CHAPTER SIX
Comparison of a Univariate Distribution with its Weighted Distribution with respect to Some Partial Orderings

6.1 INTRODUCTION

The weighted distributions arise in situations where the recorded observations can not be considered a random sample from the original distribution. The reasons behind this (as observed by Rao, 1965) are

(i) Non-observability of some events: Certain types of events, although they occur, may be unascertainable and the observed distribution would therefore be truncated. For example, if someone is interested in finding out the distribution of eggs laid by insects, then the frequency of zero eggs can not be known.

In Reliability, the distribution of the total life time of a component which has survived for \( t_0 (> 0) \) units of time, denoted by \( (X|X>t_0) \), is a weighted distribution of the life time \( X \) of the new component. Gupta and Keating (1986) point out that in order to determine the mean life of helicopter the transmissions in operation in a particular aircraft are sometimes studied on a specific date and their total life are measured. Because the transmissions studied are in operation on a particular date and have been operating for some time prior to this date, they constitute a nonrandom sample of all transmissions having longer
mean lives than the class from which they are originally drawn.

(ii) **Damage caused to original distributions**: The observations may sometimes be partially destroyed (due to accidents or something alike) or may be partially ascertained (number of eggs laid by insects). The second one is an example of observations partially produced by nature. In such cases the original distribution is destroyed.

(iii) **Adoption of unequal probability sampling**: Another important way in which the original distribution is altered is due to assignment of unequal chances to different observations of being included in the sample, which is inherent in some convenient and natural methods of sampling. In order to study the distribution of albino children in families capable of producing such children, one convenient method is first to detect an albino child and through it obtain the information about the family to which it belongs. But such a procedure may not give equal chance to all families in which albinos have occurred.

In Survival Analysis, Schotz and Zelen (1971) point out that the usual estimators of cell kinetic parameters resulting from labelled mitosis experiments are biased because cells with longer DNA synthesis periods have greater probability of being labelled.

Let X be a random variable with probability density function \( f(x) \). Using a weight function \( w(x) > 0 \), Fisher (1934) constructed
the weighted distribution corresponding to \( f(x) \), to model ascertainment bias. The probability density function of the weighted distribution has the form

\[
f^w(x) = \frac{w(x)f(x)}{E[w(X)]},
\]

(6.1.1)

where the expectation in (6.1.1) is assumed to exit. We denote the random variable having density function (6.1.1), by \( X^w \). Rao (1965, 1985) presented a unified theory of weighted distributions identifying various sampling situations which can be modeled using them.

Let \( X \) be an absolutely continuous non-negative random variable denoting the life time of a component with density function \( f(x) \) and survival function \( F(x) \). If \( Y \) denotes the life time of this component truncated on the left of time \( t_0 \), then the density of \( Y = (X | X > t_0) \) is given by

\[
g(y) = \frac{f(y)}{F(t_0)} \quad \text{if } y > t_0
\]

\[
= 0 \quad \text{otherwise},
\]

which gives the weighted distribution of \( X \) with weight function

\[
w(x) = 1 \quad \text{if } x > t_0
\]

\[
= 0 \quad \text{if } x \leq t_0.
\]
Weighted random variable $X^w$ with weight function $w(x) = x^\alpha$, $\alpha > 0$, is called size-biased of order $\alpha$. When $\alpha = 1$, it is simply called size-biased or length-biased. In reliability theory, $w(x) = x$ models size-biased sampling for life time distributions. Size-biased distributions arise when the sampling mechanism selects units with probability proportional to some measure of unit size.

In Renewal Theory, size-biased distribution arises as the limiting distribution ($t \to \omega$) of total life time of a unit which has survived for $t$ units of time (cf. Feller (1966)). The size-biased distribution also finds various applications in Biomedical areas as early detection of disease, latency periods of AIDS due to blood transfusion (cf. Zelen and Feinleib (1969)). Rao (1965) used the size-biased distribution in the study of human families and wild-life population. It has also been used in a cardiology study involving two phases (cf. Cnaan (1985)). Aerial survey involving visibility bias in wild-life ecology produces weighted distributions of the original distributions (cf. Patil, 1981). Various other important weighted distributions have been discussed in Blumenthal (1967), Scheaffer (1972), Patil and Ord (1976), Patil and Rao (1977,1978), Mahfoud and Patil (1982), Kochar and Gupta (1987) and Gupta and Kirmani (1990) among others.

It is, therefore, important to study how the different partial orderings (of the original random variable), commonly used
in Reliability and Survival Analysis, are transmitted to the weighted version depending upon the weight function used. If two random variables are ordered with respect to some partial orderings, then transmission of these partial orders to the respective weighted random variable, under certain condition on weight function, has been studied in Section 2. In this section, we have given a necessary and sufficient condition for X and X^w to have Expectation order. Since generalized equilibrium distributions can also be considered as weighted distributions with some suitable weight function, we have derived some moment inequalities regarding generalized equilibrium distributions in Section 3. A research paper based on Section 3 has appeared as follows:


6.2. SOME PARTIAL ORDERING RESULTS

If two random variables X_1 and X_2 are ordered with respect to some partial orderings, then how these partial orderings are transmitted to their respective weighted versions X^w_1 and X^w_2, have been studied in this section. One result showing how X and w(X) are correlated has also been proved.
The following two lemmas are given for proving a result regarding the ordering of residual life functions of the two weight functions \( w(X_1) \) and \( w(X_2) \) when the random variables \( X_1 \) and \( X_2 \) are ordered.

**LEMMA 6.2.1:** \( \left[ X_1 \mid X_1 > t \right] \overset{FR}{\geq} \left[ X_2 \mid X_2 > t \right] \) for all \( t > 0 \) if and only if \( X_1 \overset{FR}{\geq} X_2 \).

**Proof:** The proof of sufficiency follows from the observation that

\[
F_X(x) = \frac{F_X(x)}{F_X(t)} \quad \text{if } x < t,
\]

while the necessity follows by taking \( t = 0 \).

The following lemma is reproduced from Shaked and Shanthikumar (1994).

**LEMMA 6.2.2:** If \( X_1 \overset{FR}{\geq} X_2 \) and \( g \) is a nondecreasing function, then

\[
g(X_1) \overset{FR}{\geq} g(X_2).
\]

The following theorem shows that if two random variables \( X_1 \) and \( X_2 \) are ordered with respect to failure rate ordering, then the residual life function of \( w(X_1) \) and \( w(X_2) \), evaluated at the points
in the range of the function \( w(x) \), are also ordered with respect to the failure rate ordering, provided \( w(x) \) is nondecreasing.

**Theorem 6.2.1:** For any nondecreasing weight function \( w \),

\[
\left[ w(X_1)-w(x) \mid X_1>x \right] \overset{\text{FR}}{\geq} \left[ w(X_2)-w(x) \mid X_2>x \right]
\]

for all \( x \geq 0 \) if \( X_1 \geq X_2 \).

**Proof:** The proof follows by using Lemma 6.2.1 and Lemma 6.2.2.

Let \( X \) be a univariate nonnegative random variable with density function \( f(x) \), survival function \( \bar{F}(x) \) and failure rate \( r_F(x) \). Further let \( X^w \) be the weighted version of \( X \) with weight function \( w(x) \) and have survival function \( \bar{F}^w(x) \) and failure rate \( r_{F^w}(x) \). Then the failure rate of \( X^w \) is given by

\[
r_{F^w}(x) = \frac{w(x)r_F(x)}{w(x)+A_F(w,x)}, \quad (6.2.1)
\]

where \( A_F(w,x) = E_{\bar{F}}[w(X) - w(x) \mid X > x] \)

\[
= \frac{1}{\bar{F}(x)} \int_x^\infty \left[ w(t) - w(x) \right] dF(t),
\]

(cf. Jain et al. (1989)).
The following theorem shows that if two random variables $X$ and $Y$ are ordered with respect to their failure rate functions, then their corresponding weighted random variables will also be ordered with respect to their failure rate functions, provided the weight function is nondecreasing.

**THEOREM 6.2.2:** For any nondecreasing weight function $w(x)$,

$$X^w \preceq Y^w \quad \text{whenever} \quad X \preceq Y,$$

where $X^w$ and $Y^w$ are the respective weighted version of $X$ and $Y$ with weight function $w(x)$.

**Proof:** Let $X$ and $Y$ have respective failure rate function $r_F(x)$ and $r_G(x)$. Let us denote the failure rate function of $X^w$ and $Y^w$ by $r_{F_w}(x)$ and $r_{G_w}(x)$ respectively. Then $X^w \preceq Y^w$ if and only if

$$r_{F_w}(x) \leq r_{G_w}(x)$$

for all $x \geq 0$. which, by (6.2.1), reduces to

$$\frac{w(x)r_{F_w}(x)}{w(x)+A^F_{w}(w,x)} \leq \frac{w(x)r_{G_w}(x)}{w(x)+A^G_{w}(w,x)}$$

(6.2.3)

where $A^G_{w}(w,x)$ is obtained by replacing $F$ by $G$ in (6.2.2). Since, $X \preceq Y$, (6.2.3) holds if

$$X \preceq Y,$$
for all $x \geq 0$, which follows from Theorem 6.2.1, since failure rate ordering implies expectation ordering.

**THEOREM 6.2.3:** For any weight function $w(x)$,

$$\text{Cov} \left[ X, w(X) \right] = E(w(X)) \left( E(X^w) - E(X) \right).$$

**Proof:** Let us denote the density function of $X$ and $X^w$, by $f(x)$ and $g(x)$ respectively. Then the result follows by considering the fact that

$$E \left[ X w(X) \right] = \int_0^\infty x w(x) f(x) \, dx$$

$$= E(w(X)) \int_0^\infty x g(x) \, dx$$

$$= E(X^w) E(w(X)).$$

**COROLLARY 6.2.1:** (a) For any non-negative weight function $w(x)$,

$$\text{Cov} \left[ X, w(X) \right] \geq (=) 0 \text{ if and only if } E(X^w) \geq (=) E(X);$$

(b) If $w(x) = x$, then $E(X^w) = \mu + c \sigma$, where $\mu = E(X)$, $\sigma^2 = V(X)$ and $c = \text{coefficient of variation of } X.$
Proof: (a) follows trivially using the above theorem. Taking \( w(x) = x \), Theorem 6.2.3 gives
\[
V(X) = E(X) \left[ E(X^w) - E(X) \right]
= E(X) \cdot E(X^w) - E^2(X).
\]
This gives
\[
E(X^w) = \frac{V(X) + E^2(X)}{E(X)}
= \frac{\sigma^2 + \mu^2}{\mu}
= \mu + c \cdot \sigma,
\]
which proves (b).

REMARK 6.2.1: Corollary 6.2.1(b) is a known result (cf. Mahfoud and Patil, 1982).

6.3. PROPERTIES OF MOMENTS FOR s-ORDER EQUILIBRIUM DISTRIBUTIONS

Let \( X \) be an absolutely continuous random variable with probability density function \( f_X(x) \), survival function \( \bar{F}(x) \) and mean \( \mu_F \). Let
\[
\bar{T}_0(X,x) = f_X(x)
\]
\[ T^t(x) = F_x(x) \]
\[ V^s(x) - M^{s-1}(X) \]
\[ r_0 = \int_0^\infty T_{s-1}(x,u)du, \quad (6.3.1) \]
for \( s = 2, 3, \ldots \), where
\[ \mu_s(x) = \int_0^\infty T_s(x,x)dx, \]
for \( s = 1, 2, \ldots \).

Note that \( T_2(x,x) \) is the survival function of the equilibrium (or stationary) distribution of \( X \), which plays an important role in ageing concepts (refer to Deshpande et al. (1986), Singh (1989)) and renewal processes (refer to Cox (1962)), whereas \( T_s(x,x) \) is the survival function of the equilibrium distribution of the distribution with survival function \( T_{s-1}(x,x) \), \( s = 2, 3, \ldots \). We shall call the distribution with survival function \( T_{s+1}(x,x) \) an \( s \)-order equilibrium distribution of \( X \).

We further define, for \( s = 1, 2, \ldots \)
\[ r_s(x,x) = \frac{T_{s-1}(x,x)}{\int_0^\infty T_{s-1}(x,u)du}, \]
which represents the failure rate function corresponding to \( T_s(x,x) \). We denote the random variable having survival function \( T_s(x,x) \), by \( X_s \). Note that, for \( s = 2, 3, \ldots \), \( X_s \) has density
function

\[ f_{X(s)}(x) = \frac{T_{s-1}(X,x)}{\mu_{s-1}(X)} \]

which can be considered as a weighted distribution of \( X_{(s-1)} \) with weight function

\[ w(x) = \frac{1}{r_{s-1}(X,x)}. \]

In this section we derive some properties of moments for the \( s \)-order equilibrium distribution of \( X \). The results are expected to be useful in reliability and renewal processes.

The following theorem, which was established by Barlow et al. (1963) and Massey and Whitt (1993) in different contexts, shows that the product of the means \( \mu_i(X) \), \( i = 0,1,2,\ldots,s \), can be expressed in terms of the \( s \)-order moment of \( X \) about zero. The best thing about the result is that the proof is very straightforward and does not involve any complicated mathematics.

**THEOREM 6.3.1:** Let \( X \) be an absolutely continuous non-negative random variable. Then

\[ \mu_0(X) \cdot \mu_1(X) \cdot \mu_2(X) \cdot \ldots \cdot \mu_s(X) = \frac{E(X^s)}{s!}, \quad s = 0,1,\ldots \]  \hspace{1cm} (6.3.2)
Proof: The proof follows by using induction on $s$. For $s = 0$, the result is trivial. Assume that (6.3.2) holds for $s = s'$. Then using the definition of $\mu_s(X)$ and that of $T_s(X,x)$ repeatedly, we get

$$
\mu_{s+1}(X) = \int_0^\infty T_{s+1}(X,x)dx
$$

$$
= \int_0^\infty \frac{1}{\mu_s(X)} \int_x^\infty T_s(x,u)du dx
$$

$$
= \frac{1}{\mu_s(X)} \int_0^\infty \int_x^\infty T_s(x,u)du dx
$$

$$
= \frac{1}{\mu_s(X)\mu_{s-1}(X)\ldots\mu_0(X)} \int_0^\infty \int_x^\infty \ldots \int_x^\infty \int_0^\infty T_0(x,t)dt \ldots dx dx_s dx,
$$

which can be re-written as

$$
\mu_0(X)\mu_1(X)\ldots\mu_{s+1}(X) = \int_0^\infty \int_0^\infty \ldots \int_0^\infty f(t)dtdx_1 \ldots dx_s dx.
$$

Interchanging the order of integrations, we get

$$
\mu_0(X)\mu_1(X)\ldots\mu_{s+1}(X) = \int_0^\infty \int_0^t \ldots \int_0^x dx dx_s \ldots dx_1 f(t)dt
$$

$$
= \int_0^\infty \frac{t^{s+1}}{(s+1)!} f(t)dt
$$

$$
= \frac{E(X^{s+1})}{(s+1)!}.
$$
Another interesting expression is given in the following corollary.

**COROLLARY 6.3.1:** $\mu_s(X) = \frac{E(X^s)}{s!E(X^{s-1})}$, $s = 1, 2, \ldots$

**Proof:** From the above theorem, we have, for $s = 1, 2, \ldots$

$$\frac{E(X^s)}{E(X^{s-1})} = \frac{\mu_0(X)\mu_1(X)\ldots\mu_s(X)\cdot s!}{\mu_0(X)\mu_1(X)\ldots\mu_{s-1}(X)\cdot (s-1)!}$$

$$= \frac{s}{s!} \mu_s(X),$$

giving

$$\frac{E(X^s)}{s!E(X^{s-1})} = \mu_s(X).$$

Hence the result.

For any two absolutely continuous random variables $X_1$ and $X_2$, $X_2$ is said to be larger than $X_1$ in $s$-ST ordering, written as $X_2 \preceq_{s-ST} X_1$, if

$$\frac{T_s(X_2, x)}{T_s(X_2, 0)} \geq \frac{T_s(X_1, x)}{T_s(X_1, 0)}$$

for all $x \geq 0$. Clearly,

$$\mu_s(X_2) \preceq \mu_s(X_1) \text{ if } X_2 \preceq_{s-ST} X_1.$$
In the ensuing theorem, the implication of s-ST ordering between $X_2$ and $X_1$ has been studied on the ordering between $\mu_{s-1}(X_2)$ and $\mu_{s-1}(X_1)$.

**THEOREM 6.3.2:** For any $s = 0, 1, 2, \ldots$

$$X_2 \overset{s-ST}{\preceq} X_1 \Rightarrow \mu_{s-1}(X_2) \preceq \mu_{s-1}(X_1), \text{ with } \mu_{-1}(.) = 1.$$

**Proof:** The result is trivial for $s = 0$ and $1$.

For $s \geq 2$, $X_2 \overset{s-ST}{\preceq} X_1$ gives

$$T_s(X_2, x) \preceq T_s(X_1, x)$$

for all $x > 0$. This, using (6.3.1), can further be written as

$$\frac{1}{\mu_{s-1}(X_2)} \int_0^\infty T_{s-1}(X_2, u) du \preceq \frac{1}{\mu_{s-1}(X_1)} \int_0^\infty T_{s-1}(X_1, u) du$$

for all $x > 0$. This can equivalently be written as

$$1 - \frac{1}{\mu_{s-1}(X_2)} \int_0^\infty T_{s-1}(X_2, u) du \preceq 1 - \frac{1}{\mu_{s-1}(X_1)} \int_0^\infty T_{s-1}(X_1, u) du \quad (6.3.3)$$

for all $x > 0$, which can be written as
for all $x > 0$. Dividing both sides of (6.3.4) by $x$, it becomes

$$\frac{\int_0^x T_{s-1}(X_2, u) du}{\mu_{s-1}(X_2)} \leq \frac{\int_0^x T_{s-1}(X_1, u) du}{\mu_{s-1}(X_1)}$$

(6.3.5)

for all $x > 0$. Now taking limits as $x \to 0$ on both sides of (6.3.5), we get

$$\lim_{x \to 0} \frac{\int_0^x T_{s-1}(X_2, u) du}{x \mu_{s-1}(X_2)} \leq \lim_{x \to 0} \frac{\int_0^x T_{s-1}(X_1, u) du}{x \mu_{s-1}(X_1)}$$

which gives

$$\mu_{s-1}(X_2) \geq \mu_{s-1}(X_1).$$

Hence the result. $\blacksquare$

THEOREM 6.3.3: For $s \geq 2$, if

$$\mu_1(X_2) \geq \mu_1(X_1)$$

(6.3.6)

for all $i = 1, 2, \ldots, s-1$, then

$$\left[ \frac{E(X_1^i)}{E(X_2^i)} \right] \geq \left[ \frac{E(X_1^{s-1})}{E(X_2^{s-1})} \right]$$

(6.3.7)

for all $i = 0, 1, \ldots, s-2$. 
Proof: On using Theorem 6.3.1, (6.3.7) can be written as

\[ \frac{\mu_0(X_1)\mu_1(X_1)\ldots\mu_i(X_1)}{\mu_0(X_2)\mu_1(X_2)\ldots\mu_i(X_2)} \leq \frac{\mu_0(X_1)\mu_1(X_1)\ldots\mu_{s-1}(X_1)}{\mu_0(X_2)\mu_1(X_2)\ldots\mu_{s-1}(X_2)} \]

for all \( i = 0,1,2,\ldots,s-2 \). This further gives

\[ \mu_{i+1}(X_2)\mu_{i+2}(X_2)\ldots\mu_{s-1}(X_2) \leq \mu_{i+1}(X_1)\mu_{i+2}(X_1)\ldots\mu_{s-1}(X_1) \]

for all \( i = 0,1,\ldots,s-2 \), which follows from (6.3.6). \( \square \)

Below, we mention a few examples of random variables \( X_1 \) and \( X_2 \) satisfying (6.3.6) for every \( s \geq 1 \).

EXAMPLE 6.3.1: Suppose \( X_i \) has distribution function

\[ F_{\alpha_i}(t) = 1 - \exp(-t/\alpha_i), \]

\( t > 0, \ i = 1,2 \). Take \( \alpha_1 < \alpha_2 \). Then from Corollary 6.3.1, we have

\[ \mu_s(X_1) = \alpha_1. \]

So, \( \mu_s(X_1) < \mu_s(X_2) \), for \( s = 1,2,\ldots \). \( \square \)
EXAMPLE 6.3.2: Suppose $X_i$ has distribution function

$$F_{\alpha_i}(t) = \frac{1}{\Gamma \alpha_i} \int_0^t x^{\alpha_i-1} e^{-x} dx,$$

$t > 0$, $i = 1,2$. Take $\alpha_1 < \alpha_2$. Then from Corollary 6.3.1, we have

$$\mu_s(X_i) = \frac{\Gamma(s+\alpha_i)}{s \Gamma(s+\alpha_i-1)}$$

$$= \frac{s+\alpha_i-1}{s}.$$

Hence, $\mu_s(X_1) < \mu_s(X_2)$, for $s = 1,2,...$  \[ \Box \]

EXAMPLE 6.3.3: Suppose $X_i$ has distribution function

$$F_{\alpha_i}(t) = 1 - \exp\left(-\frac{t}{\alpha_i}\right),$$

$t > 0$, $i = 1,2$. Take $\alpha_1 < \alpha_2$. Then from Corollary 6.3.1, we have

$$\mu_s(X_i) = \frac{\alpha_i \Gamma\left(\frac{s}{\alpha_i} + 1\right)}{s \Gamma\left(\frac{s-1}{\alpha_i} + 1\right)}.$$

Hence, $\mu_s(X_1) < \mu_s(X_2)$, for $s = 1,2,...$  \[ \Box \]
THEOREM 6.3.4: Let \( s \geq 1 \). Let \( Z \) be a non-negative absolutely continuous random variable independent of \( X_1 \) and \( X_2 \). Then

\[
\left[ \frac{E(X^i_1)}{E(X^i_2)} \right] \geq \left[ \frac{E(X^{s-1}_1)}{E(X^{s-1}_2)} \right]
\]

(6.3.8)

for all \( i = 0, 1, \ldots, s-1 \) implies that

\[
\frac{\mu_1(X_1+Z)\mu_2(X_1+Z)\ldots\mu_{s-1}(X_1+Z)}{\mu_1(X_2+Z)\mu_2(X_2+Z)\ldots\mu_{s-1}(X_2+Z)} \geq \frac{\mu_1(X_1)\mu_2(X_1)\ldots\mu_{s-1}(X_1)}{\mu_1(X_2)\mu_2(X_2)\ldots\mu_{s-1}(X_2)}.
\]

(6.3.9)

Proof: (6.3.8) can equivalently be written as

\[
E(X^i_1)E(X^{s-1}_2) \geq E(X^i_2)E(X^{s-1}_1)
\]

for all \( i = 0, 1, \ldots, s-1 \). Multiplying both sides of the above expression by \( \binom{s-1}{i}E(Z^{s-1-i}) \) and summing over all \( i = 0, 1, \ldots, s-1 \), we have

\[
E(X^{s-1}_2) \sum_{i=0}^{s-1} \binom{s-1}{i}E(Z^{s-1-i})E(X^i_1) \geq E(X^{s-1}_1) \sum_{i=0}^{s-1} \binom{s-1}{i}E(Z^{s-1-i})E(X^i_2),
\]

which further gives

\[
E(X^{s-1}_2)E \left[ \sum_{i=0}^{s-1} \binom{s-1}{i}Z^{s-1-i}X^i_1 \right] \geq E(X^{s-1}_1)E \left[ \sum_{i=0}^{s-1} \binom{s-1}{i}Z^{s-1-i}X^i_2 \right],
\]
or, equivalently,
\[ E(X_2^{s-1})E(X_1+Z)^{s-1} \geq E(X_1^{s-1})E(X_2+Z)^{s-1}. \] (6.3.10)

Now, on using Theorem 6.3.1, (6.3.10) can be written as
\[
\mu_1(X_1) \ldots \mu_{s-1}(X_2) \mu_1(X_1+Z) \ldots \mu_{s-1}(X_1+Z) \\
\geq \mu_1(X_1) \ldots \mu_{s-1}(X_1) \mu_1(X_2+Z) \ldots \mu_{s-1}(X_2+Z),
\]
which gives
\[
\frac{\mu_1(X_1+Z) \mu_2(X_1+Z) \ldots \mu_{s-1}(X_1+Z)}{\mu_1(X_2+Z) \mu_2(X_2+Z) \ldots \mu_{s-1}(X_2+Z)} \geq \frac{\mu_1(X_1) \mu_2(X_1) \ldots \mu_{s-1}(X_1)}{\mu_1(X_2) \mu_2(X_2) \ldots \mu_{s-1}(X_2)}. \]