CHAPTER-I

FRACTALS

AND

FRACTAL GEOMETRY
Chapter No - I

Fractals and Fractal Geometry

“Clouds are not spheres, mountains are not cones, and coastlines are not circles and bark is not smooth. Nor does lightning travel in a straight line”

Benoit B. Mandelbrot
1.1 **Introduction to Fractals**

Fractals are rough or fragmented shape having special kind of geometrical complexity and are characterized by non-integral dimensionality, these shapes/objects have self similar structure. A part of object looks like the whole object under appropriate scaling. Fractals are mathematical sets with high degree of geometrical complexity that can model many natural Phenomenons. At the end one can say that all the natural objects can be observed as Fractals.

In nature we observe many irregular and fragmented pattern B. B. Mandelbrot called them as Fractals which can not be modeled by Euclidean geometry such as Branches of plants, weather, coastline, fluid flow, lightning, geologic activity, shapes of clouds, mountains and many more ranging from microscopic aggregates [1,2] to clusters of galaxies. There are many geometrical diagram which are fractal viz. Koch snowflakes, peano curves, Sierpiniski triangles etc. To model such things fractal geometry provides a new perspective to view the world. For centuries we have used the line as a basic building block to understand the objects around us. But a modern science now uses a different tool called fractal geometry. Fractal geometry is a new language used to describe, model and analyze complex [3] forms found in nature.

Benoit Mandelbrot gives a mathematical definition of fractal as a set for which the Hausdroff Dimension strictly exceeds the topological dimension [4] 1. According to Visecek We call a physical object fractal, if measuring its volume, surface or length, it is not possible to obtain a well converging measure for these quantities. For example the coast line of an island can be interpreted as fractal. If one wants to measure the length of a coast, then, one will see that with finer scale the coastal length get on increasing. So the length is not a suitable measure for this
object, because it depends on the scale. The physical phenomena like chaotic attractors, fractal growth patterns and other phenomena in natural sciences, in medical science, Astronomy [5] are also example of fractal.

1.2 Fractal Geometry:

Euclidian geometry is often called as ‘cold’ and ‘dry’ because there are many natural forms in science and mathematics, which are attracting human attention. The basic desire of human beings is to find simplicity and order in nature. We see that many times Euclidean geometry uses approximations or caricatures of natural forms that may be essentially complex and irregular. For example lightning, clouds, mountains etc. These are not simply straight lines, spheres or cones. To study the complexity and irregularity of natural patterns a concept of fractal theory was put forward by Benoît B. Mandelbrot in early 1980’s. For the first time he used the word “Fractal” [6, 7], which he derived from a Latin adjective “Fractus”. It was his desire to give new names to the new concept & found in fractal geometry instead of reusing old terms, which may create confusion. It is also he who declared fractal geometry as the geometry of nature. Many fractal structures go back to classical mathematics of the past, but it is true that mathematicians did not think of their creation as conceptual steps towards new perception or a new geometry. For example George Cantor (1872) constructed a set of points by considering an interval of 1/3 units, Giuseppe Peano (1890) demonstrated a famous curve known as Peano Curve. Similarly David Hilbert (1891), Helge Von Koch (1904), Waclaw Sierpinski (1916) Gaston Julia (1918) or Felix Hausdorff (1919) are a few mathematicians who produced the best creation which played a key role in developing Mandelbrot’s concepts of fractal geometry.

In recent days Fractals have got tremendous response from the scientists all over the world soon after a publication of Mandelbrot’s classic book (1982) “The
Fractal geometry of nature”, the field of fractals has been exploded by many workers. The 1991 science Citation Index listed over 400 papers with the words ‘fractals’ or power laws (the algebraic analogue of fractals) in their titles. They covered fields ranging from physical geometry e.g. the surface texture of seabeds, to the structure of continental faults. In 1992 a lively debate was organized about the distribution of intervals between earthquakes [8] and ecology. In May 1992, questions about the big bang and origin of the universe, structure of stars clusters and galaxies etc. were posed forwarded to seek the answers from these studies.

1.3 Mathematics in Fractal:

Well-known standard mathematical fractals shapes are expressed by Koch Curve Fig. 1.1 and the Sierpiniski Gasket Fig. 1.2. It is interesting to see how the mathematical sense is inured in the construction of Koch curve and Sierpiniski Gasket.

To put up Koch Curve consider a straight line of unit length as shown in Fig.1.1a. If this line is divided in three parts with the middle third of this line as base, construct an equilateral triangle and remove the base to get stage 1 as shown in Fig. 1.1b. The structure in this stage is known as “David’s star” [9]. This Figure is made up of four segment of length 1/3 each so that total length is 4/3 units. Now take middle third of each segment as base construct equilateral triangles and remove the bases to get stage 2 as shown in Fig.1.1c. This Figure is made up of $16(4^2)$ segments of length $1/ 3^2$ each so that the total length [10] is $16/3^2 [(4/3)^2]$ units. Repeating this process we get further stage as shown in Fig.1.1d. As this process goes on, the length of curve goes on increasing by a factor of $4/3$ at each stage, even though the curve is bounded. The curve becomes more and more winding and approaches limiting curve, called the Koch curve. As the size of the
ruler used to measure the length of the curve goes on decreasing, the computed length does approach a finite limit. In general if a ruler of length \( (1/3)^d = 3^{-d} \) is walked over Koch curve, it will walk over the \( d^{th} \) stage of above construction as the structure introduced after the \( d^{th} \) stage is smaller than the ruler resolution. Thus \( 4^d \) steps will be required and the length will be computed as \( (4/3)^d \). Clearly as \( d \) increases indefinitely the computed length does not approach a finite limit. Thus length of Koch curve tends towards the infinite for an infinite iteration. The end construction of Koch snowflake resembles the coastline of shore.

1.3.1 The Sierpinski Gasket

To make Sierpinski Gasket initiate with an equilateral triangle of unit side length. Join Middle point of three Side lengths. By joining the middle point of three sides, we get 4 triangles. Remove the central triangle. the figure now consist of 3 = \( 3^1 \) equilateral triangles of sides length \( 1/2 = 2^{-1} \). Now in each case of 3 equilateral triangles, the central triangle obtained by joining the middle point of that triangle is removed. At this stage this figure consists of \( 3^2 \) equilateral triangles of side length \( (1/2)/2 = 1/4 = 1/2^{-2} = 2^{-2} \). As the process is continued, at the \( n^{th} \) stage. We have a figure consisting of \( 3^n \) equilateral triangles of side length \( 2^{-n} \). The limiting figure of this process is called Sierpinski Gasket. From the above procedure one can build a table giving relation between number of equilateral triangles and side length to clear the concept. Table 1.1 Shows Stage wise relations between numbers of triangles and Side length.
Table – 1.1

Table 1.1 Shows stage wise relations between numbers of triangles and Side length in Sierpinski Gasket.

<table>
<thead>
<tr>
<th>Stage</th>
<th>No. of equilateral triangles</th>
<th>Side length</th>
</tr>
</thead>
<tbody>
<tr>
<td>First</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Second</td>
<td>$3^1$</td>
<td>$2^2$</td>
</tr>
<tr>
<td>Third</td>
<td>$3^2$</td>
<td>$2^3$</td>
</tr>
<tr>
<td>Fourth</td>
<td>$3^3$</td>
<td>$2^4$</td>
</tr>
<tr>
<td>..........</td>
<td>......</td>
<td>....</td>
</tr>
<tr>
<td>$N^{th}$ stage</td>
<td>$3^n$</td>
<td>$2^n$</td>
</tr>
</tbody>
</table>

1.3.2 The length of a Coastline

Mandelbrot began his exposition on fractal geometry by considering the question: “How long is the coast of Great Britain?”. He showed that the actual length of the coast line is greater than the actual length of a straight line joining the two points. [11] As shown in Fig 1.3

The British coastline is irregular, and the measurement of the length of the coast line depends on the need. It can be measured using a ruler of different sizes depending upon the need and ease of implementation. The length of the coastline measured using a smaller ruler will be a more realistic estimate of the actual length. However as the size of the ruler is larger and larger the measured value will depart from the true estimate. Fig 1.3 a, is a part of complete coastline of Britain. The measurement of the length of the coastline using a ruler is shown in Fig. 1.3b and the same measurement using a ruler size half of the earlier one is shown in Fig. 1.3c. It is clearly seen that the length measured in the second case, using a ruler of smaller size is more than that in the earlier case. The length measured Fig.1.3b using ruler of larger size say L is six units, where as that using smaller ruler of size L/2 Fig.1.3c is 15 units (or 7.5L), this is not 12 units (or 6L). If we halved the scale
again, we would get a similar result, a longer estimate of \( L \). In general, as the ruler gets diminishingly small, the length gets infinitely large.

The fractal dimension of coastline of a Great Britain is more than one. It is marked that its total length is more than a length of a line joining its beginning point and the end point. There are different ways of evaluating its length more accurately including the choice of different length scales.

Fractals possess a distinguishing characteristic of non-integral dimension. This non-integral dimension is most commonly referred to as fractal dimension. Fractal appears to be more popular for their aesthetic nature than they are for their mathematics. Every one who has seen fractal has admired the beauty of colorful, fascinating image. But what is the formula that makes this glitzy image? The classical Euclidean. Geometry is mainly that one learns in school is quite different than fractal geometry mainly because fractal geometry concerns with non-linear, non-integral system while Euclidean geometry is a mainly oriented around linear, integral systems. Hence Euclidean geometry is a description of lines, ellipse, circles, etc. However fractal geometry is a description of algorithms. There are two basic properties that constitute a fractal: first is self-similarity, which is to say that most magnified images of fractals are essentially indistinguishable from the unmagnified version. A fractal shape looks almost the same, no matter what size it is viewed at. This repetitive pattern gives fractals their aesthetic nature. Secondly, fractals have now integer dimensions. This means that they are entirely different from the graph of line and conic sections that form main core of fundamental Euclidean geometry. By taking the mid point of each side of an equilateral triangle and joining them, one gets an interesting fractal known as Sierpiniski Triangle [13] shown in Fig1.1
In complex notation the Sierpiniski gasket is the fix point (attractor) of equations,

\[ f_1(z) = \frac{z}{2} , \quad \ldots \ldots (1.1) \]
\[ f_2(z) = \frac{z}{2} + \frac{1}{2} \quad \ldots \ldots (1.2) \]
\[ f_3(z) = \frac{z}{2} + \frac{3 + i}{2} \quad \ldots \ldots (1.3) \]

The iterations are repeated infinite number of times and eventually a very simple fractal arises. With an introduction of the concept of fractals and fractal dimension. It is now possible to describe shape more precisely, which otherwise could not be justified with the help of Euclidean geometry. “Clouds are not spheres, mountains are not cones, and coastlines are not circles and bark is not smooth. Nor does lightning travel in a straight line”. Such situations fall in the preview of fractals and thus can be more quantitatively dealt with using the concept of fractal dimension.

Most of the fractal have fractal dimension, which are not integers and thus the main property of fractal objects is its non-integral dimension. A variety of strange and interesting shapes are obtained from non Euclidean geometry. Some fractals are plane figures; others spread their Structure in space. If we apply the iteration process even to very simple formulae using complex numbers, we then enter a fabulous world of strange shapes and some times an astonishing beauty.

Let us consider \( z \) as a complex numbers, and then it can be described as

\[ z = x + iy \quad \ldots \ldots (1.4) \]

Where \( x \) and \( y \) are real numbers and \( i \) is square root of \(-1\), \( x \) is real part of the equation and \( iy \) is imaginary part. The principle that rules the realization of most of these pictures on a computer is in fact very simple.

Whenever one graduates a coordinate axis, each division of axis can be given any conventional unit value. [14] If we state that the value of one division equals to \( i \), we will have one side of origin the representation of the numbers \( i, 2i, 3i, 4i, \ldots \). Axis of \( x \)- represent real part, \( y \)-axis represents imaginary part of the same, and
brightness and colour of each pixel is the function of number of iterations necessary for the result to match a given condition. Let us take an expression as easy as.

$$z = z^2 + c$$ ................................(1.5)

c is a complex number fixed at the start. The calculations are done for each of the z points in the complex plane. Except that, excuse the detail, instead of doing calculation once for each point, we start again, giving z the value found for z’. In short we carry out a theoretically infinite number of iteration to work out each of these points, which can be transcribed into

$$z(n+1)=z(n)^2 + c$$ ................................ (1.6)

Starting with initial value z (0) equals to the coordinates of each point of the complex plane. It is very much interesting to see towards what value this function will tend for each point of the complex plane. We notice that for many points (i.e. for initial value of z) the iteration diverges i.e. the value of z grows exceedingly large. on the other hand for some points, the result remain definitively within a limited interval, the function does not diverge, even for an infinite number of iterations. The point for which the function does not diverge gives a set called filled in Julia set’. Of course an infinite number of Julia set exists since ‘c ’ can be given any value. According to the value of c, Julia set can show fairly ordinary shapes, or quite opposite, astonishing complex pictures, some times very pleasing! Interesting point is that these Julia sets are fractal structure. If there is query in some ones mind that, do this fascinating images of fractal are dealing with the real life? the answer is amazing yes. Fractal make a large part of biological world.

In human body arteries, veins, nerves, parotid glands, ducts, and the bronchial tree, all show some type of fractal organization. Besides this, fractal can be found in the regional distribution of pulmonary blood flow, pulmonary alveolar
structure, regional myocardial blood flow etc. Understanding and mastering the concept related to fractals will untouchably lead to breakthroughs in the area. Fractal is one of most interesting branch of chaos theory, and they are beginning to become a key to understand in the world of biology and medicine. Julia sets and Mandelbrot sets are perfect and beautiful fractal sets.

1.4 Famous Standard Fractal Sets

In this section we will discuss Mandelbrot set Julia set, and cantor set. these shapes are very famous and well−known in the word of fractals.

1.4.1 The Mandelbrot set

The Mandelbrot set is widely acknowledged fractal. Mandelbrot’s discovery is outcome of his research in the area of iteration theory, also known as complex analytic dynamics. This field dates back to the investigation of P. Fatou and G.Julia in the early part of this century. A one to one correspondence between the complex number and the points in the complex plane. Repeated application of a simple function causes some of these points to run away towards infinity, while others never wander far from the origin. The Mandelbrot set is made up of connected points in the complex plane. the simple equation, which is the basis of Mandelbrot, set is,

Changing number + Fixed number = Result. 

\[ \text{Changing number} + \text{Fixed number} = \text{Result.} \quad \ldots \ldots \ (1.7) \]

In order to calculate points for a Mandelbrot set, Let us start with one of the numbers on the complex plane and put its value as “fixed number” in equation (1.7). The “changing number is repeated or iterated. This operation is repeated for infinite number of times. When iterative equations are applied to points in a certain region of complex plane, a Fractal from Mandelbrot set results. A few fractals obtained from Mandelbrot set [15] shown in Fig. 1.4a and Fig. 1.4b
1.4.2 The Julia Set.

The Julia set is named after the French mathematician Gaston Julia [16] who investigated them. Mathematicians found Julia sets in beginning as being “Pathologically complicated.” The Julia sets are now associated with points \( z = x + iy \) on the complex plane. The complexity that gives Julia sets their extraordinary beauty. Like the Mandelbrot sets, Julia sets are the product of dynamical systems based on the function.

\[
f(z) = z^2 + c
\]  \quad \text{……… (1.8)}

Where \( z \) and \( c \) are complex numbers. But for Julia sets each location in the complex plane becomes an initial value of \( z \), and same fixed value of \( c \) is used for entire set. When the function is iterated, individual points in the plane will have open neighborhoods that either converge to particular point or diverse to infinity.

Those points in convergent neighborhoods form a Fatou set. The boundary of this set is a Julia set. For each initial value \( z_0 \), the recursion \( z_{n+1} = f(z_n) \) defines a sequence of points called the forward orbit of \( z_0 \). A set of points whose forward orbits approach the same limit point is called a basin of attraction, and this limit point is known as the set's attractor. A Julia set is a set of exceptional points that separate different basins of attraction. It can be thought of as the repeller set of the iterative function, or as the attractor of the inverse relation [17]. For the quadratic function \( f(z) = z^2 + c \), the Julia set is the boundary between the set of points that iterate to infinity and those that do not. Assigning a value of 0 to \( c \) produces the function \( f(z) = z^2 \), which provides a helpful example. When this function is iterated, every point whose distance from the origin is less than 1 converges to the origin. Every point greater than 1 unit from the origin iterates to infinity. Thus the Julia set of \( f(z) = z^2 \) is the unit circle centered at the origin. For nonzero values of \( c \) the corresponding Julia sets are fractals. Fig. 1.5
1.4.3 The Cantor Set.

George cantor was a mathematician. He is best known as a creator of modern set theory. The infinite number of points on a line segment fascinated him. Cantor began to wonder what would happen when an infinite numbers of line segments were removed from an initial line interval. Cantor introduced a remarkable construction involving only real numbers between zero and one by iteratively taking away middle third of line segments. This operation created a "dust" of points; hence, the name Cantor Dust. In order to understand Cantor Dust, start with a line of unit interval [0,1]; removing the middle third from the unit interval [0,1] and leaving remaining segments [0,1/3] [2/3,1]. Now “middle thirds” of all the remaining interval/segments are removed and so on. The operation is shown below. Fig. 1.6

The Cantor set is simply the dust of points that remain. The numbers of these points are infinite, but their total length is zero. Mandelbrot considered the Cantor set as a model for the occurrence of errors in an electronic transmission line. Engineers saw periods of errorless transmission, mixed with periods when errors would come in gusts. When these gusts of errors were analysed, it was gratified that they contained error-free periods within them. As the transmissions were analysed to smaller and smaller degrees, it was determined that such dusts, as in the Cantor Dust, were indispensable in modelling intermittency.

1.5 Impact on the Sciences.

Researcher has widely been recognized that many of the structures common in their experiment possess a rather special kind of geometrical complexity. The person responsible for this kind of awareness is Benoit Mandelbrot who called attention of Researcher’s, Scientist’s and Physicist’s to the particular geometrical properties of such objects as the shape of mountains, shape of lightning, the
branches of trees and the surface of clouds. He coined the name fractal to these complex shapes and expressed that these shapes can be characterized by a non-integral dimensionality. There is a long list of examples of fractals. Which include structures from microscopic aggregates to the clusters of galaxies.

In a very common phenomena of far-from equilibrium growth in the field of science, fractals are observed. Some illustrations of these types of processes are electrodeposition of ion on an electrode, viscous fingering etc. Viscous fingering takes place when a less viscous fluid is injected into a more viscous fluid and dendritic solidification in an under cooled medium. In these cases the motion of interface is determined by the special distribution of a quantity, which satisfies the Laplace’s equation with moving boundary conditions. Like interfacial growth, aggregation of similar particle represents another important class of phenomenon producing complex Geometrical objects. [19]

Nature is filled with examples of phenomenon that exhibit apparently chaotic behavior such as air turbulence, forest fires and like. However under this behaviour it is almost possible to define self similarity i.e invariance with respect to the scale used. The structures that appear as consequences of self-similarity are known as fractals.

In Biology, The lung alveolar structure, the capillary network, and the structure of several parts of higher plant organization, such as ears, spikes, umbels, etc., are supposed to be fractals, and, in fact, mathematical functions based on fractal geometry algorithms could be developed to simulate them. However, the statement that given a biological structure is ‘fractal’, should imply that the iterative process of its construction has a real biological meaning, i.e., that its construction in nature is achieved by means of a single genetic, enzymatic, or biophysical mechanism successively repeated. Thus, such an iterative process
should not be just an abstract mathematical tool to reproduce that object. This property has not been proven at present for any biological structure, because the mechanisms that build the objects mentioned above are unknown in detail.

In geological studies, many natural geological rock formations, as well as engineered porous structure have fractal properties. They are self-similar over several length scales. Metrology provides a different kind of space-time fractals. The fractal dimension of metrological time series play a fundamental role in the development of dynamic models of metrological phenomena. There have been numerous efforts to determine the dimension of weather and climatic attractors [20].

Econophysics [21, 22] is an interdisciplinary field of research in which tools of mathematics and physics are used to study systems in economics. Applying physics to financial problems tender fresh look to existing theories of finance. Finally, computer graphics software engineer using special technique are producing amazing fractal images of great statistical complexity. Landscapes created by above technique have been used as backgrounds at various places.

1.6 The Problem.

The present work covers review of some ongoing work and advances is made in the areas where further work is needed in terms of revealing more useful information. It is intended to study the fabrication of Hale Shaw cell and under its working for different conditions. Viscous fingering can be seen between the interfaces of fluids. The development of viscous fingering is a transient phenomenon and the patterns develop rapidly and remain stable for a short time and the patterns quickly diffuse. Viscous fingering is studied with a view to attempt the growth velocity in some typical cases. The study, thus, involves fast framing photography, which is made to quantify the finger growth. The fingering patterns
thus obtained is analyzed to characterize their structures and textures using the techniques developed for determination of fractal dimensions.

Electrodeposition is electrochemical deposition. The work on electrodeposited patterns will be further extended in order to study the process in detail. Experiment will be carried under constant current and constant voltage condition and the effect of constant current and constant voltage on growth pattern will be studied. A computer controlled constant current and constant voltage power supply will be developed for electro deposition experiments with a view to provide a flexible control of the operating conditions.

In almost all the electrodeposition experiments, a constant voltage is applied across the cell containing the electrolyte and the resulting growth is studied. If the size of the cell is large compared the size of the final growth the electric field can be treated to be more or less constant. However we noticed that in real experiments the size of growth many times exceeds half the size of the cell. Thus the growth is taking place at different electric field conditions, particularly at developed stages, the effective electric field is higher than that at the initial stages. In view of this, we propose to develop a computer control system that is capable of giving output voltage under program control. It is expected that further work will make use of it to control the output voltage to maintain a constant electric field during the growth process. This in fact requires keeping track of the actual growth size and producing a voltage output that is consistent with the requirement of constant electric field and is beyond the scope of present work. Also, as the electric field changes, the current through the cell also change. This may result in a change in electric resistance of the electrodepositon cell. Thus the electric resistance of the cell will be studied by dynamic recording of the applied voltage and current
through the cell. For real time recording of the current and voltage of cell a computer based data acquisition system will be developed.

Simulation of electro deposition will be studied by using a suitable model developed to give a picture of the dendritic growth in electro deposition experiments. Simulation studies are expected to throw light on the influence of radial field. Attempt will be made to simulate growth under changing electric field conditions and to compare the simulation with the actual deposits found in the experiments. In simulation studies the effect of consecutive particles getting attached to the mass already deposited will also be studied. Emphasis will be given on, how the mass of aggregate changes and location of center of mass will be tracked. If there is any significant change of position of center of mass as the process of growth proceeds will be noted.

Many Phenomena in nature occur in some what random fashion, and at times they exhibit self-similarity in terms of the repetition of the event. The study of fractal behavior in natural or artificial events will be attempted from the point of view of time series. Applications R/S analysis to time series will be studied for weather parameters and if possible for share market etc. It is established that the Rescaled range analysis (R/S analysis) proves to be a good tool in predicting the future course of a time series of events knowing its past behavior as a time series.
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