially symmetric cosmological model with strange quark matter attached to string cloud in general relativity, which is exactly similar to the model obtained by Rao and Neelima (2012).

Also it is interesting to observe that even when \( \omega \) tends to infinity, \( r \) tends to zero and the axially symmetric cosmological model with strange quark matter attached to string cloud in Brans-Dicke theory will reduce to cosmological model in general relativity as obtained and presented above.
3.3 AXIALLY SYMMETRIC PERFECT FLUID

COSMOLOGICAL MODELS IN BRANS-DICKE

THEORY OF GRAVITATION:

The energy momentum tensor for the perfect fluid is given by

\[ T_{ij} = (\rho + p) u_i u_j - p g_{ij} \]  
(3.3.1)

where \( \rho \) is the energy density, \( p \) is the pressure,

\[ u^i \] is the four-velocity of the fluid and \( g_{ij} u^i u^j = 1 \)  
(3.3.2)

In a comoving coordinate system, we get

\[ T_1 = T_2 = T_3 = -p, \quad T_4 = \rho \quad \text{and} \quad T^i_j = 0 \ for \ i \neq j \]  
(3.3.3)

where \( \rho \) and \( p \) are functions of ‘t’ only.
The field equations (3.2.6) & (3.2.7) for the metric (3.2.1) with the help of equations (3.3.1)-(3.3.3) can be written as

\[
\frac{\ddot{A}}{A} + \frac{\dot{A}B}{AB} + \frac{\ddot{B}}{B} - \frac{1}{2} \frac{\omega \dot{\phi}^2}{\phi} + \frac{\dot{\phi}}{\phi} \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right) = -8\pi \phi^{-1} p
\] (3.3.4)

\[
\frac{2\ddot{A}}{A} + \left( \frac{\dot{A}}{A} \right)^2 - \frac{1}{A^2} \left( f'' \right) + \frac{1}{2} \frac{\omega \dot{\phi}^2}{\phi} + \frac{\dot{\phi}}{\phi} \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right) = -8\pi \phi^{-1} p
\] (3.3.5)

\[
\left( \frac{\dot{A}}{A} \right)^2 + \frac{2\dot{A}B}{AB} - \frac{1}{A^2} \left( f'' \right) - \frac{1}{2} \frac{\omega \dot{\phi}^2}{\phi} + \frac{\dot{\phi}}{\phi} \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right) = 8\pi \phi^{-1} \rho
\] (3.3.6)

\[
\phi^{k:;k} = \ddot{\phi} + \dot{\phi} \left( 2\frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right) = 8\pi (3 + 2\omega)^{-1} (\rho - 3p)
\] (3.3.7)

\[
\dot{\rho} + (\rho + p) \left( 2\frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right) = 0
\] (3.3.8)

Here the overhead dot denotes differentiation with respect to 't' and the over head dash denotes differentiation with respect to 'χ'.

**SOLUTIONS OF THE FIELD EQUATIONS:**

From (3.3.5) & (3.3.6) we can observe that it is possible to separate the terms of \( f(\chi) \) to one side and the terms of
\( A(t), B(t), \rho(t) \text{ & } p(t) \) to another side. Hence we can take each part is equal to a constant. So,

\[
\frac{f''}{f} = k^2, \quad k^2 \text{ is a constant.} \quad (3.3.9)
\]

If \( k=0 \), then \( f(x) = c_1 \chi + c_2, \quad \chi > 0 \).

where \( c_1 \) and \( c_2 \) are integrating constants.

Without loss of generality, by taking \( c_1 = 1 \) and \( c_2 = 0 \), we get \( f(\chi) = \chi \).

Now the field equations (3.3.4) to (3.3.8) will reduce to

\[
\frac{\ddot{A}}{A} + \frac{\ddot{A}B}{AB} + \frac{\ddot{B}}{B} + \frac{1}{2} \frac{\omega \dot{\phi}^2}{\phi} + \frac{\ddot{\phi}}{\phi} + \phi \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right) = -8\pi \phi^{-1} p \quad (3.3.10)
\]

\[
2\frac{\ddot{A}}{A} + \left( \frac{\dot{A}}{A} \right)^2 + \frac{1}{2} \frac{\omega \dot{\phi}^2}{\phi} + \frac{\ddot{\phi}}{\phi} + \phi \left( 2\frac{\dot{A}}{A} \right) = -8\pi \phi^{-1} p \quad (3.3.11)
\]

\[
\left( \frac{\dot{A}}{A} \right)^2 + 2\frac{\ddot{A}B}{AB} - \frac{1}{2} \frac{\omega \dot{\phi}^2}{\phi} + \phi \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right) = 8\pi \phi^{-1} \rho \quad (3.3.12)
\]

\[
\phi^k_{,k} = \ddot{\phi} + \phi \left( 2\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} \right) = 8\pi (3 + 2\omega)^{-1} (\rho - 3p) \quad (3.3.13)
\]
\[
\dot{\rho} + (\rho + p) \left( 2 \frac{A'}{A} + \frac{B'}{B} \right) = 0 \quad (3.3.14)
\]

By taking the transformation \( dt = A^2 B dT \), the above field equations can be written as

\[
\frac{A''}{A} - 2 \frac{A'^2}{A^2} + \frac{B''}{B} - \frac{B'^2}{B^2} - 2 \frac{A'B'}{AB} + 2 \frac{\omega \phi'^2}{\phi^2} + \frac{\phi''}{\phi} - \frac{A' \phi'}{A \phi} = -8 \pi \phi^{-1} p(A^4 B^2) \quad (3.3.15)
\]

\[
2 \frac{A''}{A} - 3 \frac{A'^2}{A^2} - 2 \frac{A'B'}{AB} + \frac{1}{2} \frac{\omega \phi'^2}{\phi^2} + \frac{\phi''}{\phi} - \frac{B' \phi'}{B \phi} = -8 \pi \phi^{-1} p(A^4 B^2) \quad (3.3.16)
\]

\[
\frac{A'^2}{A^2} + 2 \frac{A'B'}{AB} - \frac{1}{2} \frac{\omega \phi'^2}{\phi^2} + 2 \frac{A' \phi'}{A \phi} + \frac{B' \phi'}{B \phi} = 8 \pi \phi^{-1} \rho(A^4 B^2) \quad (3.3.17)
\]

\[
\phi'' = 8 \pi (3 + 2 \omega)^{-1} A^4 B^2 (\rho - 3p) \quad (3.3.18)
\]

\[
\rho' + (\rho + p) \left( 2 \frac{A'}{A} + \frac{B'}{B} \right) = 0 \quad (3.3.19)
\]

From here after the over head dash denotes differentiation with respect to 'T'.
The field equations (3.3.15)-(3.3.18) are only four independent equations with five unknowns $A, B, \rho, p & \phi$. Since these equations are highly non-linear in nature, in order to get a deterministic solution we take the following plausible physical condition, the shear scalar $\sigma$ is proportional to scalar expansion $\theta$, which leads to the linear relationship between the metric potentials $A$ and $B$, i.e., $A = B^n$, where $n$ is an arbitrary constant.

Using this relation, the equations (3.3.15) to (3.3.17) can be written as

\begin{align}
(n + 1) \frac{B''}{B} - (n^2 + 3n + 1) \frac{B'^2}{B^2} + \frac{1}{2} \omega \frac{\phi'^2}{\phi^2} + \frac{\phi''}{\phi} - n \frac{B' \phi'}{B \phi} &= -8\pi \phi^{-1} p B^{4n + 2} \\
(3.3.20)
\end{align}

\begin{align}
2n \frac{B''}{B} - (n^2 + 4n) \frac{B'^2}{B^2} + \frac{1}{2} \omega \frac{\phi'^2}{\phi^2} + \frac{\phi''}{\phi} - B' \frac{\phi'}{B \phi} &= -8\pi \phi^{-1} p B^{4n + 2} \\
(3.3.21)
\end{align}

\begin{align}
(n^2 + 2n) \frac{B'^2}{B^2} - \frac{1}{2} \omega \frac{\phi'^2}{\phi^2} + (2n + 1) \frac{B' \phi'}{B \phi} &= 8\pi \phi^{-1} \rho B^{4n + 2} \\
(3.3.22)
\end{align}
From (3.3.20) - (3.3.22), we get

\[
\frac{B''}{B} - \left( \frac{n^2 + 4n + 1}{2n + 1} \right) \frac{B'^2}{B^2} + \frac{\omega}{2(2n+1)} \left( \frac{\phi'^2}{\phi} - 2 \frac{\phi''}{\phi} \right) = 0
\]  
(3.3.23)

From (3.3.23) with \( A = B^n \), we get

\[
B = (aT + b)^{-k_1}
\]

\[
A = (aT + b)^{-nk_1}
\]

\[
\phi = (aT + b)^m
\]  
(3.3.24)

where \( a \neq 0, b \), m are arbitrary constants and

\[
k_1 = \frac{(1 - 4c_1c_2)^{\frac{1}{2}} + 1}{2c_1} \text{ with } c_1 = \frac{n^2 + 2n}{2n + 1}, c_2 = \frac{\omega m(m-2)a^2}{2(2n+1)},
\]

\( n \neq 0, \frac{1}{2} \& -2, \text{ and } (1 - 4c_1c_2)^{\frac{1}{2}} > 0. \)

From (3.3.22), we get the proper density

\[
8\pi\rho = a^2 \left[ k_1^2 (n^2 + 2n) - k_1 (2nm + m) - \frac{1}{2} \omega m^2 \right] (aT + b)^k
\]  
(3.3.25)
From (3.3.20) & (3.3.21), we get the proper pressure

\[ 8\pi p = a^2 \left( -nk_1^2(n+2) + \frac{k_1}{2}(3n+1+m(n+1)) \right) + \frac{1}{2}((\omega + 2)m^2 - 2m) (aT + b)^{k_2} \]  

(3.3.26)

where \( k_2 = (4n+2)k_1 + (m-2) \).

The metric (3.2.1) can now be written as

\[ ds^2 = (aT + b)^{-(2n+1)k_1} dT^2 - (aT + b)^{-2nk_1} (d\chi^2 + \chi^2 d\phi^2) - (aT + b)^{-2k_1} dz^2 \]  

(3.3.27)

Since \( m \& n \) are arbitrary constants, for different values of \( m \& n \), we will get different cosmological models. But in this chapter we will present some important cosmological models, as special cases.

**Axially symmetric perfect fluid cosmological model, in isotropic form, in Brans-Dicke theory of gravitation:**

If \( m = -1 \) and \( n = 1 \), the model (3.3.27) will reduce to an axially symmetric perfect fluid isotropic cosmological model in Brans-Dicke theory of gravitation given by
\[ ds^2 = (aT + b)^{-3k_1} dT^2 - (aT + b)^{-2k_1} (d\chi^2 + \chi^2 d\phi^2 + dz^2) \]

(3.3.28)

where \( A = (aT + b)^{-k_1} \),

\[ B = (aT + b)^{-k_1} , \]

\[ \phi = (aT + b)^{-1} , \]

\[ 8\pi \rho = a^2 \left( 3k_1^2 + 3k_1 - \frac{\omega}{2} \right) (aT + b)^{6k_1-3} \]

and

\[ 8\pi p = a^2 \left( -3k_1^2 + k_1 + 3 + \frac{\omega}{2} \right) (aT + b)^{6k_1-3} \]

with \( k_1 = \frac{((1-2\omega a^2)^{\frac{1}{2}} + 1)}{2} \).

and the constants \( k_1 \) and \( \omega \) are related by

\[ 12k_1^3 - 2k_1^2 - 2(\omega + 6)k_1 + \omega = 0. \]
Axially symmetric vacuum cosmological model in Brans-Dicke theory of gravitation:

If \( m = 1 \) and \( n = 1 \), the model (3.3.27) will reduce to an axially symmetric vacuum cosmological model in Brans-Dicke theory of gravitation given by

\[
ds^2 = (aT + b)^{-3k_1} dT^2 - (aT + b)^{-2k_1} (d\chi^2 + \chi^2 d\phi^2 + dz^2)
\]

(3.3.29)

where \( A = (aT + b)^{-k_1} \),

\[ B = (aT + b)^{-k_1}, \]

\[ \phi = (aT + b) \]

with \( k_1 = \frac{\left( \frac{3+2\omega a^2}{3} \right)^{\frac{1}{2}} + 1}{2} \).

In this case, the density \( \rho \) and the pressure \( p \) will become zero.
Also we can observe that if either \( f(\chi) \propto \sin k\chi \) or \( f(\chi) \propto \sinh k\chi \), we will get only vacuum cosmological models in Brans-Dicke theory of gravitation.

**Axially symmetric perfect fluid cosmological model in general relativity:**

If \( m = 0 \), the model (3.3.27) will reduce to an axially symmetric perfect fluid cosmological model in general relativity given by

\[
ds^2 = (aT + b)^{- (2n + 1)k_1} dT^2 - (aT + b)^{- 2nk_1} (d\chi^2 + \chi^2 d\phi^2) - (aT + b)^{- 2k_1} dz^2
\]

where \( A = (aT + b)^{- nk_1} \),

\( B = (aT + b)^{- k_1} \),

\( \phi = \text{const} \tan t \),

\[
8\pi\rho = \frac{2(2n - 1)a^2}{(3n - 1)} (aT + b)^{k_2}, \quad n \neq \frac{1}{3}
\]

\[
8\pi\rho = \frac{2na^2}{(1 - 3n)} (aT + b)^{k_2}
\]
where \( k_2 = \frac{2(1-n)}{(1-3n)} \) and

\[ n \text{ satisfies the relation } 2n^2 + n + 1 = 0. \]

### 3.4 AXIALLY SYMMETRIC STRING

#### COSMOLOGICAL WITH BULK VISCOSITY IN SELF

#### CREATION THEORY OF GRAVITATION:

Barber’s second self creation theory field equations are

\[
G_{ij} = -8\pi\phi^{-1}T_{ij}\]

(3.4.1)

and the scalar field \( \phi \) satisfies the equation

\[
\square \phi = \frac{8}{3}\pi\eta T
\]

(3.4.2)

where \( G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij} \) is an Einstein tensor, \( T_{ij} \) is the stress energy tensor of the matter, \( \phi \) is the Barber’s scalar and \( \eta \) is a coupling constant to be determined from experiments. The measurement of the deflection of light restricts the value of the coupling to \( |\eta| < 10^{-1} \). In the limit \( \eta \to 0 \), the theory approaches Einstein’s theory in every respect.
In the investigation of cosmological models we, usually,
choose the energy momentum tensor for a bulk viscous fluid
containing one dimensional string as

\[ T_{ij} = (\rho + p)u_i u_j + \overline{p}g_{ij} - \lambda x_i x_j \]  \hspace{1cm} (3.4.3)

and \( \overline{p} = p - 3\xi H \), \hspace{1cm} (3.4.4)

is the total pressure which includes the proper pressure, \( \rho \)
is the rest energy density of the system, \( \lambda \) is tension in the
string, \( \xi(t) \) is the coefficient of bulk viscosity, \( 3\xi H \) is usually
known as bulk viscous pressure, \( H \) is the Hubble parameter,
\( u^i \) is the four velocity vector and \( x^i \) is a space-like vector
which represents the anisotropic directions of the string.

Here \( u^i \) and \( x^i \) satisfy the equations

\[ g_{ij}u^i u^j = -1, \]

\[ g_{ij}x^i x^j = 1, \]

and \( u^i x_i = 0. \) \hspace{1cm} (3.4.5)
We assume that the string to be lying along the $z$-axis. The one dimensional strings are assumed to be loaded with particles and the particle energy density is $\rho_p = \rho - \lambda$.

In a comoving coordinate system, we get

$$T^1_1 = T^2_2 = \bar{\rho}, \; T^3_3 = \bar{\rho} - \lambda, \; T^4_4 = -\rho$$

(3.4.6)

where $\rho, \lambda, \bar{\rho}$ and $\phi$ are functions of time ‘t’ only.

The field equations (3.4.1) & (3.4.2) for the metric (3.2.1) with the help of equations (3.4.3) - (3.4.6) can be written as

$$\frac{\ddot{A}}{A} + \frac{\dot{A} \dot{B}}{AB} + \frac{\ddot{B}}{B} = -8\pi \phi^{-1} \bar{\rho}$$

(3.4.7)

$$\frac{2\ddot{A}}{A} + \left(\frac{\dot{A}}{A}\right)^2 - \frac{1}{A^2} \left(\frac{f''}{f}\right) = -8\pi \phi^{-1} (\bar{\rho} - \lambda)$$

(3.4.8)

$$\left(\frac{\dot{A}}{A}\right)^2 + \frac{2\dot{A} \dot{B}}{AB} - \frac{1}{A^2} \left(\frac{f''}{f}\right) = 8\pi \phi^{-1} \rho$$

(3.4.9)

$$\dot{\phi} + \dot{\phi} \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) = \frac{8\pi}{\eta} (3\bar{\rho} - \rho - \lambda)$$

(3.4.10)
Here the over head dot denotes differentiation with respect to \( t \) and over head dash denotes differentiation with respect to \( \chi \).

**SOLUTIONS OF THE FIELD EQUATIONS:**

From (3.4.8) & (3.4.9), we can observe that it is possible to separate the terms of \( f(\chi) \) to one side and the terms of \( A(t), B(t), \rho(t), \bar{p}(t) \) & \( \lambda(t) \) to another side. Hence we can take each part is equal to a constant.

So, \( \frac{f''}{f} = k^2 \), \( k^2 \) is a constant. \( (3.4.11) \)

If \( k=0 \), then \( f(x) = c_1 \chi + c_2, \chi > 0 \).

where \( c_1 \) and \( c_2 \) are integrating constants.

Without loss of generality, by taking \( c_1 = 1 \) and \( c_2 = 0 \), we get \( f(\chi) = \chi \).

Now the field equations (3.4.7) to (3.4.10) will reduce to

\[
\frac{\ddot{A}}{A} + \frac{\dot{A}\dot{B}}{AB} + \frac{\ddot{B}}{B} = -8\pi\phi^{-1} \bar{P}
\]

\( (3.4.12) \)
\[
\frac{2\ddot{A}}{A} + \left(\frac{\dot{A}}{A}\right)^2 = -8\pi\phi^{-1}(\bar{p} - \lambda) \quad (3.4.13)
\]

\[
\left(\frac{\dot{A}}{A}\right)^2 + \frac{2\ddot{A}}{A} = 8\pi\phi^{-1}\rho \quad (3.4.14)
\]

\[
\ddot{\phi} + \dot{\phi}\left(2\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) = \frac{8\pi}{3}\eta(3\bar{p} - \rho - \lambda) \quad (3.4.15)
\]

Here the over head dot denotes differentiation with respect to \(t\).

Using the transformation \(dt = A^2BdT\), the field equations (3.4.12) to (3.4.15) reduce to

\[
\frac{A''}{A} - 2\frac{A'^2}{A^2} + \frac{B''}{B} - \frac{B'^2}{B^2} - 2\frac{A'B'}{AB} = -8\pi\phi^{-1}\bar{p}(A^4B^2) \quad (3.4.16)
\]

\[
2\frac{A''}{A} - 3\frac{A'^2}{A^2} - 2\frac{A'B'}{AB} = -8\pi\phi^{-1}(\bar{p} - \lambda)(A^4B^2) \quad (3.4.17)
\]

\[
\frac{A'^2}{A^2} + \frac{2A'B'}{AB} = 8\pi\phi^{-1}\rho(A^4B^2) \quad (3.4.18)
\]

\[
\phi'' = \frac{8\pi}{3}\eta(A^4B^2)(3\bar{p} - \rho - \lambda) \quad (3.4.19)
\]
The field equations (3.4.16) to (3.4.19) are only four independent equations with seven unknowns $A, B, \rho, \lambda, \bar{p}, \phi & \xi$, which are functions of $T$. Here after the over head dash denotes differentiation with respect to $T$. Since these equations are highly non-linear in nature, in order to get a deterministic solution we take the following plausible physical conditions:

(1) The shear scalar $\sigma$ is proportional to scalar expansion $\theta$, so that we can take a linear relationship between the metric potentials $A$ and $B$, i.e.

$$A = B^n$$

where $n$ is an arbitrary constant.

(2) A more general relationship between the proper rest energy density $\rho$ and string tension density $\lambda$ is taken to be

$$\rho = r\lambda$$

where $r$ is an arbitrary constant which can take both positive and negative values. The negative value of $r$ leads to the absence of strings in the universe and the positive
value show the presence of one dimensional strings in the cosmic fluid.

The energy density of the particles attached to the strings is

$$\rho_p = \rho - \dot{\lambda} = (r - 1)\dot{\lambda} \quad (3.4.22)$$

(3) For a barotropic fluid, the combined effect of the proper pressure and the barotropic bulk viscous pressure can be expressed as

$$\bar{p} = p - 3\zeta H = (\varepsilon \rho) \quad (3.4.23)$$

where $\varepsilon = \varepsilon_0 - \zeta$ and $p = \varepsilon_0 \rho \quad (0 \leq \varepsilon_0 \leq 1), \quad (3.4.24)$

$\varepsilon$ and $\zeta$ are arbitrary constants.

Using (3.4.20), the field equations (3.4.16) to (3.4.19) can be written as

$$(n + 1) \left( \frac{B'}{B} \right)' - (n^2 + 2n) \frac{B'^2}{B^2} = -8\pi\phi^{-1}\bar{p}B^{4n+2} \quad (3.4.25)$$

$$2n \left( \frac{B'}{B} \right)' - (n^2 + 2n) \frac{B'^2}{B^2} = -8\pi\phi^{-1}(\bar{p} - \lambda)B^{4n+2} \quad (3.4.26)$$

$$(n^2 + 2n) \frac{B'^2}{B^2} = 8\pi\phi^{-1}\rho B^{4n+2} \quad (3.4.27)$$
\begin{align*}
\phi'' &= \frac{8\pi}{3} \eta B^4 n^2 (3\bar{p} - \rho - \lambda) \\
\text{From equations (3.4.25) \& (3.4.26), we get} \\
(1-n) \left( \frac{B'}{B} \right)' &= -8\pi \phi^{-1} \lambda B^4 n + 2
\end{align*} 

(3.4.29)

From equations (3.4.21), (3.4.27) \& (3.4.29), we get

\begin{align*}
C_1 \left( \frac{B''}{B} \right) - C_2 \frac{B'^2}{B^2} &= 0
\end{align*} 

(3.4.30)

where \( C_1 = r(n-1) \) and \( C_2 = r(n-1) + (n^2 + 2n) \).

From equation (3.4.30), we get

\begin{align*}
B &= \left[ C_3 (aT + b) \right]^{1/C_3}, \quad C_3 \neq 0
\end{align*} 

(3.4.31)

From equations (3.4.20) \& (3.4.31), we get

\begin{align*}
A &= \left[ C_3 (aT + b) \right]^{n/C_3}
\end{align*} 

(3.4.32)

where \( C_3 = 1 - \frac{C_2}{C_1} = \frac{n(n+2)}{r(1-n)}, n \neq 0 \& 1, -2, r \neq 0 \).

From equations (3.4.25)-(3.4.28) \& (3.4.31), we get

\begin{align*}
(aT + b)^2 \phi'' - \kappa a^2 \phi &= 0
\end{align*} 

(3.4.33)
where \( K = \frac{2\eta r(1-n)}{3n(n+2)}[(2n+1) + r(1-n)]. \)

From equation (3.4.33), we get

\[
\phi = (aT + b)^m
\]  

(3.4.34)

where \( m = \frac{1 \pm \sqrt{1+4K}}{2} \).

Since Barber’s theory has to go to Einstein theory in all respects when \( \eta \to 0 \), \( \phi = \) constant. \( m \) should be of the form

\[
m = \frac{1 - \sqrt{1+4K}}{2}.
\]

From equations (3.4.27), (3.4.31) & (3.4.34) we get the energy density

\[
8\pi\rho = n(n+2)a^2 K_1 (aT + b)^K_2
\]  

(3.4.35)

From equations (3.4.23) & (3.4.35) we get the total pressure

\[
8\pi\overline{p} = \varepsilon n(n+2)a^2 K_1 (aT + b)^K_2
\]  

(3.4.36)

From equations (3.4.24) & (3.4.35) we get the proper pressure

\[
8\pi\rho = \varepsilon_0 n(n+2)a^2 K_1 (aT + b)^K_2
\]  

(3.4.37)
From equations (3.4.22), (3.4.31), (3.4.34) & (3.4.36) 

we get the string tension density

\[ 8\pi \lambda = \left[ \frac{r(\varepsilon - 1)(1 - n) - 2n}{r(1 - n)} \right] n(n + 2)a^2 K_1(aT + b) K_2 \]  \hspace{1cm} (3.4.38)

The Coefficient of bulk viscosity is given by

\[ \xi = \frac{\zeta an(n + 2)}{8\pi(2n + 1)} K_1(aT + b) K_2^n \hspace{0.5cm} n \neq \frac{-1}{2} \]  \hspace{1cm} (3.4.39)

where

\[ K_1 = \left[ \frac{n(n + 2)}{r(1 - n)} \right]^{-\frac{2r(1-n)(2n+1)+n(n+2)}{n(n+2)}} \hspace{0.5cm} \text{and} \hspace{0.5cm} K_2 = \frac{n(m - 2)(n + 2) - 2r(1-n)(2n+1)}{n(n+2)} \]

The components of Hubble Parameter $H_1, H_2$ are given by

\[ H_1 = \frac{2A'}{A} = \frac{2r(1-n)a}{(n+2)(aT + b)} \hspace{0.5cm} H_2 = \frac{B'}{B} = \frac{ar(1-n)}{n(n + 2)(aT + b)}. \]

Therefore the generalized mean Hubble parameter $(H)$ is

\[ H = \frac{1}{3}(H_1 + H_2) = \frac{r(1-n)(2n+1)a}{3n(n+2)(aT + b)} \] \hspace{1cm} (3.4.40)
The metric (3.2.1) can now be written as

\[
ds^2 = -\left[\frac{n(n+2)}{r(1-n)}(aT + b)\right]^{\frac{r(1-n)(4n+2)}{n(n+2)}} dT^2 - \left[\frac{n(n+2)}{r(1-n)}(aT + b)\right]^{\frac{2r(1-n)}{(n + 2)}}
\]

\[
\times (dx^2 + \chi^2 d\phi^2) - \left[\frac{n(n+2)}{r(1-n)}(aT + b)\right]^{\frac{2r(1-n)}{(n + 2)}} dz^2
\]

(3.4.41)

Thus (3.4.41) together with (3.4.35), (3.4.36), (3.4.38) & (3.4.39) constitutes an axially symmetric string cosmological model with bulk viscosity in Barber’s (1982) second self-creation theory of gravitation.

**THE COSMOLOGICAL MODEL IN THE ABSENCE OF BULK VISCOSITY:**

It is interesting to note that in the absence of bulk viscosity, by taking \( \zeta = 0 \) in equation (3.4.39), we get the axially symmetric perfect fluid string cosmological model and if we assign the value zero to \( \epsilon \) & \( \epsilon_0 \), the present model reduces to string cosmological model in Barber’s (1982) second self-creation theory of gravitation.
3.5. SOME IMPORTANT FEATURES OF THE MODELS WITH CONCLUSIONS:

Axially symmetric cosmological model with strange quark matter attached to string cloud in Brans - Dicke theory of gravitation:

The spatial volume of the model (3.2.33) is given by

\[ V = \sqrt{-g} = \chi \left[ k_3 \left( k_1 t + k_2 \right)^{\frac{2n+1}{k_3}} \right], \quad k_3 = \frac{n^2 + n + 1}{n + 1} \]  

(3.5.1)

The expansion scalar \( \theta \), calculated for the flow vector \( u^i \) is given by

\[ \theta = \frac{(2n + 1)(n + 1)k_1}{(n^2 + n + 1)(k_1 t + k_2)} \]  

(3.5.2)

The shear scalar \( \sigma \) is given by

\[ \sigma^2 = \frac{1}{2} \sigma_{ij} \sigma^{ij} = \frac{7}{18} \frac{(2n + 1)^2(n + 1)^2 k_1^2}{(n^2 + n + 1)^2 (k_1 t + k_2)^2} \]  

(3.5.3)

The mean Hubble parameter \( H \) is given by

\[ H = \frac{(2n + 1)(n + 1)k_1}{3(n^2 + n + 1)(k_1 t + k_2)} \]  

(3.5.4)
The mean anisotropy parameter $A_m$ is given by

$$A_m = \frac{1}{3} \sum_{i=1}^{3} \left( \frac{H_i - H}{H} \right)^2 = \frac{2(n-1)^2}{(2n+1)^2}$$

(3.5.5)

The density parameter $\Omega$ is given by

$$\Omega = \frac{\rho}{3H^2} = \left[ \frac{3[n(n+2) + (n+1)rk_3 - \frac{\varphi}{2} r^2 k_3^2]}{16\pi(n-1)^2} \right] (k_1 t + k_2)^r A_m$$

(3.5.6)

The deceleration parameter is given by

$$q = -3\theta^{-2}[\theta_{,ij}u^a + \frac{1}{3} \theta^2] = \frac{n^2 + 2}{(2n+1)(n+1)}, n \neq -1, -\frac{1}{2}$$

(3.5.7)

The tensor of rotation

$$W_{ij} = u_{i,j} - u_{j,i}$$

is identically zero and hence this universe is non-rotational.

**Conclusions:**

We observe that the model (3.2.33) has no singularity at

$$t = \frac{-k_2}{k_1} \text{ for } n > 0.$$ 

It is also observed that at $t = \frac{-k_2}{k_1}$ the proper volume will be zero, whereas when $t \to \infty$, the spatial volume
becomes infinitely large. At $t = \frac{-k_z}{k_1}$, the expansion scalar $\theta$, shear scalar $\sigma$ tend to infinity whereas when $t \to \infty$, expansion scalar $\theta$, shear scalar $\sigma$ tend to zero. The Hubble parameter $H$ increases with time for $n > 0$. Equation (3.5.6) shows that the density parameter $\Omega$ depends on the anisotropy parameter $A_m$. Also since the mean anisotropy parameter $A_m \neq 0$ for $n \neq 1$, the model does not approach isotropy for $n \neq 1$. But for $n = 1$, the mean anisotropy parameter is zero. Hence the model (3.2.33) will become isotropic for $n = 1$. It is interesting to note that the model represents the decelerating universe for $n > 0$ and accelerating universe for $-1 < n < \frac{-1}{2}$. For our model, we get $\frac{\sigma}{\theta} \approx 0.6236$ which is greater than the present upper limit (10)$^{-5}$ of $\frac{\sigma}{\theta}$ obtained by Collins et al. (1980) from indirect arguments concerning the isotropy of the primordial black body radiation. This fact implies that our solution also represents the early stages of evolution of the universe. Also, we noticed that the presence of scalar field doesn’t affect the geometry of the space-time but changes the
matter distribution and it is always possible to obtain
axially symmetric cosmological model with strange quark
matter attached to string cloud in general relativity, as a
special case, when either \( r \) equals to zero or \( \omega \) tends to
infinity.

**Axially symmetric perfect fluid cosmological model in Brans-Dicke theory of gravitation:**

The spatial volume of the model (3.3.27) is given by

\[
V = \tilde{\chi}(aT + b)^{-(2n+1)k_1}
\]  

(3.5.8)

The expression for the expansion scalar \( \theta \), calculated for the
flow vector \( u^i \) is given by

\[
\theta = -(2n+1)a(aT + b)^{(2n+1)k_1 - 1}
\]  

(3.5.9)

and the shear \( \sigma \) is given by

\[
\sigma^2 = \frac{7}{18}\left((2n+1)^2 a^2 (aT + b)^{(4n+2)k_1 - 2}\right)
\]  

(3.5.10)
The deceleration parameter is given by

\[ q = \frac{3}{(2n+1)^2} [(2n+1)k_i - 1][aT + b]^{-2} - 1 \]  

(3.5.11)

The tensor of rotation

\[ W_{ij} = u_{i,j} - u_{j,i} \]

is identically zero and hence this universe is non-rotational.

The density parameter \( \Omega \) is given by

\[ \Omega = \frac{\rho}{3H^2} = \left[ \frac{(k_i^2 n(n+2) - k_i m(2n+1) - \frac{1}{3} \omega m^2)}{8\pi k_i^2 (2n-1)^2} (aT + b)^{k_i^2 + 2} \right] A_m \]  

(3.5.12)

The Hubble parameter \( H \) is given by

\[ H = \frac{-(2n+1)k_i a}{3(aT + b)} \]  

(3.5.13)

The average anisotropy parameter is

\[ A_m = \frac{1}{3} \sum_{i=1}^{3} \left( \frac{\Delta H_i}{H} \right)^2 = \frac{2(n-1)^2}{9(2n+1)^2} \]  

(3.5.14)

where \( \Delta H_i = H_i - H \) (\( i = 1,2,3 \)).
Conclusions:

It is well known that cosmological models with in the framework of scalar-tensor theories of gravitation play a vital role for better understanding of the early stages of evolution of the universe. Also we know that the quantitative formulation of the laws of physics, i.e. a mathematical formulation becomes an essential component for the understanding of physics. Also beyond the most elementary components, physics simply cannot be developed into a science without a quantitative treatment, i.e. without mathematics. Hence, in this paper we have presented axially symmetric perfect fluid cosmological models in Brans - Dicke theory of gravitation and also in general theory of relativity without taking any relationship between \( \rho \) and \( p \). The model (3.3.27) has singularity at \( T = -\frac{b}{a} \) when \( n \in (-2, \infty) \) except \( n = 0, \frac{-1}{2} \). We observe that at \( T = -\frac{b}{a} \), the spatial volume vanishes if \( n > 0 \) and \( n < -2 \) and \( n < 0 \) and \( n > -2 \) while all other parameters diverge, if \( n > 0 \) and \( n < -2 \) and \( n < 0 \) and \( n > -2 \). The Hubble parameter \( H \)
always decreases with time. Equation (3.5.12) shows that the density parameter $\Omega$ depends on the anisotropy parameter $A_m$ and the average anisotropy parameter $A_m = 0 \text{ for } n = 1$. Hence all the models will represent an isotropic universe for $n = 1$. Here $q$ is less than zero for either $T$ increases to $+\infty$ or decreases to $-\infty$. The negative value of the deceleration parameter ‘$q$’ shows that the model is accelerating. Finally we can conclude that the axially symmetric cosmological models in Brans-Dicke theory are anisotropic, accelerating and non-rotating for $n \neq 1$.

**Axially symmetric string cosmological model**

**with bulk viscosity in self creation theory of gravitation:**

The spatial volume for the metric (3.4.41) is

$$V = (-g)^{\frac{1}{2}} = x \left[ \left( \frac{n(n+2)}{r(1-n)} \right)(aT + b) \right] \frac{r(2n+1)(1-n)}{n(n+2)} \quad \text{(3.5.15)}$$
The expression for expansion scalar $\theta$ calculated for the flow vector $u^i$ is given by

$$\theta = - (2n + 1) a \left[ \frac{n(n + 2)}{r(1 - n)} (aT + b) \right]^{\frac{r(1 - n)(2n + 1) + n(n + 2)}{n(n + 2)}} \tag{3.5.16}$$

and the shear $\sigma$ is given by

$$\sigma^2 = \frac{3}{18} (2n + 1)^2 a^2 \left[ \frac{n(n + 2)}{r(1 - n)} (aT + b) \right]^{\frac{-2r(1 - n)(2n + 1) + n(n + 2)}{n(n + 2)}} \tag{3.5.17}$$

The tensor of rotation

$$W_{ij} = u_{i,j} - u_{j,i}$$

is identically zero and hence this universe is non-rotational.

The deceleration parameter

$$q = \left( \frac{-3r(1 - n)(2n + 1) + n(n + 2)}{n(n + 2)(2n + 1)} \right) \left[ \frac{n(n + 2)}{r(1 - n)} (aT + b) \right]^{\frac{r(2n + 1)(1 - n)}{n(n + 2)}} - 1 \tag{3.5.18}$$
The density parameter $\Omega$ is given by

$$\Omega = \frac{\rho}{3H^2} = \left[ \frac{3n^3(n+2)^3 K_1}{16\pi r^2 (n-1)^4} (aT + b)^{K_2+2} \right] A_m$$

(3.5.19)

The average anisotropy parameter is defined by

$$A_m = \frac{1}{3} \sum_{i=1}^{3} \left( \frac{\Delta H_i}{H} \right)^2 = \frac{2(n-1)^2}{(2n+1)^2}$$

(3.5.20)

where $\Delta H_i = H_i - H \ (i = 1,2,3)$.

**Conclusions:**

We observe that the model (3.4.41) at $T = \frac{-b}{a}$, the spatial volume vanishes. The model has singularity at $T = \frac{-b}{a}$, for $r > 0 \& n > 1$. Also the model has no singularity at $T = \frac{-b}{a}$, for $r > 0 \& 0 < n < 1$. The energy density, the total pressure and the coefficient of bulk viscosity increases as time $T$ increases if $K_2 > 0$. Also the energy density, the total pressure and the coefficient of bulk viscosity decreases as time $T$ increases if $K_2 < 0$. The Hubble parameter $H$ decreases with the increase of time and increases with the decrease of time. The expansion scalar $\theta$ and the shear
scalar $\sigma$ for this model will tend to infinity as $T$ approaches to zero. Also the expansion scalar $\theta$ and the shear scalar $\sigma$ will tend to zero as $T$ approaches to infinity. Equation (3.5.19) shows that the density parameter $\Omega$ depends on the anisotropy parameter $A_m$. After assigning suitable values to the constants $n$ & $r$, we can observe that deceleration parameter $q$ is always negative. Hence the present model always represents an accelerating model of the universe. we can conclude that the present model in self-creation theory is anisotropic, accelerating and also non-rotating for large values of $T$. 