CHAPTER 1

Publications based on this Chapter

• Some results on Lorentzian $\alpha$-Sasakian manifolds, Communicated.

• On Generalized Recurrent Lorentzian $\alpha$-Sasakian manifolds, Communicated.
Chapter 1

On Lorentzian $\alpha$-Sasakian manifolds

1.1 Preliminaries

A $(2n+1)$-dimensional Lorentzian manifold $M$ is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric $g$, that is, $M$ admits a smooth symmetric tensor field $g$ of type $(0,2)$ such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \to \mathbb{R}$ is a non-degenerate inner product of signature $(-,+,\ldots,+,+)$, where $T_pM$ denotes the tangent space of $M$ at $p$ and $\mathbb{R}$ is the real number space.

A differentiable manifold $M$ of dimension $(2n+1)$ is called Lorentzian $\alpha$-Sasakian manifold if it admits a $(1,1)$-tensor field $\phi$, a contravariant vector field $\xi$, a covariant vector field $\eta$ and a Lorentzian metric $g$ satisfying ([2], [28])

\begin{align*}
\eta(\xi) &= -1, \\
\phi^2 &= I + \eta \otimes \xi, \\
g(\phi X, \phi Y) &= g(X, Y) + \eta(X)\eta(Y), \\
g(X, \xi) &= \eta(X), \\
\phi \xi &= 0, \quad \eta(\phi X) = 0,
\end{align*}

(1.1.1)-(1.1.5)
for all $X, Y \in TM$.

Also the Lorentzian $\alpha$-Sasakian manifold $M$ satisfies

$$(a) \nabla_X \xi = -\alpha \phi X, \quad (b) \ (\nabla_X \eta)(Y) = -\alpha g(\phi X, Y),$$

where $\nabla$ denotes the operator of covariant differentiation with respect to the Lorentzian metric $g$.

Further on Lorentzian $\alpha$-Sasakian manifold $M$, the following relations holds ([114],[58]):

$$\eta(R(X,Y)Z) = \alpha^2[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]$$

$$R(\xi,X)Y = \alpha^2[g(X,Y)\xi - \eta(Y)X]$$

$$R(X,Y)\xi = \alpha^2[\eta(Y)X - \eta(X)Y]$$

$$R(\xi,X)\xi = \alpha^2[\eta(X)\xi + X]$$

$$(\nabla_X \phi)(Y) = \alpha[g(X,Y)\xi - \eta(Y)X]$$

$$S(X,\xi) = 2\alpha^2 \eta(X)$$

$$S(\phi X, \phi Y) = S(X,Y) + 2\alpha^2 \eta(X)\eta(Y)$$

$$g(R(\xi,X)Y,\xi) = -\alpha^2[g(X,Y) + \eta(X)\eta(Y)]$$

for all vector fields $X, Y, Z$, where $S$ is the Ricci curvature and $Q$ is the Ricci operator given by $S(X,Y) = g(QX,Y)$. 
Ricci operator, Ricci tensor and Curvature tensor in a 3-dimensional Lorentzian $\alpha$-Sasakian manifold are given respectively by

\[ QX = \left[ \frac{r}{2} + \alpha^2 \right] X + \left[ \frac{r}{2} + 3\alpha^2 \right] \eta(X)\xi \]  
(1.1.15)

\[ S(X, Y) = \left[ \frac{r}{2} + \alpha^2 \right] g(X, Y) + \left[ \frac{r}{2} + 3\alpha^2 \right] \eta(X)\eta(Y), \]  
(1.1.16)

\[ R(X, Y)Z = \left[ \frac{r}{2} + 2\alpha^2 \right] [g(Y, Z)X - g(X, Z)Y] \]
\[ + \left[ \frac{r}{2} + 3\alpha^2 \right] [\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \]  
(1.1.17)

### 1.2 Objectives

To study:

- $W_2$-Semisymmetric $\alpha$-Sasakian manifold.
- Generalized recurrent Lorentzian $\alpha$-Sasakian manifold.
- Nature of the 1-forms $P$ and $Q$ on a Generalized Ricci Recurrent Lorentzian $\alpha$-Sasakian manifold.
- Lorentzian $\alpha$-Sasakian manifold satisfying $S(X, \xi) \cdot R = 0$.
- 3-dimensional Lorentzian $\alpha$-Sasakian manifold with Cyclic-Parallel Ricci tensor.
1.3 $W_2$-Semisymmetric Lorentzian $\alpha$-Sasakian manifold

In [44], Pokhariyal and Mishra defined a new curvature called $W_2$-semisymmetric curvature tensor and is given by

$$W_2(X, Y, Z, U) = R(X, Y, Z, U) + \frac{1}{2n}[g(X, Z)S(Y, U) - g(Y, Z)S(X, U)], \quad (1.3.1)$$

where $S$ is a Ricci tensor of type $(0,2)$. In this section we study $W_2$-semisymmetric Kenmotsu manifold.

Definition 4. An $(2n + 1)$-dimensional Lorentzian $\alpha$-Sasakian manifold is called $W_2$-semisymmetric if it satisfies

$$R(X, Y)W_2 = 0, \quad (1.3.2)$$

where $R(X, Y)$ is to be considered as a derivation of the tensor algebra at each point of the manifold for tangent vectors $X, Y$.

It can be easily shown that in a Lorentzian $\alpha$-Sasakian manifold the $W_2$-curvature tensor satisfies the condition

$$W_2(X, Y, Z, \xi) = 0. \quad (1.3.3)$$

Now consider $W_2$-semisymmetric Lorentzian $\alpha$-Sasakian manifold, then from (1.3.2) we have

$$R(X, Y)W_2(Z, U)V - W_2(R(X, Y)Z, U)V$$

$$- W_2(Z, R(X, Y)U)V - W_2(Z, U)R(X, Y)V = 0.$$
which implies

\[ g(R(X, Y)W_2(Z, U)V, \xi) - g(W_2(R(X, Y)Z, U)V, \xi) 
- g(W_2(Z, R(X, Y)U)V, \xi) - g(W_2(Z, U)R(X, Y)V, \xi) = 0. \] (1.3.4)

Putting \( X = \xi \) in (1.3.4), we have

\[ (R(\xi, Y)W_2(Z, U)V, \xi) - W_2(R(\xi, Y)Z, U)V, \xi) 
- W_2(Z, R(\xi, Y)U)V, \xi) - W_2(Z, U)R(\xi, Y)V, \xi) = 0. \]

Using (1.1.8) and (1.3.3) in the above equation, we obtain

\[ \eta(Y)\eta(W_2(Z, U)V) - W_2(Z, U, V, Y) = 0. \] (1.3.5)

Further using (1.1.3) in (1.3.5), we get

\[ R(X, Y, Z, U) = \frac{1}{2n} [g(Y, Z)S(X, V) - g(X, Z)S(Y, V)]. \] (1.3.6)

Contracting (1.3.6) we have

\[ S(Y, Z) = \frac{r}{2n + 1} g(Y, Z). \] (1.3.7)

From (1.3.6) and (1.3.7), we obtain

\[ R(X, Y, Z, U) = \frac{r}{2n(2n + 1)} [g(Y, Z)g(X, V) - g(X, Z)g(Y, V)]. \] (1.3.8)

From the above result, we state that

**Theorem 1.3.1.** \( W_2 \)-semisymmetric Lorentzian \( \alpha \)-Sasakian manifold is of constant curvature.
1.4 Generalized recurrent Lorentzian $\alpha$-Sasakian manifold

A Riemannian manifold is called a generalized recurrent Riemannian manifold [38] if the curvature tensor $R$ satisfies the condition:

$$(\nabla_X R)(Y, Z, W) = P(X)R(Y, Z, W) + Q(X)[g(Z, W)Y - g(Y, W)Z],$$

(1.4.1)

where $P$ and $Q$ are two 1-forms, $Q$ is non-zero and these are defined by

$$P(X) = g(X, A), \quad Q(X) = g(X, B),$$

(1.4.2)

where $A$ and $B$ are the vector fields associated with 1-forms $P$ and $Q$ respectively.

Taking $Y = W = \xi$ in (1.4.1), we have

$$(\nabla_X R)(\xi, Z, \xi) = P(X)R(\xi, Z, \xi) + Q(X)[g(Z, \xi)\xi - g(\xi, \xi)Z]$$

(1.4.3)

The left hand side of (1.4.3), clearly can be written in the form

$$(\nabla_X R)(\xi, Z, \xi) = XR(\xi, Z, \xi) - R(\nabla_X Z, \xi)$$

$$- R(\xi, \nabla_X Z, \xi) - R(\xi, Z, \nabla_X \xi),$$

which in view of (1.1.6) and (1.1.10), we have

$$(\nabla_X R)(\xi, Z, \xi) = \alpha^2 X[\eta(Z)\xi + Z] + \alpha(\phi X, Z, \xi) - \alpha^2(\eta(\nabla_X Z)\xi + \nabla_X Z) + \alpha R(\xi, Z, \phi X).$$

Next, using (1.1.4) and (1.1.8) in the above equation, we get

$$(\nabla_X R)(\xi, Z, \xi) = \alpha^2 g(\nabla_X Z, \xi)\xi + \alpha R(\phi X, Z, \xi) + \alpha^3 g(Z, \phi X)\xi.$$
Using (1.1.5), we obtain
\[(\nabla_X R)(\xi, Z, \xi) = \alpha R(\phi X, Z, \xi)\]

While the right hand side of (1.4.3) gives
\[P(X)R(\xi, Z, \xi) + Q(X)[g(Z, \xi)\xi - g(\xi, \xi)Z]\]
\[= P(X)\alpha[\eta(Z)\xi + Z] + Q(X)[\eta(Z)\xi + Z]\]
\[= \alpha[P(X) + Q(X)][\eta(Z)\xi + Z].\]

Therefore
\[R(\phi X, Z, \xi) = [\alpha P(X) + Q(X)][\eta(Z)\xi + Z]. \quad (1.4.4)\]

Taking \(Z = \xi\) in (1.4.4) and then using (1.1.1) and (1.1.10), we get
\[\alpha(\phi X) = [\alpha P(X) + Q(X)][-\xi + \xi].\]

So by virtue of (1.1.5) in above, we get
\[\alpha(\phi X) = 0.\]

In view of (1.1.6), we have
\[\nabla_X \xi = 0.\]

Thus we have the following result:

**Theorem 1.4.1.** If the Lorentzian \(\alpha\)-Sasakian manifold is generalised recurrent, then the associated vector field \(\xi\) is constant.

Permuting equation (1.4.1) twice with respect to \(X,Y,Z\); adding the resulting three
equations and using Bianchi's second identity, we have

\[ P(X)R(Y, Z, W) + Q(X)[g(Z, W)Y - g(Y, W)Z] \]
\[ + P(Y)R(Z, X, W) + Q(Y)[g(X, W)Z - g(Z, W)X] \]
\[ + P(Z)R(X, Y, W) + Q(Z)[g(Y, W)X - g(X, W)Y] = 0. \]  \hspace{1cm} (1.4.5)

Contracting (1.4.5) with respect to \( Y \), we get

\[ P(X)S(Z, W) + (n - 1)Q(X)g(Z, W) + R(Z, X, W, A) \]
\[ + Q(Z)g(X, W) - Q(X)g(Z, W) - P(Z)S(X, W) \]
\[ - (n - 1)Q(Z)g(X, W) = 0. \] \hspace{1cm} (1.4.6)

In view of \( S(Y, Z) = g(Q_1 Y, W) \), equation (1.4.6) reduces to

\[ P(X)g(Q_1 Z, W) + (n - 1)Q(X)g(Z, W) - g(R(Z, X, A), W) \]
\[ + Q(Z)g(X, W) - Q(X)g(Z, W) - P(Z)g(Q_1 X, W) \]
\[ - (n - 1)Q(Z)g(X, W) = 0. \] \hspace{1cm} (1.4.7)

Factoring off \( W \), we get from (1.4.7),

\[ P(X)Q_1 Z + (n - 1)Q(X)Z - R(Z, X, A) + Q(Z)X \]
\[ - Q(X)Z - P(Z)Q_1 X - (n - 1)Q(Z)X = 0. \] \hspace{1cm} (1.4.8)

Contracting (1.4.8) with respect to \( \dot{Z} \), we get

\[ P(X)r + n(n - 1)Q(X) - S(X, A) \]
\[ - (n - 1)Q(X) - S(X, A) - (n - 1)Q(X) = 0 \]

or

\[ P(X)r + (n - 1)(n - 2)Q(X) - 2S(X, A) = 0. \] \hspace{1cm} (1.4.9)
Taking $X = \xi$ and then using (1.1.12) and (1.4.2), we get

$$\eta(A) r + (n - 1)(n - 2)\eta(B) - 2(n - 1)\eta(A) = 0$$

or,

$$r = -(n - 1) \left[ (n - 2)\frac{\eta(B)}{\eta(A)} + 2 \right]$$

(1.4.10)

Hence we can state the following:

**Theorem 1.4.2.** If the Lorentzian $\alpha$-Sasakian manifold is generalized recurrent, then the scalar curvature $r$ is related in terms of constant forms $\eta(A)$ and $\eta(B)$ as given by (1.4.10).

### 1.5 Nature of The 1-form $P$ and $Q$ on a generalized Ricci recurrent Lorentzian $\alpha$-Sasakian manifold

A Riemannian manifold is generalized Ricci recurrent manifold [38], if

$$(\nabla X S)(Y, Z) = P(X)S(Y, Z) + (n - 1)Q(X)g(Y, Z).$$

(1.5.1)

Taking $Z = \xi$ in (1.5.1), we have

$$(\nabla X S)(Y, \xi) = P(X)S(Y, \xi) + (n - 1)Q(X)g(Y, \xi).$$

(1.5.2)

The left hand side of (1.5.2), clearly can be written in the form

$$(\nabla X S)(Y, \xi) = XS(Y, \xi) - S(\nabla X Y, \xi) - S(Y, \nabla X \xi),$$

which in view of (1.1.4), (1.1.6) and (1.1.12) gives

$$= (n - 1)\nabla X \eta(Y) - (n - 1)\eta(\nabla X Y) + \alpha S(Y, \phi X)$$

$$= (n - 1)[\nabla X g(Y, \xi) - g(\nabla X Y, \xi)] + \alpha S(Y, \phi X)$$

$$= -(n - 1)\alpha g(Y, \phi X) + \alpha S(Y, \phi X)$$
while the right hand side of (1.5.2) equals

\[ P(X)S(Y, \xi) + (n - 1)Q(X)g(Y, \xi) \]

\[ = (n - 1)P(X)\eta(Y) + (n - 1)Q(X)\eta(Y). \]

Hence

\[ -\alpha(n - 1)g(Y, \phi X) + \alpha S(Y, \phi X) = \alpha(n - 1)P(X)\eta(Y) + (n - 1)Q(X)\eta(Y) \]  

(1.5.3)

Taking \( Y = \xi \) in (1.5.3) and then using (1.1.1), (1.1.4) and (1.1.12), we get

\[ \alpha P(X) + Q(X) = 0. \]  

(1.5.4)

for all \( X \). This leads us to the following:

Theorem 1.5.1. There exists no generalized Ricci recurrent Lorentzian \( \alpha \)-Sasakian manifold if \( \alpha P + Q \) is not everywhere zero.

### 1.6 Lorentzian \( \alpha \)-Sasakian manifold satisfying

\[ S(X, \xi) \cdot R = 0 \]

We now consider an Lorentzian \( \alpha \)-Sasakian manifold satisfying the condition

\[ (S(X, \xi) \cdot R)(U, V)Z = 0. \]  

(1.6.1)

By definition we have

\[ (S(X, \xi) \cdot R)(U, V)Z = ((X \wedge_S \xi) \cdot R)(U, V)Z \]  

(1.6.2)

\[ = (X \wedge_S \xi)R(U, V)Z + R((X \wedge_S \xi)U, V)Z + R(U, (X \wedge_S \xi)V)Z + R(U, V)(X \wedge_S \xi)Z, \]
where the endomorphism $X \wedge S Y$ is defined by

$$(X \wedge S Y)Z = S(Y, Z)X - S(X, Z)Y \tag{1.6.3}$$

Using (1.6.3) in (1.6.2) we get by virtue of (1.1.12), that

$$(S(X, \xi) \cdot R)(U, V)Z$$

$$= 2na^2[\eta(R(U, V)Z)X + \eta(U)R(X, V)Z$$

$$+ \eta(V)R(U, X)Z + \eta(Z)R(U, V)X]$$

$$- S(X, R(U, V)Z)\xi - S(X, U)R(\xi, V)Z$$

$$- S(X, V)R(U, \xi)Z - S(X, Z)R(U, V)\xi$$

Which in view of (1.1.1) and (1.6.4) we have

$$2na^2[\eta(R(U, V)Z)X + \eta(U)R(X, V)Z$$

$$+ \eta(V)R(U, X)Z + \eta(Z)R(U, V)X]$$

$$- S(X, R(U, V)Z)\xi - S(X, U)R(\xi, V)Z$$

$$- S(X, V)R(U, \xi)Z - S(X, Z)R(U, V)\xi = 0.$$
On Lorentzian $\alpha$-Sasakian manifolds

Putting $U = Z = \xi$ in (1.6.6) and then using (1.1.1)-(1.1.7), we get

$$S(X, V) = -2n\alpha^2 g(X, V) - 4n\alpha^2 \eta(X)\eta(V).$$  

(1.6.7)

which says that the manifold is $\eta$-Einstein. This leads to the following:

**Theorem 1.6.1.** An Lorentzian $\alpha$-Sasakian manifold satisfying the condition $S(X, \xi) \cdot R = 0$ is a $\eta$-Einstein manifold.

### 1.7 3-dimensional Lorentzian $\alpha$-Sasakian manifold with Cyclic-Parallel Ricci tensor

Let us suppose that 3-dimensional Lorentzian $\alpha$-Sasakian manifold satisfies the cyclic-Parallel Ricci tensor. Then we have [11],

$$(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0.$$  

(1.7.1)

Putting $Y = Z = e_i$ in (1.7.1) and taking summation over $i$, $1 \leq i \leq 3$, we get

$$(\nabla_X S)(e_i, e_i) + 2(\nabla_{e_i} S)(e_i, X) = 0.$$  

(1.7.2)

Now

$$(\nabla_X S)(e_i, e_i) = \nabla_Y S(e_i, e_i) - 2S(\nabla_{e_i} e_i, e_i).$$  

(1.7.3)

We know that the scalar curvature $r = \sum_i S(e_i, e_i)$. Also in local co-ordinates $\nabla_X e_i = X^j \Gamma^a_{ji} e_a$, where $\Gamma^a_{ji}$ are the christoffel symbols. Since $e_i$ are orthonormal basis, the metric tensor $g_{ij} = \delta_{ij}$, where $\delta_{ij}$ is the Kronecker delta and hence the Christoffel symbols are zero. Therefore, $\nabla_X e_i = 0.$
Hence from (1.7.3) it follows that

\[(\nabla_X S)(e_i, e_i) = \nabla_X r = dr(X). \tag{1.7.4}\]

Let \(Q\) be the Ricci operator defined by \(g(QX, Y) = S(X, Y)\), that is, \(Q\) is the \((1,1)\) Ricci tensor. Then

\[(\nabla Z S)(X, Y) = g((\nabla Z Q)(X), Y). \tag{1.7.5}\]

Taking \(Y = Z = e_i\) in (1.7.5) and taking summation over \(i\), \(1 \leq i \leq 3\), we get

\[(\nabla_{e_i} S)(X, e_i) = g((\nabla_{e_i} Q)(X), e_i). \tag{1.7.6}\]

We know that

\[(\text{div} Q)(X) = tr(Z \rightarrow (\nabla Z Q)(X))\]

\[= \sum_i g((\nabla_{e_i} Q)(X), e_i) \tag{1.7.7}\]

But it is known that \((\text{div} Q)(X) = \frac{1}{2} dr(X)\).

Hence,

\[(\nabla_{e_i} S)(X, e_i) = \frac{1}{2} dr(X). \tag{1.7.8}\]

Now using (1.7.4) and (1.7.8) in (1.7.2) we obtain,

\[dr(X) = 0, \text{ for all } X, \tag{1.7.9}\]

which implies that \(r\) is constant.

From (1.1.16), we have

\[(\nabla Z S)(X, Y) = \frac{dr(Z)}{2} [g(X, Y) - \eta(X)\eta(Y)]\]

\[= \left[\frac{r}{2} + 3\right] [\eta(Y)(\nabla Z \eta)(X) + \eta(X)(\nabla Z \eta)(Y)] \tag{1.7.10}\]
Using (1.7.9) in (1.7.10), we have

\[(\nabla_z S)(X, Y) = - \left[ \frac{r}{2} + 3 \right] \left[ \eta(Y)(\nabla_{\eta} \eta)(X) + \eta(X)(\nabla_{\eta} \eta)(Y) \right] \tag{1.7.11} \]

By virtue of (1.7.11), we get from (1.7.1) that

\[\left[ \frac{r}{2} + 3 \right] \left[ \eta(Z)g(\phi X, Y) + \eta(Y)g(\phi X, Z) + \eta(X)g(\phi Y, Z) \right. \]
\[\left. - \eta(Z)g(\phi Y, X) + \eta(Y)g(\phi Z, X) + \eta(X)g(\phi Z, Y) \right] = 0 \tag{1.7.12} \]

Putting \(Y = Z = e_i\) in (1.7.12), we get

\[ [r + 6\alpha^2] \eta(X) = 0, \]

which implies that \(r = -6\alpha^2\).

This leads the following theorem

**Theorem 1.7.1.** If a 3-dimensional Lorentzian \(\alpha\)-Sasakian manifold satisfies the condition (1.7.1), then the manifold is of constant negative curvature.

### 1.8 Conclusion

- A \(W_t\)-semisymmetric Lorentzian \(\alpha\)-Sasakian manifold is of constant curvature.

- If the Lorentzian \(\alpha\)-Sasakian manifold is generalised recurrent, then the associated vector field \(\xi\) is constant.

- If the Lorentzian \(\alpha\)-Sasakian manifold is generalized recurrent, then the scalar curvature \(r\) is related in terms of constant forms \(\eta(A)\) and \(\eta(B)\) as given by (1.4.10).

- There exists no generalized Ricci recurrent Lorentzian \(\alpha\)-Sasakian manifold if \(\alpha P + Q\) is not everywhere zero.
• An Lorentzian $\alpha$-Sasakian manifold satisfying the condition $S(X, \xi) \cdot R = 0$ is an $\eta$-Einstein manifold.

• If a 3-dimensional Lorentzian $\alpha$-Sasakian manifold satisfies the condition (1.7.1), then the manifold is of constant negative curvature.