Chapter 1

Introduction

The theory of valuations was started in 1912 by the Hungarian mathematician Josef Kürschák who formulated the valuation axioms. These were real valuations. His main motivation was to provide a solid foundation for the theory of $p$-adic fields introduced by Kurt Hensel. In 1932, Krull gave a more general definition of valuation which turned out to be applicable in other mathematical branches such as algebraic geometry and functional analysis.

In this thesis, by a valuation $v$ of a field $K$ we mean a Krull valuation, i.e., $v$ is a mapping from $K$ onto $\Gamma \cup \{\infty\}$, where $\Gamma$ is a totally ordered (additively written) abelian group, such that for all $a, b$ in $K$, the following properties are satisfied:

(i) $v(a) = \infty$ if and only if $a = 0$;
(ii) $v(ab) = v(a) + v(b)$;
(iii) $v(a + b) \geq \min\{v(a), v(b)\}$.

The pair $(K, v)$ is called a valued field and $\Gamma$ the value group of $v$. The subring $R_v = \{a \in K | v(a) \geq 0\}$ of $K$ is called the valuation ring of $v$. It has a unique maximal ideal given by $M_v = \{a \in K | v(a) > 0\}$. $R_v/M_v$ is called the residue field of $v$. The order type of the chain of all convex subgroups different from $\Gamma$ of the value group $\Gamma$ of $v$ is called the rank of $v$. Rank one valuations will be referred to as real valuations in view of the fact that a totally ordered abelian group $\Gamma$ is of rank one if and only if it is order isomorphic to a non-trivial subgroup of $(\mathbb{R}, +)$ (see [En-Pr, Proposition 2.1.1]). The valuations which occur in number theory have
value group isomorphic to the group $\mathbb{Z}$ of integers; such valuations are known as
discrete valuations.

Let $v$ be a valuation of a field $K$. A valuation $v'$ of an overfield $K'$ of $K$ which coincides with $v$ on $K$ is called an extension or a prolongation of $v$ to $K'$; in this situation the residue field $R_v/M_v$ may be regarded as a subfield of $R_{v'}/M_{v'}$ and the degree of the extension $R_{v'}/M_{v'}$ over $R_v/M_v$ is called the residual degree of $v'/v$. If $\Gamma \subseteq \Gamma'$ are the value groups of $v, v'$ respectively, then the index of $\Gamma$ in $\Gamma'$ is called
the index of ramification of $v'/v$. A valued field $(K, v)$ is said to be henselian if $v$ has a unique prolongation to the algebraic closure of $K$. Henselian fields got their name from the fact that Hensel’s Lemma holds in such fields; moreover, these fields are characterized by the validity of this lemma which has several equivalent statements (cf. [Rib1], [En-Pr, Theorem 4.1.3]). Henselian fields form a very important class of valued fields and have been widely studied. It was Ostrowski [see [Roq, p.342]) who introduced ‘Henselian fields’ though he termed these as ‘relatively complete’. The main development of valuation theory before Ostrowski was influenced by its applications to number fields. Thus completions played an important role in this theory. Ostrowski observed that for studying the algebraic properties of valuations, it is not completeness that is important but rather the validity of Hensel’s Lemma.

Complete discrete valued fields satisfy a crucial property that if $(K, v)$ is such a field and $(K', v')$ is a finite extension of $(K, v)$, then the degree of the extension $K'/K$ is the product of the index of ramification and the residual degree of $v'/v$ (cf. [Rib2, §6.1, Theorem A], [End2, Theorem 18.8], [Iya, Chapter 2, Theorem 4.7]). It is also known that this property does not hold for finite extensions of complete non-discrete rank one valued fields in general (see [Rib2, Chapter 6, Example 2]). This led to the notion of defectless extensions. A finite extension $(K', v')$ of a henselian valued field $(K, v)$ is said to be defectless if $[K' : K] = ef$ where $e, f$ are the index of ramification and the residual degree of $v'/v$. In this thesis, we shall study properties of simple defectless extensions of $(K, v)$, i.e., those defectless extensions which are generated over $K$ by a single element.
Valuations have been extensively used to prove the irreducibility of polynomials over the field of rational numbers and over general valued fields and to study the properties of these polynomials (cf. [Col], [Pa-St], [Po-Za], [Kh-Sa], [Bh-Kh], [Zah], [F-K-P], [Bro2], [Bro3], [En-Kh], [Br-Me], [Sh-Ti], [Bo-Za]). In this thesis, we investigate certain interesting problems arising out of the study of henselian valued fields and also work on some problems of irreducibility of polynomials over valued fields.

Let \((K, v)\) be a henselian valued field of arbitrary rank which is not separably closed. Let \(k\) be a subfield of \(K\) of finite codimension (i.e., \(K/k\) is a finite extension) and \(v_k\) be the valuation obtained by restricting \(v\) to \(k\). In Chapter 2, we give some necessary and sufficient conditions for \((k, v_k)\) to be henselian. The motivation behind this problem is a paper of Bevelacqua and Motley [Be-Mo] published in 2006 wherein they characterize those complete rank one valued fields \((K, v)\) whose each subfield of finite codimension is complete. This led us to consider a similar problem for henselian valued fields. Recall that a valuation \(w\) of a field \(K\) with valuation ring \(R_w\) is called a proper coarsening of a valuation \(v\) of \(K\) if \(R_w \supsetneq R_v\). In this situation, the valuation \(v\) of \(K\) is said to be composed of \(w\) and the valuation \(\bar{v}\) induced by \(v\) on the residue field \(\overline{K_w}\) of \(w\); in fact the valuation ring of \(\bar{v}\) is \(R_v/M_w\) and \(v\) is written as \(w \circ \bar{v}\) (cf. [En-Pr, §2.3])

With the above notations, the following is the main result of Chapter 2.

**Theorem 1.1.** Let \(v\) be a henselian valuation of arbitrary rank of a field \(K\) which is not separably closed. Let \(k\) be a subfield of \(K\) of finite codimension with valuation denoted by \(v_k\) obtained by restricting the given valuation to \(k\).

Then the following statements are equivalent:

(i) \((k, v_k)\) is henselian.

(ii) \(k\) is dense in the henselization \(k^h\) of \((k, v_k)\).

(iii) For each valuation \(w\) of \(K\) which is a proper coarsening of \(v = w \circ \bar{v}\), the residue field \(\overline{K_w}\) of \(w\) restricted to \(k\), is henselian with respect to the restriction of \(\bar{v}\) to \(\overline{K_w}\).
Whenever $w$ is a proper coarsening of $v$ such that the residue field of $w$ is an algebraically closed field of characteristic zero, then the restriction of $\bar{v}$ to $\overline{K}_w$ has a unique prolongation to $\overline{K}_w$.

The results stated below are obtained as applications of Theorem 1.1.

**Corollary 1.2.** Let $(K, v)$ be as in Theorem 1.1. If there exists no proper coarsening $w$ of $v$ such that the residue field of $w$ is an algebraically closed field of characteristic zero, then each finite codimensional subfield of $(K, v)$ is henselian.

**Corollary 1.3.** Let $(K, v)$ be as above. If the characteristic of $K$ is $p > 0$ or rank $v$ is one, then every finite codimensional subfield of $(K, v)$ is henselian.

The paper [Kh-Kh2] containing the proofs of Theorem 1.1 and its corollaries has appeared in *Colloquium Mathematicum*, Vol. 120, No. 1 (2010), 157-163.

Chapters 3, 4 and 5 deal with the irreducibility problem of polynomials over valued fields. It is well-known that if $f(x) = x^d + a_1 x^{d-1} + \cdots + a_d$ belonging to $K[x]$ is irreducible over a complete field $K$ equipped with a real valuation $v$, then there exists a positive real number $\epsilon$ (depending on $f$) such that any monic polynomial $g(x) = x^d + b_1 x^{d-1} + \cdots + b_d$ belonging to $K[x]$ with $v(b_j - a_j) > \epsilon$ for $1 \leq j \leq d$, is also irreducible over $K$ (see [Art, Chapter 2, Theorem 11]). In 2004, Zaharescu [Zah] proved a more general result in this direction for polynomials with coefficients in valued fields of characteristic zero that are not necessarily complete but where the lack of completeness is compensated by the presence of a secondary valuation which satisfies a certain property in connection with the given primary valuation. In Chapter 3, we have extended Zaharescu’s result to polynomials with coefficients in arbitrary valued fields without assuming any condition on the characteristic of $K$. The difficulty arising in the characteristic $p > 0$ case has been overcome with the help of a result of Ershov proved in 2006.

Precisely stated, we have proved

**Theorem 1.4.** Let $K$ be a field equipped with two non-archimedean valuations $v_1$ and $v_2$ of arbitrary rank having value groups $\Gamma_1$ and $\Gamma_2$ respectively. Let $A$ be a
subring of $K$ with field of fractions $K$ which is integrally closed in $K$ and $\tilde{A}$ be the integral closure of $A$ in the algebraic closure $\tilde{K}$ of $K$. Let $\tilde{v}_1$ and $\tilde{v}_2$ be valuations on $\tilde{K}$ whose restrictions to $K$ coincide with $v_1$ and $v_2$ respectively. Assume that for any $\beta \in \tilde{A} \setminus A$ and $\lambda_2 \in \Gamma_2$, there exists an element $\lambda_1 \in \Gamma_1$ such that

$$\tilde{v}_1(u - \beta) \leq \lambda_1 \text{ for all } u \in A \text{ with } v_2(u) \geq \lambda_2.$$ 

Then for any polynomial $f(x) = x^d + a_1x^{d-1} + \cdots + a_d \in A[x]$ which is irreducible over $K$ and any $\lambda_2 \in \Gamma_2$, there corresponds $\lambda_1 \in \Gamma_1$ depending upon $f$ and $\lambda_2$ such that for any $b_1, b_2, \ldots, b_d \in A$ satisfying

$$v_1(b_1 - a_i) \geq \lambda_1, \quad 1 \leq i \leq d,$$

and

$$v_2(b_1) \geq \lambda_2, \quad 1 \leq i \leq d,$$

the polynomial $g(x) = x^d + b_1x^{d-1} + \cdots + b_d$ is irreducible over $K$.

As an application of Theorem 1.4, the following result has been deduced.

**Theorem 1.5.** Let $K_0$ be a field complete with respect to a real valuation $v_0$. Let $f(x, y) = x^d + P_1(y)x^{d-1} + \cdots + P_d(y)$ be an irreducible polynomial in two variables over $K_0$. Let $v_0^g$ denote the Gaussian extension of $v_0$ to $K_0(y)$ defined by $v_0^g(\sum a_iy^i) = \min\{v_0(a_i) | a_i \in K_0\}$. Then given any integer $M$, there exists $N > 0$ (depending upon $f$ and $M$) such that whenever $Q_i(y), \ 1 \leq i \leq d$, are polynomials over $K_0$ satisfying (i) degree $Q_i(y) \leq M$, (ii) $v_0^g(\sum Q_i(y) - P_i(y)) > N$, then $g(x, y) = x^d + Q_1(y)x^{d-1} + \cdots + Q_d(y)$ is irreducible over $K_0$.

Theorems 1.4 and 1.5 are proved in the paper [Kh-Kh1] which has appeared in Indian Journal of Pure and Applied Mathematics, Vol. 41, No. 1 (2010), 67-75.

Chapter 4 deals with generalizations of Eisenstein, Schönemann Irreducibility Criteria. One of the results generalizing Eisenstein Irreducibility Criterion states that if $\phi(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$ is a polynomial with coefficients from the ring $\mathbb{Z}$ of integers such that $a_s$ is not divisible by a prime $p$ for some $s \leq n$, then
each $a_i$ is divisible by $p$ for $0 \leq i \leq s - 1$ and $a_0$ is not divisible by $p^2$, then $\phi(x)$ has an irreducible factor of degree at least $s$ over the field of rational numbers (see [Es-Mu, §3.3, p.37]). We observed that if $\phi(x)$ is as above, then it has an irreducible factor $g(x)$ of degree $s$ over the ring of $p$-adic integers such that $g(x)$ is an Eisenstein polynomial with respect to $p$. In the fourth chapter, we have proved an analogue of the above result which extends Generalized Schönemann Irreducibility Criterion (stated below) and yields some more irreducibility criteria.

The classical Schönemann Irreducibility Criterion states that if $f(x)$ is a monic polynomial with coefficients from the ring $\mathbb{Z}$ which is irreducible modulo a prime number $p$ and if $g(x)$ belonging to $\mathbb{Z}[x]$ is a polynomial of the form $g(x) = f(x)^s + pM(x)$ where $M(x)$ belonging to $\mathbb{Z}[x]$ has degree less than that of $g(x)$ and is relatively prime to $f(x)$ modulo $p$, then $g(x)$ is irreducible over the field $\mathbb{Q}$ of rational numbers. Observe that with $g(x)$ as above, the $f(x)$-expansion of $g(x)$ obtained on dividing it by successive powers of $f(x)$ given by

$$g(x) = \sum_{i=0}^{s} A_i(x)f(x)^i, \; \text{deg } A_i(x) < \text{deg } f(x),$$

satisfies (i) $A_s(x) = 1$, (ii) $p$ divides the content of each polynomial $A_i(x)$ for $0 \leq i \leq s - 1$ and (iii) $p^3$ does not divide the content of $A_0(x)$. Conversely, it is clear that any polynomial $g(x)$ belonging to $\mathbb{Z}[x]$, whose $f(x)$-expansion has the above three properties, satisfies the conditions of Schönemann Irreducibility Criterion. The above observation led to the extension of this criterion to polynomials with coefficients in arbitrary valued fields (see [Kh-Sa]). In 2008, Ron Brown [Bro2] gave a simple proof of the most general version of the criterion which will be stated after introducing some notations.

For an element $c$ in the valuation ring $\mathcal{R}_v$ of a valued field $(K, v)$ with maximal ideal $\mathcal{M}_v$, $\bar{c}$ will denote its $v$-residue, i.e., the image of $c$ under the canonical homomorphism from $\mathcal{R}_v$ onto its residue field $\mathcal{R}_v/\mathcal{M}_v$. For $f(x)$ belonging to $\mathcal{R}_v[x]$, $\bar{f}(x)$ will stand for the polynomial (over $\mathcal{R}_v/\mathcal{M}_v$) obtained by replacing each coefficient of $f(x)$ by its $v$-residue. We shall denote by $v^*$ the Gaussian valuation of the field $K(x)$ of rational functions in an indeterminate $x$ which extends a given valuation $v$
of $K$ and is defined on $K[x]$ by

$$v^f(\sum_i a_i x^i) = \min_i \{v(a_i)\}, \ a_i \in K. \quad (1.1)$$

The following generalization of Schönemann Irreducibility Criterion is proved in [Bro2, Lemma 4].

**Theorem 1.A.** Let $v$ be a Krull valuation of a field $K$ with value group $\Gamma$ and valuation ring $R_v$. Let $f(x)$ belonging to $R_v[x]$ be a monic polynomial of degree $m$ such that $\tilde{f}(x)$ is irreducible over the residue field of $v$. Assume that $g(x)$ belonging to $R_v[x]$ is a monic polynomial whose $f(x)$-expansion

$$E \sum_{i=0}^{n} A_i(x)f(x)^i$$

satisfies (i) $A_0(x) \neq 0$, $A_n(x) = 1$, (ii) $\frac{v^f(A_i(x))}{n-i} \geq \frac{v^f(A_0(x))}{n} > 0$ for $0 \leq i \leq n-1$ and (iii) $v^f(A_0(x)) \notin d\Gamma$ for any number $d > 1$ dividing $n$. Then $g(x)$ is irreducible over $K$.

A polynomial $g(x)$ satisfying conditions (i),(ii), (iii) of the above theorem will be referred to as a Generalized Schönemann polynomial with respect to $v$ and $f(x)$. Note that in case $v$ is a discrete valuation of $K$ with value group $\mathbb{Z}$, then condition (iii) of Theorem 1.A says that $v^f(A_0(x))$ and $n$ are coprime. Hence in this case, it is immediate from the above theorem that a polynomial $g(x)$ having $f(x)$-expansion

$$f(x)^n + \sum_{i=0}^{n-1} A_i(x)f(x)^i$$

with $v^f(A_0(x)) = 1$, $v^f(A_i(x)) > 0$ for $0 \leq i \leq n-1$, is irreducible over $K$; such a polynomial is called a Schönemann polynomial with respect to the discrete valuation $v$ and $f(x)$.

The theorem stated below which extends Theorem 1.A is the main result of Chapter 4.

**Theorem 1.6.** Let $v$ be a henselian Krull valuation of a field $K$ with value group $\Gamma$ and valuation ring $R_v$ having maximal ideal $\mathcal{M}_v$. Let $f(x)$ belonging to $R_v[x]$ be a monic polynomial of degree $m$ such that $\tilde{f}(x)$ is irreducible over $R_v/\mathcal{M}_v$ and $\phi(x)$ belonging to $R_v[x]$ be a monic polynomial with $f(x)$-expansion $\sum_{i=0}^{n} A_i(x)f(x)^i$. Assume that there exists $s \leq n$ such that (i) $A_0(x) \neq 0$, $v^f(A_s(x)) = 0$,
(ii) $\frac{v^i(A(x))}{s-i} > 0$ for $0 \leq i \leq s-1$ and (iii) $v^i(A_0(x)) \notin dI$ for any number $d > 1$ dividing $s$. Then $\phi(x)$ has an irreducible factor $g(x)$ of degree $sm$ over $K$ such that $g(x)$ is a Generalized Schönemann polynomial with respect to $v$ and $f(x)$; moreover the $f(x)$-expansion of $g(x) = f(x)^s + B_{s-1}(x)f(x)^{s-1} + \cdots + B_0(x)$ satisfies $v^d(B_0(x)) = v^s(A_0(x))$.

The following corollaries are deduced from Theorem 1.6. Corollary 1.7 extends Schönemann Irreducibility Criterion [Rib2, §3.1, Theorem D]. Corollary 1.8 extends Akira’s criterion (cf. [Aki],[Pa-St]).

**Corollary 1.7.** Let $v$ be a discrete valuation of $K$ with value group $\mathbb{Z}$ and $\pi$ be an element of $K$ with $v(\pi) = 1$. Let $f(x), m$ be as in Theorem 1.6. Let $F(x)$ belonging to $R_v[x]$ be a monic polynomial having $f(x)$-expansion $\sum_{i=0}^{n} A_i(x) f(x)^i$. Assume that there exists $s \leq n$ such that $\pi$ does not divide the content of $A_s(x)$, $\pi$ divides the content of each $A_i(x), 0 \leq i \leq s-1$ and $\pi^2$ does not divide the content of $A_0(x)$. Then $F(x)$ has an irreducible factor of degree $sm$ over the completion $(\overline{K}, \overline{v})$ of $(K, v)$ which is a Schönemann polynomial with respect to $\overline{v}$ and $f(x)$.

**Corollary 1.8.** Let $(K, v), \pi$ be as above and $F(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial over $R_v$ satisfying the following conditions for an index $s \leq n - 1$.

(i) $\pi|a_i$ for $0 \leq i \leq s - 1$, $\pi^2 \nmid a_0$, $\pi \nmid a_s$.

(ii) The polynomial $x^{n-s} + \tilde{a}_{n-1}x^{n-s-1} + \cdots + \tilde{a}_s$ is irreducible over the residue field of $v$.

(iii) $\tilde{d} \neq \tilde{a}_s$ for any divisor $d$ of $a_0$ in $R_v$.

Then $F(x)$ is irreducible over $K$.

Theorem 1.6 and its corollaries have been proved in [Kh-Kh3] which has been accepted for publication in *Manuscripta Mathematica*.

Chapter 5 deals with the irreducibility of the polynomial $P_{n,k}(x) = \sum_{j=0}^{k} \binom{n}{j} x^j$ obtained by truncating the binomial expansion of $(1 + x)^n$ at the $k^{th}$ stage for pos-
itive integers $k$ and $n$ with $k \leq n - 1$. In 2007, Filaseta, Kumchev and Pasechnik considered the problem of irreducibility of $P_{n,k}(x)$ over the field $\mathbb{Q}$ of rational numbers. In [F-K-P], Filaseta et al. pointed out that when $k = n - 1$, then $P_{n,k}(x)$ is irreducible over $\mathbb{Q}$ if and only if $n$ is a prime number. They also proved that for any fixed integer $k \geq 3$, there exists an integer $n_0$ depending on $k$ such that $P_{n,k}(x)$ is irreducible over $\mathbb{Q}$ for every $n \geq n_0$. In Chapter 5, we have proved the irreducibility of $P_{n,k}(x)$ for all $n, k$ such that $2 \leq 2k \leq n < (k + 1)^3$. In fact we have considered the irreducibility of the polynomial $P_{n,k}(x - 1) = \sum_{i=0}^{k} c_i x^i$ (say). Our method works to give the irreducibility of the polynomials

$$F_{n,k}(x) = \sum_{i=0}^{k} a_i c_i x^i,$$

(1.2)

where $a_0, a_1, \ldots, a_k$ are non-zero integers and each $a_i$ has all of its prime factors $\leq k$.

The following two theorems are the main results of Chapter 5.

**Theorem 1.9.** Let $k$ and $n$ be positive integers such that $2k \leq n < (k + 1)^3$. Then $P_{n,k}(x)$ is irreducible over $\mathbb{Q}$.

**Theorem 1.10.** Let $k$ and $n$ be positive integers such that $8 \leq 2k \leq n < (k + 1)^3$ and $F_{n,k}(x)$ be as in (1.2). Then $F_{n,k}(x)$ is irreducible over $\mathbb{Q}$ except possibly when $(n,k)$ belongs to the set $\{(8,4), (10,5), (12,6), (16,8)\}$.

In the course of the proof of the above theorem, we have proved the following result which is handy for putting constraints on the degrees of possible factors of $F_{n,k}(x)$.

**Theorem 1.11.** Let $k, n$ be integers such that $n \geq k + 2 \geq 4$. Suppose there exists a prime $p > k$, $p$ divides a number $n - l, 1 \leq l \leq k - 1$, with exact power $e \geq 1$ such that $\gcd(e, l) \leq 2$ and $\gcd(e, k - l) \leq 2$. If $l_1 < k/2$ is a positive integer such that $l \notin \{l_1, 2l_1, k - l_1, k - 2l_1\}$, then $F_{n,k}(x)$ cannot have a factor of degree $l_1$ over $\mathbb{Q}$.

The paper [K-K-L] containing the proofs of Theorems 1.9 to 1.11 has been sub-
The last chapter deals with invariants associated to a certain class of irreducible polynomials (viz., defectless polynomials defined below) with coefficients in a henselian valued field \((K, v)\) of arbitrary rank. In what follows, \(\tilde{v}\) is the unique prolongation of \(v\) to the algebraic closure \(\overline{K}\) of \(K\). Each subfield of \(\overline{K}\) is assumed to have valuation obtained by restricting \(\tilde{v}\). A monic irreducible polynomial \(g(x)\) belonging to \(K[x]\) is said to be a defectless polynomial if \(K\{0\}\) is a defectless extension of \((K, v)\) for any root \(\theta\) of \(g(x)\). Popescu, Zaharescu [Po-Za], Ota [Ota], Aghigh and Khanduja [Ag-Kh2] have shown that one can associate several invariants to a defectless polynomial by means of complete distinguished chains defined below.

A pair \((\theta, \alpha)\) of elements of \(\overline{K}\) is called a distinguished pair (more precisely a \((K, \tilde{v})\)-distinguished pair) if the following three conditions are satisfied: (i) \([K(\theta) : K] > [K(\alpha) : K]\); (ii) \(\tilde{v}(\theta - \beta) \leq \tilde{v}(\theta - \alpha)\) for every \(\beta\) in \(\overline{K}\) with \([K(\beta) : K] < [K(\theta) : K]\); (iii) whenever \(\beta\) belonging to \(\overline{K}\) is such that \([K(\beta) : K] < [K(\alpha) : K]\), then \(\tilde{v}(\theta - \beta) < \tilde{v}(\theta - \alpha)\).

A chain \(\theta = \theta_0, \theta_1, \ldots, \theta_n\) of elements of \(\overline{K}\) will be called a complete distinguished chain for \(\theta\) (with respect to \((K, v)\)) if \((\theta_i, \theta_{i+1})\) is a \((K, \tilde{v})\)-distinguished pair for \(0 \leq i \leq n - 1\) and \(\theta_n\) belongs to \(K\). Popescu and Zaharescu were the first to introduce the notion of distinguished pairs and distinguished chains. In 1995, they proved the existence of a complete distinguished chain for each \(\theta \in \overline{K} \setminus K\) in case \((K, v)\) is a complete discrete rank one valued field. In 2005, taking \((K, v)\) to be a henselian valued field of arbitrary rank, Aghigh and Khanduja [Ag-Kh2] proved that an element \(\theta\) belonging to \(\overline{K} \setminus K\) has a complete distinguished chain with respect to \((K, v)\) if and only if \(K(\theta)\) is a defectless extension of \((K, v)\). They also showed that complete distinguished chains for an element \(\theta \in \overline{K} \setminus K\) give rise to several invariants associated with the minimal polynomial of \(\theta\) over \(K\) (see [Ag-Kh2, Theorems 1.4, 1.5]).

Recently Ron Brown jointly with J. Merzel also studied certain invariants (cf. [Br-Me], [Bro3]) of defectless polynomials by a different approach using the notion...
of a strict system of polynomial extensions, defined in Chapter 6, §1 of this thesis. They also developed some connections between the two approaches. In Chapter 6, we have established a one-to-one correspondence between the two.

Indeed our key result in this direction is the following.

Theorem 1.12. Let \((g_0, w_0, \gamma_0), (g_1, w_1, \gamma_1), \ldots, (g_{n+1}, w_{n+1}, \gamma_{n+1})\) be a strict system of polynomial extensions over a henselian valued field \((K, v)\) of arbitrary rank. Then for each \(i\), one can choose a root \(\theta_{n+i-1}\) of \(g_i\) such that \(\theta_0, \theta_1, \ldots, \theta_{n+1}\) is a complete distinguished chain with respect to \((K, v)\).

The converse of the above result is given by the theorem stated below.

Theorem 1.13. Let \(\theta_0, \theta_1, \ldots, \theta_{n+1}\) be a complete distinguished chain with respect to \((K, v)\) and \(g_i\) be the minimal polynomial of \(\theta_{n+i-1}\) over \(K\), \(0 \leq i \leq n + 1\). If \(w_i\) denotes the extension of \(v\) to \(K[x]\) defined for any \(q(x)\) in \(K[x]\) by \(w_i(q(x)) = \tilde{v}(q(\theta_{n+i-1}))\) and \(\gamma_0 = -\infty, \gamma_{i+1} = w_i(g_{i+1})\), then \((g_0, w_0, \gamma_0), (g_1, w_1, \gamma_1), \ldots, (g_{n+1}, w_{n+1}, \gamma_{n+1})\) is a strict system of polynomial extensions over \((K, v)\).

With the help of this correspondence, we have given a new characterization of an invariant \(\lambda_g\) (defined in Theorem 1.14 and denoted by \(\gamma_g\) in [Bro1], [Bro3]) associated to any defectless polynomial \(g(x)\) with coefficients in \((K, v)\). The invariant \(\lambda_g\) is the smallest with the property that whenever \(\tilde{v}(g(\beta)) > \lambda_g\) with \(\beta\) a tamely ramified\(^1\) extension of \((K, v)\), then \(K(\beta)\) contains a root of \(g(x)\). Hence the condition \(\tilde{v}(g(\beta)) > \lambda_g\) is weaker than the analogous condition \(\tilde{v}(g(\beta)) > 2\tilde{v}(g'(\beta))\) in Hensel’s Lemma for guaranteeing the existence of a root of \(g(x)\) in a tamely ramified extension \(K(\beta)\) of \((K, v)\). For characterizing \(\lambda_g\) in the following theorem, we have used the notion of the main invariant \(\delta_K(\theta)\) associated to any element \(\theta\) belonging to \(\bar{K} \setminus K\) defined by

\[
\delta_K(\theta) = \sup\{\tilde{v}(\theta - \alpha) \mid \alpha \in \bar{K}, [K(\alpha) : K] < [K(\theta) : K]\}. \tag{1.3}
\]

\(^1\)A finite defectless extension \((K', v')/(K, v)\) is said to be tamely ramified if the residue field of \(v'\) is a separable extension of the residue field of \(v\) and the ramification index of \(v' / v\) is not divisible by the characteristic of the residue field of \(v\).
Theorem 1.14. Let $K(\theta)$ be a defectless extension of a henselian valued field $(K,v)$ and $g(x)$ be the minimal polynomial of $\theta$ over $K$. Let $\theta = \theta_0, \theta_1, \ldots, \theta_{n+1}$ be a complete distinguished chain for $\theta$ and $\beta$ be an element of $\bar{K}$ with $\bar{v}(g(\beta)) > \bar{v}(g(\theta_1))$. Then there exists a $K$-conjugate $\theta'$ of $\theta$ such that $\bar{v}(\theta' - \beta) > \delta_K(\theta')$. Moreover the constant $\lambda_g = \bar{v}(g(\theta_1))$ depends only on $g(x)$ and is the least element $\lambda$ of $\bar{v}(\bar{K})$ such that for any $\beta$ in $\bar{K}$ with $\bar{v}(g(\beta)) > \lambda$, there exists a $K$-conjugate $\theta'$ of $\theta$ satisfying $\bar{v}(\theta' - \beta) > \delta_K(\theta)$.

The following result which is proved by a different method in [Bro3, Theorem 1] is quickly deduced from the above theorem.

Corollary. Let the hypothesis be as in Theorem 1.14. Assume in addition that either $K(\theta)$ or $K(\beta)$ is a tamely ramified extension of $(K,v)$. Then $K(\beta)$ contains a root of $g(x)$.

The paper [Kh-Kh4] containing the proofs of Theorems 1.12-1.14 and their applications has been accepted for publication in Communications in Algebra.