Chapter 1

Introduction
1.1 History and Survey of Thermoelasticity

Thermoelasticity is the change in the size and shape of a solid object as the temperature of that object fluctuates. Materials that are more elastic will expand and contract more than those materials that are more inelastic. Scientists use their understanding of thermoelasticity to design materials and objects that can withstand fluctuations in temperature without breaking.

Scientists have understood the equations that describe thermoelasticity for over 100 years but have only recently begun stress testing materials in order to determine how thermoelastic they are. By subjecting materials to rising and falling temperatures, engineers are able to predict how much these materials will expand or contract at different temperatures. This knowledge is important when building machines or weight bearing structures with pieces that need to fit closely together. Understanding the principles of thermoelasticity helps engineers design things that maintain their structural integrity for a range of temperatures.

The principles of thermoelasticity have affected the way engineers design a number of different objects. Knowing that concrete expands when it is heated, for instance, is the reason that sidewalks are designed with small spaces between the slabs. Without these spaces, the concrete would have no room to expand, causing a great
deal of stress on the material, and leading to cracks, breaks or holes. Likewise, bridges are designed with expansion joints to allow for the components to expand as they are heated.

The theory of thermoelasticity has long history, its foundation has been laid in the starting of 19th century by Duhamel and Neumann. The main aspects and problems of thermoelasticity are started appearing in the scientific literature just four decades ago. The study of the effect of the heat stress for deformable solids started at 1838 with Duhamel articles [1]-[2]. Similar approach was used by Neumann [3] forty seven years later. The main offset of these two theories is their not acceptance of the deformation influence on the thermal conduction. Effort in this direction was made by Voigt [4] and also by Jeffreys [5], but they used the equilibrium thermodynamics and consequently their theories are limited for quasi equilibrium processes. The classical theory of the thermoelasticity, based on the irreversible thermodynamics of the Groot [6], was offered by Biot [7]. However the theory of Biot is founded on the Fourier law [8] for heat conduction and therefore allows for infinite speed of propagation of thermal perturbations, which is physically unrealistic. An attempt, to clean this paradox, was done by Green and Lindsay [11] in their model for thermoelastic media.

A challenging problem faced by applied mathematicians & engineers is to find the solution of the equations arising in their fields of study,
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which in a large number of cases are partial differential equations. There could be nothing more desirable than to find exact solution of this equations. However due to the complexity of the exact solution one as to adopt numerical techniques to obtain approximate solution. One such field of applied mathematics is thermoelasticity. The mathematical foundation of thermoelasticity based on the famous paper entitled “Memorie sur le calcul des actions moteculairs developpees par les changement de temperature dans les corps solid” by Duhamel [1]. In his memoire he first formulated the fundamental equation of thermoelasticity taking into consideration the deformation produced by the temprature changes. Duhamel’s analysis showed that the stresses produced by the temperature gradient can be calculated separately from those produced by mechanical forces and that the total stresses can be obtained by superposition. This seem to be the first use of superposition in stress analysis. Duhamel applied his famous fundamental equations to several specific cases, including the circular cylinder and the subjected to radially varying temperature.

The conventional quasi-static approach to thermoelastic problems in the presence of time dependent temperature fields (and under stationary loading condition) rest on the assumption that the inertia terms may be neglected in the governing field equation. This hypothesis, which goes back to Duhamel [2], is known to yield to useful results in a wide variety of applications. It is evident whoever, that
the quality of the approximation must depend both on the size of the relevant intrinsic inertia parameters and on the nature of the time variations inherent in the temperature distribution. If, in particular, the temperature field exhibits sufficiently steep time-gradient, the dynamic effects disregarded in the traditional treatment of the problem may be expected to become significant. Moreover, when the inertia terms are taken into account, the entire character of the problem is altered, the process of transmission of the thermal stresses is then no longer purely diffusive but involves propagation of elastic waves.

The first attempt to examine the inertia effects in a transient thermoelastic problem is apparently due to Danilovskaya [12]. The particular problem considered in [12] concerns an elastic half space that is free from loading is subjected to a suddenly applied, and thereafter steadily maintained, uniform change of the surface temperature over its entire plane boundary. Throughout the interior of the medium, the temperature distribution is assumed to obey the heat-conduction equation, in the absence of thermoelastic coupling. The ensuing thermal stresses are determined rigorously within classical is constrained against lateral displacement. Mura [13] independently arrived at the same results, while Danilovskaya [12] generalized her previous solution to accommodate convective boundary conditions. None of the investigations cited so far includes the determination of the accompanying thermal displacements the conclusion reached
in [12] and [13] involve qualitative and sizable quantitative, inertia corrections, which invalidate Duhamel’s for the problem discussed there.

**Sternberg** and **Charavorty** [14] obtained the missing displacements in the closed form as in [12] and showed that the dynamic effects induced by gradual heating or cooling of the boundary are substantially different from those encountered in [12] and [13], even at extremely high, but finite, time gradients of the surface temperature.

Inertia terms has been taken into account in several thermoelastic investigations since the appearance of Danilovskaya’s original paper [12]. Thus **Boley** and **Weiner** [15] studied thermally induced beam and plate vibration. A particularly important contribution to the subject under discussion is due to **Nowacki** [16] who obtained several closed exact solution to the uncoupled three-dimensional thermoelastic equations of motion, corresponding to the time-dependent heat source in the interior medium which occupies the entire space. Here the mention a purely formal dynamic treatment by **Ignaczak** [17] of the thermoelastic problem for the half-space, in the presence of time-dependent heat source at the boundary will not be out of context.
The theory of thermoelasticity begin with \textbf{uncoupled} thermoelasticity. This classical uncoupled thermoelasticity predicts two phenomena not compatible with physical observations. First, the equation of heat conduction of this theory does not contain any elastic terms. Second, the heat equation is of parabolic type, predicting infinite speeds of propagation for heat waves. Biot \cite{7} introduced the theory of \textbf{coupled} thermoelasticity to overcome the first shortcoming. The governing equation of this theory is coupled, eliminating the first paradox of classical theory. However both theories share the second shortcoming. Due to the advancement of pulsed lasers, fast burst nuclear reactors and particle accelerators etc. which can supply heat supply heat pulses with very fast time rise the generalized thermoelasticity theory receive serious attention. Mainly two different models of generalized thermoelasticity are extensively used, one proposed by Lord and Shulman \cite{18} called as \textbf{L-S theory} and the other proposed by Green and Lindsay \cite{11} called as \textbf{G-L theory}. L-S theory suggests one relaxation time and according to this theory, only Fourier’s heat conduction equation is modified, while G-L theory suggest two relaxation times and both the energy equation and equation of motion are modified.

The thermoelastic problems are specialize mainly in two types direct thermoelastic problems and inverse thermoelastic problems. Depending upon external and internal boundary conditions of heat transfer under consideration respectively. This direct and inverse
problem includes in the uncoupled thermoelasticity. These problems consist of determination of unknown temperature, heat flux, displacement, deflection, strain and thermal stress functions of solids when the conditions of temperature, displacement, deflection and stress are known at the some points of the solid under consideration. In a great number of the papers on thermoelasticity, an approach was taken to derive the temperature, displacement and thermal stresses in different types solids due to heating. The problems of thermoelasticity which consists of conditions of temperature on outer surface of solid under consideration in view to solve the heat conduction equation called as **Direct Thermoelastic Problems**, such type of problems have been studied by many authors \[19\]–[63]. Also the problems of thermoelasticity which consists of conditions of temperature in interior of solid under consideration in view to solve the heat conduction equation are called as **Inverse Thermoelastic Problems**. This inverse problems have begun to attract deal of attention in the recent literature as in case of heat conduction. The inverse heat conduction problem is one of the most frequently encountered problems by scientists. The wide varieties of problems that are covered under conduction also make it one of the most researched and thought about problems in the field of engineering and technology. This variety of problems can thus be solved in a variety of methods. This inverse problem is very important in view of it’s relevant to various industrial machinery such as main shaft of lathe, turbine and
roll of rolling mills under the heating. Several authors have investigated the this type of thermoelastic problems for various bodies in [64]-[87].

Some authors [88]-[97] were discussed a thermoelastic problems of composite materials. Further more some problems of theory of generalized thermoelasticity was discussed in [98]-[108]. Jaeger et al. [109], Ozisik [110] and Noda et al. [111] have been studied varieties of thermoelastic problems and solved by different methods. Marchi and Zgrablich [112], Sneddon [113], Davies [114] and Miles [115] have put several Integral transforms and discussed their applications to solved various boundary value problems of heat conduction.

During the second half of the twentieth century, non isothermal problems of the theory of Elasticity became increasingly important. This is due to mainly to their many applications in diverse fields. First, the high velocity of modern aircraft give rise to an aerodynamic heating, which produce intense thermal stresses, reducing the strength of aircrafts structure, secondly in the nuclear field, the extremely high temperature and temperature gradients, originating inside nuclear reactor influence their design and operation. The theory of generalized thermoelasticity is more applicable in the aspects of pulsed lasers, fast burst nuclear reactors and particle accelerators in various industries and in scientific research. The problems of thermoelasticity are very important in view of its relevance to various
industrial machines subjected to heating, such as main shaft of lathe, turbines and the roll of rolling mills, analysis experimental data and the measurement of aerodynamic heating.

1.2 Some Useful Results

1.2.1 Laplace Transform

Let $f(t)$ be a continuous function of $t$ defined for $t > 0$, then the Laplace transform of $f(t)$ is defined by

$$
\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt
$$

(1.2.1)

if integral in RHS of (1.2.1) is exists, then it is denoted by $\mathcal{L}\{f(t)\} = \mathcal{L}\{f\}$, and

$$
f(t) = \mathcal{L}^{-1}\{\mathcal{L}\{f(t)\}\} = \int_0^\infty e^{-st} \mathcal{L}\{f\} ds
$$

(1.2.2)

$f(t)$ is called the inverse Laplace transform of $\mathcal{L}\{f(t)\}$, where $s$ is the Laplace transform parameter.

Theorem 1.2.1. If $\mathcal{L}\{f(t)\}$ is an analytic function of complex variable $s$ and is of order $O(s^{-k})$ in some half plane $R(s) \geq c$, where $c, k$ are real and $k > 1$, then the integral

$$
\frac{1}{2\pi i} \lim_{\beta \to \infty} \int_{\gamma-i\beta}^{\gamma+i\beta} e^{st} \mathcal{L}\{f\} ds
$$

(1.2.3)
along any line \( R(s) = \gamma \geq c \) converges to a function \( f(t) \) which is independent of \( \gamma \) and whose Laplace transform is \( \mathcal{L}(f) \). Furthermore, the function \( f(t) \) is a continuous for each \( t > 0 \), \( f(0) = 0 \) and \( f(t) \) is of order \( O(e^{\gamma t}) \) for all \( s \geq 0 \).

**Theorem 1.2.2. Convolution Theorem for Laplace Transform:**

Suppose \( \mathcal{L}(f) \) and \( \mathcal{L}(g) \) are the Laplace transform of the function \( f(t) \) and \( g(t) \) then,

\[
\frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \mathcal{L}(f) \mathcal{L}(g) e^{st} \, ds = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} \mathcal{L}(f) \, ds \int_0^{\infty} g(u) e^{-su} \, du \\
= \int_0^{\infty} g(u) du \times \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{s(t-u)} \mathcal{L}(f) \, ds
\]

but \( \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{s(t-u)} \mathcal{L}(f) \, ds = f(t-u) \),

so that \( \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \mathcal{L}(f) \mathcal{L}(g) e^{st} \, ds = \int_0^{\infty} g(u) f(t-u) du \).

Now \( f(t-u) = 0 \) if \( t-u < 0 \), that is if \( u > t \) and hence the integrand vanishes if \( u > t \) and finally obtains

\[
\frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \mathcal{L}(f) \mathcal{L}(g) e^{st} \, ds = \int_0^{\infty} g(u) f(t-u) du \quad (1.2.4)
\]
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Changing the variable in the integral on the right hand side of equation (1.2.4) from $u$ to $\eta = t - u$ one obtains

$$\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} f(s)\overline{g}(s)e^{st}ds = \int_{0}^{\infty} f(\eta)g(t - \eta)d\eta \quad (1.2.5)$$

1.2.2 Finite Fourier Transform

1.2.2.1 Finite Fourier sine Transform

If $f(x)$ satisfies Dirichelet’s conditions in the interval $(0, a)$, then its finite Fourier sine transform in this interval $(0, a)$ is defined as

$$\overline{f}_s(m) = \int_{0}^{a} f(x) \sin \left( \frac{m\pi x}{a} \right) dx \quad (1.2.6)$$

then at each point of $(0, a)$ at which $f(x)$ is continuous,

$$f(x) = \frac{2}{a} \int_{0}^{a} \overline{f}_s(m) \sin \left( \frac{m\pi x}{a} \right) dx \quad (1.2.7)$$

1.2.2.2 Finite Fourier cosine Transform

If $f(x)$ satisfies Dirichelet’s conditions in the interval $(0, a)$, then its finite Fourier cosine transform in this interval $(0, a)$ is defined as

$$\overline{f}_c(m) = \int_{0}^{a} f(x) \cos \left( \frac{m\pi x}{a} \right) dx \quad (1.2.8)$$

then at each point of $(0, a)$ at which $f(x)$ is continuous,

$$f(x) = \frac{\overline{f}_c(0)}{a} + \frac{2}{a} \int_{0}^{a} \overline{f}_c(m) \cos \left( \frac{m\pi x}{a} \right) dx \quad (1.2.9)$$
1.2.2.3 Properties of Finite sine and cosine Transforms

1. \[ \int_0^a \frac{\partial f}{\partial x} \sin \left( \frac{m\pi x}{a} \right) \, dx = -\frac{m\pi}{a} \mathcal{F}_c(m) \]

2. \[ \int_0^a \frac{\partial f}{\partial x} \cos \left( \frac{m\pi x}{a} \right) \, dx = (-1)^m f(a) - f(0) + \frac{m\pi}{a} \mathcal{F}_s(m) \]

3. \[ \int_0^a \frac{\partial^2 f}{\partial x^2} \sin \left( \frac{m\pi x}{a} \right) \, dx = \frac{m\pi}{a} \left[ (-1)^{m+1} f(a) - f(0) \right] - \frac{m^2\pi^2}{a^2} \mathcal{F}_s(m) \]

4. \[ \int_0^a \frac{\partial^2 f}{\partial x^2} \cos \left( \frac{m\pi x}{a} \right) \, dx = (-1)^m f'(a) - f'(0) - \frac{m^2\pi^2}{a^2} \mathcal{F}_c(m) \]

1.2.3 Finite Hankel Transform

If \( f(x) \) satisfies the Dirichlet’s conditions in the interval \((0, a)\) then its finite Hankel transform in the range \((0, a)\), is defined to be,

\[ \mathcal{F}_\mu(\xi_i) = \int_0^a x f(x) J_\mu(x\xi_i) \, dx \]  \hspace{1cm} (1.2.10)

where \( \xi_i \) is the root of the transcendental equation

\[ J_\mu(a\xi_i) = 0 \]  \hspace{1cm} (1.2.11)
then, at any point of \((0, a)\) at which the function \(f(x)\) is continuous,

\[
f(x) = \frac{2}{a^2} \sum_i J_\mu(\xi_i) \frac{J_\mu(x\xi_i)}{[J_\mu'(a\xi_i)]^2}
\]  

(1.2.12)

where the sum is taken over all the positive roots of the equation \((1.2.11)\)

**Properties of Hankel transform in (1.2.10)**

If \(f(x)\) satisfies the Dirichlet’s conditions in the interval \((0, a)\) then,

1. Finite Hankel transform of \(\frac{\partial f}{\partial x}\) i.e,

\[
H \left[ \frac{\partial f}{\partial x} \right] = \int_0^a \frac{\partial f}{\partial x} x J_\mu(x\xi_i) dx
= \frac{\xi_i}{2\mu} [ (\mu - 1) H_{\mu+1} \{ f(x) \} - (\mu + 1) H_{\mu-1} \{ f(x) \} ]
\]

2. \(H_\mu \left[ \frac{\partial^2 f}{\partial x^2} + \frac{1}{x} \frac{\partial f}{\partial x} \right] = \frac{\xi_i}{2} \left[ -H_{\mu-1} \{ \frac{\partial f}{\partial x} \} + H_{\mu+1} \{ \frac{\partial f}{\partial x} \} \right]
\]

If \(f(x)\) satisfies the Dirichlet’s conditions in the range \(b \leq x \leq a\) and if its finite Hankel transform in that range is defined to be

\[
H[f(x)] = \overline{f}_\mu(\xi_i)
= \int_b^a f(x) [ J_\mu(x\xi_i) G_\mu(a\xi_i) - J_\mu(a\xi_i) G_\mu(x\xi_i) ] dx
\]  

(1.2.13)

where \(\xi_i\) is the root of the transcendental equation

\[
[J_\mu(\xi_i b) G_\mu(\xi_i a) - J_\mu(\xi_i a) G_\mu(\xi_i b)] = 0
\]  

(1.2.14)
then at which the function is continuous

\[
    f(x) = \sum_i \frac{2\xi_i^2 J^2_\mu(b\xi_i) J_\mu(\xi_i)}{J^2_\mu(a\xi_i) - J^2_\mu(b\xi_i)^2} [J_\mu(x\xi_i)G_\mu(a\xi_i) - J_\mu(a\xi_i)G_\mu(x\xi_i)]
\]

(1.2.15)

Property of Hankel transform defined in (1.2.13)

\[
    \int_a^b f(x) \left[ \frac{\partial^2 f}{\partial x^2} + \frac{1}{x} \frac{\partial f}{\partial x} \right] [J_\mu(x\xi_i)G_\mu(a\xi_i) - J_\mu(a\xi_i)G_\mu(x\xi_i)] \, dx
\]

\[
    = -\xi_i^2 J_\mu(\xi_i) + a [J_\mu(x\xi_i)G_\mu(a\xi_i) - J_\mu(a\xi_i)G_\mu(x\xi_i)]_{x=a}
\]

\[
    + b [J_\mu(x\xi_i)G_\mu(a\xi_i) - J_\mu(a\xi_i)G_\mu(x\xi_i)]_{x=b}
\]

\[
    = -\xi_i^2 J_\mu(\xi_i)
\]

(1.2.16)

1.2.4 Finite Marchi-Fasulo Transform

The finite Marchi-Fasulo integral transform of \( f(z) \), in \(-h < z < h\) as in [73] is defined to be

\[
    \overline{F}(n) = \int_{-h}^{h} f(z) P_n(z) \, dz
\]

(1.2.17)

then at each point of \((-h, h)\) at which \( f(z) \) is continuous

\[
    f(z) = \sum_{n=1}^{\infty} \frac{\overline{F}(n)}{\lambda_n} P_n(z)
\]

(1.2.18)

where \( P_n(z) = Q_n \cos(a_n z) - W_n \sin(a_n z) \)

\[
    Q_n = a_n (\alpha_1 + \alpha_2) \cos(a_n h) + (\beta_1 - \beta_2) \sin(a_n h)
\]
\[ W_n = (\beta_1 + \beta_2) \cos(a_n h) + a_n(\alpha_2 - \alpha_1) \sin(a_n h) \]

\[ \lambda_n = \int_{-h}^{h} P_n^2(z)dz = h[Q_n^2 + W_n^2] + \frac{\sin(2a_n h)}{2a_n}[Q_n^2 - W_n^2] \]

### 1.2.5 Finite Marchi-Zgrablich Integral Transform

Let us consider the Bessel’s differential equation of order \( p \) as

\[
\frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + \left[ \mu^2 - \frac{p^2}{x^2} \right] y = 0 \quad (1.2.19)
\]

with the boundary conditions

\[
y(a) + k_1 y'(a) = 0; \quad y(b) + k_2 y'(b) = 0 \quad (1.2.20)
\]

The general solution of equation (1.2.19) is

\[
y(x) = C_1 J_p(\mu x) + C_2 G_p(\mu x) \quad (1.2.21)
\]

where \( J_p(\mu x) \) and \( G_p(\mu x) \) are the Bessel’s functions of first and second kind respectively of order \( p \).

Using equation (1.2.20) and equation (1.2.21) one can obtains

\[
C_1 J_p(k_1, \mu a) = -C_2 G_p(k_1, \mu a); \quad C_1 J_p(k_2, \mu b) = -C_2 G_p(k_2, \mu b)
\]

(1.2.22)

where

\[
J_p(k_i, \mu x) = J_p(\mu x) + k_i \mu J_p'(\mu x); \quad G_p(k_i, \mu x) = G_p(\mu x) + k_i \mu G_p'(\mu x)
\]

(1.2.23)
From equation (1.2.22) we see that solution is exist only if

\[ J_p(k_1, \mu_a) G_p(k_2, \mu_b) - J_p(k_2, \mu_b) G_p(k_1, \mu_a) = 0 \quad (1.2.24) \]

where \( \mu_n \) are the positive roots of equation (1.2.24). Substituting equation (1.2.22) in (1.2.21) one obtains,

\[
Y_n(x) = \frac{C_1}{G_p(k_1, \mu_n a)} \{ J_p(\mu_n x) G_p(k_1, \mu_n a) - G_p(\mu_n x) J_p(k_1, \mu_n a) \}
\]

and

\[
Y_n(x) = \frac{C_2}{G_p(k_2, \mu_n b)} \{ J_p(\mu_n x) G_p(k_2, \mu_n b) - G_p(\mu_n x) J_p(k_2, \mu_n b) \}
\]

Using equation (1.2.25) and (1.2.26) one obtains,

\[
S_p(k_1, k_2, \mu_n x) = J_p(\mu_n x) \{ G_p(k_1, \mu_n a) + G_p(k_2, \mu_n b) \} - G_p(\mu_n x) \{ J_p(k_1, \mu_n a) + J_p(k_2, \mu_n b) \}
\]

which are the solution of Bessel’s differential equation of order \( p \) and satisfy the boundary condition (1.2.20), because of this fact, such functions are orthogonal in the interval (a, b).

Now let us define the finite integral transform as in [112]

\[
\overline{f}_p(n) = \int_a^b x f(x) S_p(k_1, k_2, \mu_n x) dx \quad (1.2.28)
\]
and its inversion as

\[ f(x) = \sum_{n=1}^{\infty} \frac{\overline{f}_p(n) S_p(k_1, k_2, \mu_n x)}{C_n} \quad (1.2.29) \]

where

\[ C_n = \frac{b^2}{2} \left\{ S_p^2(k_1, k_2, \mu_n b) - S_{p-1}(k_1, k_2, \mu_n b) \cdot S_{p+1}(k_1, k_2, \mu_n b) \right\} \]

\[ - \frac{a^2}{2} \left\{ S_p^2(k_1, k_2, \mu_n a) - S_{p-1}(k_1, k_2, \mu_n a) \cdot S_{p+1}(k_1, k_2, \mu_n a) \right\} \quad (1.2.30) \]

Equations (1.2.28) and (1.2.29) define the finite Marchi-Zgrablich integral transform of order \( p \) and its inverse transform respectively.

Further we include an operational property

\[ \int_a^b x \left\{ \frac{\partial^2 f}{\partial x^2} + \frac{1}{x} \frac{\partial f}{\partial x} - \frac{p^2}{x^2} f \right\} S_p^2(k_1, k_2, \mu_n x) \]

\[ = \frac{b}{k_2} S_p(k_1, k_2, \mu_n b) \left\{ f + k_2 \frac{\partial f}{\partial x} \right\}_{x=b} \]

\[ - \frac{a}{k_1} S_p(k_1, k_2, \mu_n a) \left\{ f + k_1 \frac{\partial f}{\partial x} \right\}_{x=a} \]

\[ - \mu_n^2 \overline{f}_p(n) \quad (1.2.31) \]
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