Chapter 10

Entire Semitotal-Point Domination in Graphs

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ABSTRACT

Let $G = (V, E)$ be a graph. Then the semitotal-point graph is denoted by $T_2(G)$. Let the vertices and edges of $G$ be the elements of $G$. An entire dominating set $X$ of a graph $T_2(G)$ is an entire semitotal-point (ESP) dominating set if every element not in $X$ is either adjacent or incident to at least one element in $X$. An ESP domination number $\varepsilon_{tp}(G)$ of $G$ is the minimum cardinality of an ESP dominating set of $G$. In this chapter, many bounds on $\varepsilon_{tp}(G)$ are obtained in terms of elements of $G$. Also its relationship with other domination parameters are investigated.
10.1 Introduction.

The graphs considered in this chapter are finite, nontrivial and simple. The vertices and edges of a graph $G$ are called the elements of $G$.

Let $G = V, E$ be a graph. A dominating set $X$ of a graph $G$ is called an entire dominating set of $G$, if every element not in $X$ is either adjacent or incident to at least one element in $X$. The entire domination number $\varepsilon(G)$ of $G$ is the minimum cardinality of an entire dominating set of $G$ [40].

In this chapter, we study the entire domination in semitotal-point graphs. Which is defined as follows.

**Definition 10.1.** An entire dominating set $X$ of a graph $T_2(G)$ is an entire semitotal-point(ESP) dominating set if every element not in $X$ is either adjacent or incident to at least one element in $X$, an ESP domination number $\varepsilon_{tp}(G)$ of $G$ is the minimum cardinality of an ESP dominating set of $G$.

The Figure 10.1 shows the $G$ and $T_2(G)$ and formation of $\varepsilon(G)$ and $\varepsilon_{tp}(G)$.
In Figure 10.1, $V(G) = p = 4$ and $E(G) = q = 4$.

In $T_2(G)$, $V(T_2(G)) = p + q$ and $E(T_2(G)) = 3q$.

The minimal entire dominating set in $G$ is $X = \{v_2, e_3\}$.

Therefore, $\varepsilon(G) = |X| = |\{v_2, e_3\}| = 2$.

The minimal semitotal-point entire dominating set is $T_2(G)$ is $X' = \{v_2, e_1, v_3, v_4\}$.

Therefore, $\varepsilon_{tp}(G) = |X'| = |\{v_2, e_1, v_3, v_4\}| = 4$. 
10.2 Results

We need the following theorems for our further results.

Theorem 10.A [40]. For any connected graph $G$ of order $p$,

$$\varepsilon(G) \leq \lceil \frac{p}{2} \rceil$$

Further, equality holds for $G = K_p$; $p \geq 2$.

Theorem 10.B [27]. For any connected graph $G$ of order $p$,

$$\gamma(G) \geq \lceil \frac{p}{\Delta(G)+1} \rceil.$$

Where $\Delta(G)$ is the maximum degree of $G$.

Theorem 10.C [9]. For any tree $T$, $\gamma_{tp}(T) \geq \gamma(T)$.

Theorem 10.D [45]. For any graph $G$ with an pendant vertex,

$$\gamma(G) = \gamma_s(G).$$

Theorem 10.E [59]. If $T$ is a tree with $p \geq 3$, then $\gamma_c(T) = p - e$.

Where $e$ is the number of pendant vertices in a tree.

Theorem 10.F [35]. Let $G$ be a $(p,q)$ graph with edge domination number $\gamma'(G)$. Then $\gamma'(G) \leq \lfloor \frac{p}{2} \rfloor$.

Proposition 10.1. For any graph $G$, $\gamma(G) \leq \varepsilon(G)$.

Proposition 10.2. If $G = K_p$; $p \geq 2$, then $\varepsilon_{tp}(K_p) = p$.

Observation 10.1. In this chapter, $\gamma(T_2(G))$ and $\gamma_{tp}(G)$ both denote
the domination number of the semitotal-point graph $T_2(G)$.

**Theorem 10.1** For any graph $G$ of order $p$, $\varepsilon_{tp}(G) = p$.

**Proof.** Let $G$ be a $(p,q)$ graph. We consider the following cases:

**Case 1.** When $q \leq p$. Let $X = \{v_1, v_2, \cdots, v_k\}$ be the set of vertices in $T_2(G)$ and by definition, $V(T_2(G)) = p + q$. By Theorem 10.A,

$$\varepsilon_{tp}(G) \leq \left\lceil \frac{p+q}{2} \right\rceil \leq p.$$ 

Let $F = \{v_1, v_2, \cdots, v_m\}$ be the maximum independent set of $T_2(G)$. Since every maximum independent set is a minimal dominating set, therefore $\gamma(T_2(G)) \leq \frac{p+q}{2}$. Let $\{e_1, e_2, \cdots, e_q\}$ be the set of edge vertices in $T_2(G)$. Let $F' = \{e_1', e_2', \cdots, e_n'\}$ be the edge subset of $E(T_2(G))$ such that no edge in $F'$ is incident with a vertex in $F$. By Theorem 10.F, $|F'| = \left\lfloor \frac{p+q}{2} \right\rfloor$. Clearly $F \cup F'$ is an entire dominating set of $T_2(G)$.

Therefore,

$$\varepsilon_{tp}(G) \leq |F \cup F'|$$

$$= \frac{p+q}{2} + \left\lfloor \frac{p+q}{2} \right\rfloor$$

$$\geq p.$$ 

Thus, $\varepsilon_{tp}(G) = p$.

**Case 2.** When $q > p$, then $G$ must be either $K_p$ or $K_p - x_i$, where $x_i; i \geq 1$ denotes the edges of $K_p$. Suppose $G = K_p$, then by Propo-
sition 10.2, the result follows. If $G \neq K_p$ and $q > p$, then by Case 1, $\varepsilon_{tp}(G) = p$.

Next, theorem gives the relation between $\varepsilon(G)$ and $\varepsilon_{tp}(G)$.

**Theorem 10.2** For any graph $G$, $\varepsilon(G) < \varepsilon_{tp}(G)$.

**Proof.** Let $D$ and $D'$ be minimal entire dominating sets of $G$ and $T_2(G)$ respectively. By Theorem 10.1, $\varepsilon_{tp}(G) = p$ and by Theorem 10.A, we have $\varepsilon(G) \leq \lceil \frac{p}{2} \rceil$. Hence from these two results we get the required result.

Next, we give the lower bound for $\varepsilon_{tp}(G)$.

**Theorem 10.3** For any $(p, q)$ graph $G$ with maximum degree $\Delta$

$$\frac{p+q}{2\Delta(G)+1} \leq \varepsilon_{tp}(G).$$

**Proof.** By Theorem 10.B, we have $\gamma(G) \geq \lceil \frac{p}{\Delta(G)+1} \rceil$. Therefore, $\gamma_{tp}(G) \geq \lceil \frac{p+q}{2\Delta(G)+1} \rceil$. Since $\gamma_{tp}(G) \leq \varepsilon_{tp}(G)$, therefore $\varepsilon_{tp}(G) \geq \lceil \frac{p+q}{2\Delta(G)+1} \rceil$.

**Corollary 10.1.** For any $(p, q)$ graph $G$

$$\frac{p}{\Delta(G)+1} < \varepsilon_{tp}(G).$$

The next result relates $\varepsilon_{tp}(G)$ with $\beta_0(G)$.

**Theorem 10.4** For any graph $G \neq K_p$ and tree $T$ with $p \geq 3$,

$$2\beta_0 \leq \varepsilon_{tp}(G).$$
Where $\beta_0$ is the vertex independence number of $G$.

**Proof.** Let $B$ be the independent set of $G$ which is also a dominating set of $G$. Then $\gamma(G) \leq |B| \leq \frac{p}{2}$. Now,

$$2\beta_0(G) = 2|B|$$

$$\leq 2 \times \frac{p}{2}$$

$$= p.$$  

Also by using Theorem 10.1, we get the required result. 

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**Theorem 10.5** For any graph $G$ with $\Delta(G) < p - 1$,

$$\frac{\gamma(G)+\beta_0(G)}{2} < \varepsilon_{tp}(G) \leq \gamma(G) + \beta_0(G) + 1$$

**Proof.** By Theorem 10.1, we have $\varepsilon_{tp}(G) = p$. Also for any graph $G$ without isolated vertices we have $\gamma(G) \leq \frac{p}{2}$ and $\beta_0(G) \leq \frac{p}{2}$. Therefore from these the lower bound is attained.

The upper bound is obvious. 

In the following result we establish the relation between $\varepsilon_{tp}(G)$ and diameter of $G$.

**Theorem 10.6** For any graph $G$,

$$\frac{\text{diam}(G)+1}{3} < \varepsilon_{tp}(G).$$
Where $\text{diam}(G)$ is the diameter of $G$.

**Proof.** In [27], we have $\left\lceil \frac{\text{diam}(G)+1}{3} \right\rceil \leq \gamma(G)$ and by Proposition 10.1 and Theorem 10.2, the result follows. □

Next, we obtain the relation between $\varepsilon_{tp}(G)$, $\alpha_0(G)$ and $\beta_0(G)$.

**Theorem 10.7** For any graph $G$,

$$
\varepsilon_{tp}(G) = \alpha_0(G) + \beta_0(G)
$$

**Proof.** The result follows from Theorem 10.1 and also from the fact that $\alpha_0(G) + \beta_0(G) = p$. □

The immediate consequence of the above theorem is as follows.

**Theorem 10.8** For any graph $G$,

$$
\varepsilon_{tp}(G) = \alpha_1(G) + \beta_1(G).
$$

Where $\alpha_1(G)$ and $\beta_1(G)$ denote the edge covering number and edge independence number of $G$.

**Theorem 10.9** If a graph $G$ and its complement $\overline{G}$ are connected, then

$$
\varepsilon_{tp}(\overline{G}) = p
$$

**Proof.** Let $G$ and its complement $\overline{G}$ be connected. Then the proof follows from Theorem 10.1. □
10.3 Comparison of $\varepsilon_{tp}(G)$ with other domination parameters

In the following theorem we obtain the relation between $\gamma_s(G)$ and $\varepsilon_{tp}(G)$.

**Theorem 10.10** For any graph $G$ with a pendant vertex

$$\gamma_s(G) \leq \varepsilon_{tp}(G).$$

The equality holds for $G = K_p; \ p \geq 2$.

**Proof.** For any graph $G$, we have $\gamma(G) \leq \varepsilon(G)$ and also by Theorem 10.2, $\gamma(G) \leq \varepsilon(G) < \varepsilon_{tp}(G)$. Therefore by from Theorem 10.D, we get $\gamma_s(G) < \varepsilon_{tp}(G)$.

Since $\gamma_s(K_p) = p$, therefore by Theorem 10.1, the equality follows.

The next result gives the relation between $\gamma_c(T)$ and $\varepsilon_{tp}(T)$.

**Theorem 10.11** For any non trivial tree $T$,

$$\varepsilon_{tp}(T) = \gamma_c(T) + e.$$

Where $e$ is the number of pendant vertices in $T$.

**Proof.** Let $G$ be any non trivial tree $T$. Then by Theorem 10.E, $\gamma_c(T) = p - e$. Substituting for $\gamma_c(T)$ in the required result, the result follows.
Next, we obtain the relationship between $\gamma_m(G)$ and $\varepsilon_{tp}(G)$

**Theorem 10.12** For any connected graph $G$ of order at least 2,

$$\gamma_m(G) \leq \varepsilon_{tp}(G).$$

Further the equality holds for $G = K_p$

**Proof.** Let $G$ be a $(p, q)$ graph $G$ which is not $K_p$. Let $D$ be a maximal dominating set of $G$. Then $V - D$ contains at least one vertex $v_i$ which does not form a dominating set of $G$. Hence $|V - D| < |V(G)|$. Thus by Theorem 10.1, $\gamma_m(G) < \varepsilon_{tp}(G)$.

For equality, suppose $G = K_p$, then $\gamma_m(K_p) = p$. Hence by Theorem 10.1, we get $\gamma_m(G) = \varepsilon_{tp}(G)$. \qed

**Definition 10.2.** Let $S$ be the set of elements of a graph $G$ and $X$ be the minimum entire dominating set of $G$. If $S - X$ contains an entire dominating set say $X'$ then $X'$ is called inverse entire dominating set of $G$ with respect to $X$. The *inverse entire domination number* $\varepsilon^{-1}(G)$ of $G$ is the minimum number of elements in an inverse entire dominating set of $G$ [52].

**Theorem 10.13** For any graph $G$,

$$\varepsilon^{-1}(G) < \varepsilon_{tp}(G).$$
Proof. Since every inverse entire dominating set is an entire dominating set, therefore $\varepsilon(G) \leq \varepsilon^{-1}(G)$. By Proposition 10.1 and Theorem 10.2, we have $\varepsilon^{-1}(G) < \varepsilon_{tp}(G)$.

In the next theorem we establish the relation between $\gamma_{ns}(G)$ and $\varepsilon_{tp}(G)$.

Theorem 10.14 Let $D$ be a $\gamma_{ns}$-set of a connected graph $G$. If no two vertices in $V - D$ are adjacent to a common vertex in $D$, then

$$\gamma_{ns}(G) + \xi(T) \leq \varepsilon_{tp}(G).$$

Where $\xi(T)$ is the maximum number of pendant vertices in any spanning tree $T$ of $G$.

Proof. Let $G$ be a graph such that no two vertices in $V - D$ are adjacent to a common vertex in $D$. By Theorem 10.1, we have $\varepsilon_{tp}(G) = p$.

Let $D$ be a $\gamma_{ns}$-set of $G$. Since for any two vertices $u, v \in V - D$, there exists no vertices $u_1, v_1 \in D$ such that $u_1$ is adjacent to $u$ but not $v$ and $v_1$ is adjacent to $v$ but not to $u_1$. This implies that there exists a spanning tree $T$ of $\langle V - D \rangle$ in which each vertex of $V - D$ is adjacent to a vertex of $D$. This shows that $\xi(T) \geq |V - D|$.

Thus from above two results we get, $\gamma_{ns}(G) + \xi(T) \leq \varepsilon_{tp}(G)$.

Definition 10.3. Let $D$ be a minimum dominating set in $G = (V, E)$. 

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If \( V - D \) contains a dominating set \( D' \) of \( G \) then \( D' \) is called an inverse dominating set with respect to \( D \). **The inverse domination number** \( \gamma^{-1}(G) \) of \( G \) is the cardinality of a smallest inverse dominating set of \( G \) [39].

**Theorem 10.15** Let \( T \) be a tree with every nonpendant vertex adjacent to at least one pendant vertex. Then

\[
\gamma(T) + \gamma^{-1}(T) = \varepsilon_{tp}(T)
\]

**Proof.** Let \( T \) be a tree with every nonpendant vertex is adjacent to at least one pendant vertex. If every nonpendant vertex is adjacent to at least two pendant vertices, then the set of all nonpendant vertices is a minimum dominating set and the set of all pendant vertices is a minimum inverse dominating dominating set. Suppose there are nonpendant vertices which are adjacent to exactly one pendant vertex. Let \( D \) and \( D' \) denote the minimum dominating and inverse dominating sets respectively. Let \( u \) be a nonpendant vertex adjacent to exactly one pendant vertex \( v \). If \( u \in D \) then \( v \in D' \) and if \( u \in D' \) then \( v \in D \). Therefore \( |D| + |D'| = p \). Also by Theorem 10.1, \( \varepsilon_{tp}(T) = p \). Thus \( \gamma(T) + \gamma^{-1}(T) = \varepsilon_{tp}(T) \).
Next, we establish Nordhaus-Gaddum type results.

**Theorem 10.16** For any \((p, q)\) graph \(G\)

\[
(i) \quad \varepsilon_{tp}(G) + \varepsilon_{tp}(\overline{G}) \leq 2p
\]

\[
(ii) \quad \varepsilon_{tp}(G)\varepsilon_{tp}(\overline{G}) \leq p^2.
\]

*Further, the equality holds for \(G = C_5\) or \(P_4\).*

**Proof.** From Theorem 10.1 and Theorem 10.9 the result follows. □