Preface

Life is Good for only two Thing,

Discovering Mathematics and Teaching Mathematics.

Siemon Denis Poisson
1781 - 1840.

The purpose of this study is to give a presentation of some known and some new results on ideals of a lattice $L$ with the least element $0$. This thesis initiates the study of extending ideals in a lattice $L$. We mainly concentrate on dimension theory and extending property of ideals of a lattice $L$ with the least element $0$. We also give few important characterizations when $L$ is a modular lattice.

The thesis consists of two parts. In the first part we concentrate on the dimension theory of lattices and an analogue of extending modules for ideals of a lattice. In the second part we consider pseudo rank functions on multiplicative lattices.

In the first chapter, some background material is given together with some basic definitions and results of lattice theory, which are used in subsequent sections.

The second chapter is concerned with the structure and properties of essential extensions of ideals of a lattice $L$ with 0. In this chapter we define an essential ideal, a max-semicomplement of an ideal and a closed ideal in a lattice $L$. We show that every ideal has a maximal essential extension and a max-semicomplement in a lattice $L$. We prove that the direct sum of an ideal $I$ and its max-semicomplement $J$ is essential in $L$. In addition, we have also shown that an ideal $I$ in a lattice $L$ is closed in $L$ if and only if $I$ is a max-semicomplement in $L$.

The third chapter focuses on finite uniform dimension of lattices with respect to its ideals. In the second section, we define an uniform ideal in a lattice $L$. We show
that if \( I, J \) are ideals of a lattice \( L \) such that \( I \subseteq J \) and \( I \) is uniform then \( J \) is an uniform ideal of \( L \).

In the third section, we define finite uniform dimension or finite Goldie dimension of a lattice with respect to its ideals. We show that if \( L \) is modular and has finite uniform dimension then every nonzero ideal of \( L \), contains an uniform ideal. We prove a structure theorem for finite Goldie dimension of a modular lattice, which states that “a lattice \( L \) with finite Goldie dimension contains a finite number of uniform ideals \( I_1, I_2, \ldots, I_n \) whose sum is direct and essential in \( L \). We also give some characterizations for finite Goldie dimension of a modular lattice.

In the fourth chapter chapter, we formulate the concept of \( J \)-injective ideal, extending ideals and \( J \)-ojective ideals in lattices.

In the second section, we define \( J \)-injective ideals and study some of its properties. In the third section, we study extending ideals in lattices. We show that every direct summand of an extending ideal is extending. In the fourth section, we define an exchangeable decomposition and internal exchange property for direct summands of an ideal of a lattice.

In the fifth section, we investigate \( J \)-ojective ideal in a lattice. We prove that mutual ojectivity is necessary and sufficient condition for a direct sum of extending ideals to be extending in a modular lattice. Some other characterizations are given for \( J \)-ojective ideals in lattices.

In the fifth chapter, we continue our study of extending ideals. In the second section, we introduce a generalization of \( J \)-ojective ideals, namely \( J \)-jective ideal in a lattice. We also give some basic results on \( J \)-jective ideals. In the third section, we discuss and study a generalization of extending ideals, namely generalized extending ideals.

In the sixth chapter, we mainly concentrate on Goldie extending ideals. In the second section, we define socle of an ideal in a lattice. We also prove analogous of some results on socle of modules for ideals of a lattice.

In the third section, we study a generalization of \( J \)-injective ideals, namely \( J \)-jective ideal in a lattice and prove some of its characterizations. In forth section, we define Goldie extending ideals, which is a generalization of extending ideal. We give some important characterizations of Goldie extending ideals in a modular lattice.

This concludes Part-I.
In the **seventh chapter**, the only chapter in Part-II, we formulate the concept of a pseudo rank function on a multiplicative lattice. It is shown that a pseudo rank function on a multiplicative lattice $L$ can be constructed from some given ones.

In the third section, we consider the set $\mathbb{P}(L)$ of all pseudo rank functions on $L$, as a subset of the real vector space $\mathbb{R}^{L} = \{ f \mid f : L \rightarrow \mathbb{R} \}$ equipped with the product topology. We show that the set $\mathbb{P}(L)$ is a compact convex subset of $\mathbb{R}^{L}$. We give some characterizations of the set of all pseudo rank functions on a multiplicative lattice $L$.

Papers based on the results of Chapter 3, 4, 5 and 6 are communicated and a paper based results of Chapter 7 is published in *Bulletin of Calcutta Mathematical Society*, 104 (2) (2012), 147 - 160.

Propositions, Lemmas, Theorems, Remarks etc. are all numbered section wise in each chapter. Figures are numbered serially and chapter wise. References are given at the end of the Thesis and are listed alphabetically by last name of the author. The end of the proof is indicated by rectangular box ( ◯ ).

An index of definitions and list of publications is given after references.