CHAPTER THREE
3.1 INTRODUCTION

If $X$ is a random variable with absolutely continuous distribution function $F$ and density $f$, then the reversed (or retro, or backward) hazard rate of $X$ at the point $x$, is defined as

$$
\mu_F(x) = \frac{d}{dx} \ln F(x) = \frac{f(x)}{F(x)}, \text{ where defined}, \quad (3.1.1)
$$

(refer to Shaked and Shanthikumar, 1994). Suppose that $X$ is the lifetime of a unit which could be a living organism or a mechanical component. Given that the unit has already failed by time $t$, then the probability that it survived up to time $t - \epsilon$ (where $\epsilon$ is very small positive number) is approximately given by $\epsilon \mu_F(t)$. The reversed hazard rate function plays an important role in the analysis of left censored data (cf. Anderson et al., 1993).

Let $X$ and $Y$ be two random variables with absolutely continuous distribution function $F$ and $G$ and reversed hazard rate function $\mu_F$ and $\mu_G$ respectively. Then if

$$
\mu_F(x) \geq \mu_G(x) \quad (3.1.2)
$$
for all $x \geq 0$, $X$ is said to be larger than $Y$ in the reversed RHR _ RHR _ hazard rate ordering (written as $X \geq Y$, or $F \geq G$), where $F$ and $G$ denote the respective survival functions. It can be verified that (3.1.2) holds if and only if

$$\frac{F(x)}{G(x)} \text{ is nondecreasing in } x \geq 0.$$  

(3.1.3)

The condition (3.1.3) was first discussed by Keilson and Sumita (1982). They used the terminology "$Y$ is uniformly smaller than $X$ in the negative direction". One can see that (3.1.3) is equivalent to

$$\left[ X \mid X \leq t \right] \overset{\text{ST}}{\geq} \left[ Y \mid Y \leq t \right]$$

for all $t \geq 0$ (see also Shaked and Shanthikumar, 1994). Thus, reversed hazard rate order is stronger than usual stochastic order for random variables, yet is weaker than the likelihood ratio (LR) ordering.

Let us consider a system $C$ consisting of $n$ components $C_1, C_2, \ldots, C_n$. Let $h(p_1, p_2, \ldots, p_n)$ denote the reliability function of the structure $\phi$ (see Barlow and Proschan (1981) for definition). We assume that each of the $n$ components can dichotomously be divided into two states: a functioning state and a failed state. In order to indicate the state of the $i$th
component $C_i$, we define an indicator random variable $X_i(t)$ as

$$X_i(t) = \begin{cases} 1 & \text{if } C_i \text{ is functioning at time } t \\ 0 & \text{if } C_i \text{ has failed at time } t \end{cases}$$

$i = 1, 2, \ldots, n$. We further assume that the state of the system is determined completely by the states of the components. So we can write $\phi(X(t)) = \phi(X_1(t), X_2(t), \ldots, X_n(t))$. The function $\phi$ is called the structure function of the system. Let $\phi$ be a deterministic binary function taking values zero or one according as the system is functioning or it has failed. A system which functions if and only if all its components function, is known as a series system. So its structure function will be $\phi(x) = \min(x_1, x_2, \ldots, x_n)$. Similarly, a system which functions if and only if at least one of its components function, is known as a parallel system. Its structure function is given by $\phi(x) = \max(x_1, \ldots, x_n)$.

A $k$-out-of-$n$ system is defined as a system which functions if and only if at least $k$ of its components out of $n$ components function. Clearly, an $n$-out-of-$n$ system is a series system whereas a $1$-out-of-$n$ system is a parallel system.

Let us assume that $T_i$, the lifetime of $C_i$, is a continuous random variable with distribution function $F_i$ and density $f_i$, $i = 1, 2, \ldots, n$. Note that

$$X_i(t) = \begin{cases} 1 & \text{if and only if } T_i > t \end{cases}$$
and

\[ P(X_i(t) = 1) = 1 - F_i(t) = F_i(t), \]

the reliability of \( C_i \) at time \( t \). If \( \tau \) denotes the life time of the system, then

\[
P(\tau > t) = P[\tau(T_1, T_2, \ldots, T_n) > t] = P[\phi(X(t)) = 1] = h(F_1(t), F_2(t), \ldots, F_n(t)) = h(F(t)).
\]

\( h \) is known as the system reliability function. If the components are identically distributed, we write \( h(F(t)) \) in place of \( h(F(t)) \).

Throughout, in this chapter, unless otherwise specified, we assume that the components are independent and the system \((C, \phi)\) is a coherent system (a system whose structure function \( \phi \) is nondecreasing in each argument and that each component is relevant). Since, it is quite unnatural to consider a system which deteriorates by improvement of its component performance and it is of not much interest to consider a system whose state does not depend on the component states, the coherent systems get much importance in reliability theory.
Lynch et al. (1987) have shown that if the failure rate (or hazard rate) of one type of component is higher than that of a second type of component, then certain systems have a higher failure rate if components of the first type are used rather than components of the second type. In Section 2, the same type of result has been proved for reversed hazard rate. We have established sufficient conditions on structure function of a coherent system of independent and identically distributed (iid) components to preserve reversed hazard rate ordering and likelihood ratio ordering. As a particular case, we have shown that the result is true for any k-out-of-n system. A class of structures satisfying a sufficient condition for preservation of reversed hazard rate ordering is also constructed. Section 3 deals with the systems constructed out of independent but not necessarily identical components. We provide bounds on system reversed hazard rate. It is shown that for a k-out-of-n system, the bounds on system reversed hazard rate are nonincreasing if the component reversed hazard rates are nonincreasing. We also prove that if each of a set of independent and not necessarily identical components has smaller (larger) reversed hazard rate than the components of second type (of iid components), then some coherent systems have less (more) reversed hazard rate if first set of components are used rather than components of the second type, and as a particular case, it is true for any k-out-of-n system. It is well known that for a series system, the system hazard rate is the sum of the individual component hazard rates. We have shown that
the converse is also true. This characterizes a series system. 
Same type of characterization, by means of reversed hazard rate 
functions, has also been provided for a parallel system. Finally, 
in Section 4, we investigate the closure property of reversed 
hazard rate ordering under a Poisson shock model. A research paper 
based on this chapter has been accepted for publication as 
follows:

Some Partial Orderings under the Formation of Coherent Systems. 
Statistics and Probability Letters, To Appear.

3.2. PRESERVATION OF SOME PARTIAL ORDERINGS UNDER THE FORMATION OF 
COHERENT SYSTEMS WITH IID COMPONENTS

Let us consider two coherent systems $C_1$ and $C_2$ each 
consisting of $n$ iid components. Suppose the life time of the 
components from system $C_1$ has distribution function $F$ and that 
from system $C_2$ has distribution function $G$.

The following theorem shows that if the reversed hazard rate 
of one type of component is higher (lower) than that of a second 
type of components, then certain systems have a higher (lower) 
reversed hazard rate if the components of the first type are used 
rather than the components of the second type.
**THEOREM 3.2.1:** Let $h(p)$ be the reliability function of a coherent system of $n$ iid components having derivative $h'(p)$ for $p \geq 0$. If \[
\frac{(1-p)h'(p)}{1-h(p)}
\] is nondecreasing in $p$, then

$$RHR_{h(F)} \preceq (s) h(G),$$

whenever $F \preceq (s) G$.

**Proof:** Let $\mu_{h(F)}(t)$ be the reversed hazard rate function of the coherent system with reliability function $h$. Then

$$\mu_{h(F)}(t) = \frac{h'(F(t)).f(t)}{1-h(F(t))}$$

$$= \frac{f(t)}{F(t)}.F(t).\frac{h'(F(t))}{1-h(F(t))}$$

$$= \mu_F(t).(1-F(t)).\frac{h'(F(t))}{1-h(F(t))}$$

$$\preceq (s) \mu_G(t).(1-G(t)).\frac{h'(G(t))}{1-h(G(t))}$$

$$= \mu_{h(G)}(t).$$

The above inequality follows from the hypothesis using the fact that $\mu_F(t) \preceq (s) \mu_G(t)$ implies $F(t) \preceq (s) G(t)$.

The following corollary shows that the above type of preservation result always holds for a $k$-out-of-$n$ system.
COROLLARY 3.2.1: For a k-out-of-n system,

\[ \text{RHR} \quad h(F) \preceq h(G) \quad \text{if} \quad \overline{F} \preceq \overline{G}. \]

Proof: It is well known that for a k-out-of-n system, \( ph'(p)/h(p) \) is decreasing in \( p \) (cf. Barlow and Proschan (1965)). We show that for such a system

\[ \frac{(1-p)h'(p)}{1-h(p)} \text{ is nondecreasing in } p. \]

For a k-out-of-n system,

\[
h(p) = \sum_{i=k}^{n} \binom{n}{i} p^i (1-p)^{n-i}
\]

\[ = \frac{\Gamma(n+1)}{\Gamma(k) \cdot \Gamma(n-k+1)} \int_0^p t^{k-1} (1-t)^{n-k} dt,
\]

giving

\[ h'(p) = \frac{\Gamma(n+1)}{\Gamma(k) \cdot \Gamma(n-k+1)} p^{k-1} (1-p)^{n-k}
\]

and

\[ 1-h(p) = \frac{\Gamma(n+1)}{\Gamma(k) \cdot \Gamma(n-k+1)} \int_0^1 t^{k-1} (1-t)^{n-k} dt.
\]

Hence

\[
\frac{1-h(p)}{(1-p)h'(p)} = (1-p)^{-1} \int_0^1 \left( \frac{t}{p} \right)^{k-1} \left( \frac{1-t}{1-p} \right)^{n-k} dt
\]
Note that \( \frac{1-u(1-p)}{p} \) is nonincreasing in \( p \) for any \( u \in (0,1) \) and hence the result follows.

**THEOREM 3.2.2:** Consider a coherent structure with reliability function \( h(p) \) and each component life independently distributed according to distribution function \( F \), survival function \( \bar{F}(t) = 1-F(t) \) and density \( f \). Then

(a) \( \frac{\mu_h(F)(t)}{\mu_F(t)} = \frac{(1-\bar{F}(t))h'(\bar{F}(t))}{1-h(F(t))} \)

(b) \( \frac{(1-p)h'(p)}{1-h(p)} \) is nondecreasing in \( p \) if and only if \( \frac{\mu_h(F)(t)}{\mu_F(t)} \) is nonincreasing in \( t \);

(c) if \( \mu_F(t) \) is a nonincreasing function of \( t \) and \( \frac{(1-p)h'(p)}{1-h(p)} \) is nondecreasing in \( p \), then \( \mu_h(F)(t) \) is nonincreasing in \( t \).

**Proof:** We know that \( \mu_h(F)(t) = \frac{h'(\bar{F}(t)).f(t)}{1-h(F(t))} \) and \( \mu_F(t) = \frac{f(t)}{\bar{F}(t)} \).

Hence the proof of (a) follows by taking the ratio, and the proofs of (b) and (c) follow from (a).
REMARK 3.2.1: Several systems other than the k-out-of-n systems satisfy the condition that 
\((1-p)h'(p)/(1-h(p))\) is nondecreasing in \(p\). A simple example is a structure where two components are in 
series and a third component is in parallel with these two. However, this condition is not satisfied for a general k-out-of-m 
system where each component is a \(k_i\)-out-of-\(n_i\) system, \(i = 1, 2, \ldots, m\).

The reliability of a system, where two components are in 
series and a third component is in parallel with these two, is 
given by

\[
h(p) = p + p^2 - p^3,
\]

where \(p\) is the component reliability. Hence

\[
1 - h(p) = (1-p)(1-p^2).
\]

This gives

\[
\frac{(1-p)h'(p)}{1-h(p)} = \frac{1+2p-3p^2}{1-p^2}
\]

\[
= g(p), \text{ say.}
\]

A simple calculation shows that

\[
g'(p) = 2/(1+p)^2,
\]
which is strictly positive. Hence

\[
\frac{(1-p)h'(p)}{1-h(p)} \text{ is increasing in } p.
\]

This establishes the first part of the remark.

To support the second part, let us take the system where two components are in parallel and a third component is in series with these two. Then the reliability of the system is given by

\[ h(p) = 2p^2 - p^3, \]

where \( p \) is the component reliability, giving

\[ 1-h(p) = (1-p)(1+p-p^2). \]

Hence,

\[
\frac{(1-p)h'(p)}{1-h(p)} = \frac{4p-3p^2}{1+p-p^2}
\]

\[ = g(p), \text{ say.} \]

It can easily be checked that \( g(p) \) increases for \( 0 \leq p < 3.5^{1/2} \) and decreases for \( 3.5^{1/2} < p \leq 1. \)

**REMARK 3.2.2:** We can generate new structures with the property that \( \frac{(1-p)h'(p)}{1-h(p)} \) is nondecreasing in \( p \), by composition of the structures having the same property. Under composition, we form a super structure where each element consists of copies of a given
structure. If $h = f \circ g$ with $g'(p) \neq 0$, then since

$$\frac{(1-p)h'(p)}{1-h(p)} = \frac{(1-g(p))f'(g(p))}{1-f(g(p))} \cdot \frac{(1-p)g'(p)}{1-g(p)},$$

the property is closed under composition.

The following is a preservation theorem for a $k$-out-of-$n$ system of iid components.

**Theorem 3.2.3:** If the lifetime of each of the independent and identically distributed components has decreasing reversed hazard rate function, then a $k$-out-of-$n$ system formed by these components also has a decreasing reversed hazard rate function.

**Proof:** Let $X$, the lifetime of a component, have decreasing reversed hazard rate. So, $\mu_F(t)$ is nonincreasing in $t$ (assuming $X$ follows $F$). From Theorem 3.2.2(a) we have, for any coherent system

$$\mu_h(F)(t) = \frac{(1-F(t))h'(F(t))}{1-h(F(t))}.\mu_F(t).$$

In the proof of Corollary 3.2.1, it is shown that for a $k$-out-of-$n$ system of independent like components $\frac{(1-p)h'(p)}{1-h(p)}$ is nondecreasing in $p$. Thus, $\frac{(1-F(t))h'(F(t))}{1-h(F(t))}$ is nonincreasing in $t$. Hence, $\mu_h(F)(t)$ is nonincreasing in $t$. \qed
In the following theorem, the preservation of likelihood ratio ordering is established for a coherent system with identically distributed components. (Also see Bapat and Kochar (1994) for more results on likelihood ratio ordering of order statistics).

**THEOREM 3.2.4:** Let \( h(p) \) be the reliability function of a coherent system of \( n \) iid components having first and second derivatives \( h'(p) \) and \( h''(p) \) respectively. If \( \frac{ph''(p)}{h'(p)} \) is nonincreasing in \( p \), then

\[
\text{if } LR_{h(F)} \geq LR_{h(G)}, \quad \text{whenever } F \leq G.
\]

**Proof:** \( h(F) \succeq h(G) \) if and only if

\[
\frac{f(t)h'(F(t))}{g(t)h'(G(t))}
\]

is nondecreasing in \( t > 0 \).

Assuming \( f(t) \) and \( g(t) \) are differentiable, the above is equivalent to

\[
g(t)h'(G(t)) \left[ f'(t)h'(F(t)) - f'(t)h''(F(t)) \right] -
\]

\[
f(t)h'(F(t)) \left[ g'(t)h'(G(t)) - g'(t)h''(G(t)) \right] \succeq 0.
\]
This can further be written as

\[
\begin{bmatrix}
    f'(t)g(t) - g'(t)f(t) \\
    g(t)h''(\bar{G}(t))h'(\bar{F}(t)) - f(t)h'(\bar{F}(t))h''(\bar{G}(t))
\end{bmatrix} \cdot h'(\bar{F}(t))h'(\bar{G}(t)) + f(t)g(t).
\]

\[
\left[ g(t)h''(\bar{G}(t))h'(\bar{F}(t)) - f(t)h'(\bar{F}(t))h''(\bar{G}(t)) \right] \geq 0.
\]

It is well known that the reliability function of a coherent system is strictly increasing in \( p \), so by dividing throughout by \( h'(\bar{F}(t))h'(\bar{G}(t)) \), the above expression reduces to

\[
\left[ f'(t)g(t) - g'(t)f(t) \right] + 
\frac{f(t)g(t)}{h'(\bar{G}(t))} \left[ g(t)h''(\bar{G}(t)) - f(t)h'(\bar{F}(t))h''(\bar{G}(t)) \right] \geq 0.
\]

This further can be written as

\[
\left[ f'(t)g(t) - g'(t)f(t) \right] + f(t)g(t).
\]

\[
\left[ g(t)h''(\bar{G}(t)) - f(t)h'(\bar{F}(t))h''(\bar{G}(t)) \right] \geq 0. \tag{3.2.1}
\]

The first term of the above expression is non-negative due to the fact that \( \bar{F} \preceq \bar{G} \). Further, \( \bar{F} \preceq \bar{G} \) gives
This means, for all \( t \geq 0 \),

\[
\frac{f(t)}{F(t)} \leq \frac{g(t)}{G(t)} \tag{3.2.2}
\]

and

\[
F(t) \geq G(t). \tag{3.2.3}
\]

Thus, from (3.2.3), on using the given condition, we have

\[
\frac{G(t)h''(G(t))}{h'(G(t))} \geq \frac{F(t)h''(F(t))}{h'(F(t))}. \tag{3.2.4}
\]

Hence, by (3.2.2) and (3.2.4), the second term within parenthesis in (3.2.1) is non-negative. This establishes the result.

The result of Chan et al. (1990) that likelihood ratio ordering is always preserved under the formation of \( k \)-out-of-\( n \) systems follows as a corollary to the above theorem.

**COROLLARY 3.2.2:** For a \( k \)-out-of-\( n \) system,

\[
\text{LR} \quad h(F) \geq h(G) \quad \text{if} \quad F \geq G.
\]
Proof: It is enough to show that for a \( k \)-out-of-\( n \) system,

\[
\frac{ph''(p)}{h'(p)} \text{ is nonincreasing in } p.
\]

Since, for a \( k \)-out-of-\( n \) system with iid components,

\[
h(p) = \sum_{i=k}^{n} \binom{n}{i} p^i (1-p)^{n-i}
\]

\[
= \frac{\Gamma(n+1)}{\Gamma(k) \cdot \Gamma(n-k+1)} \int_0^p t^{k-1} (1-t)^{n-k-1} dt,
\]

we have

\[
h'(p) = \frac{\Gamma(n+1)}{\Gamma(k) \cdot \Gamma(n-k+1)} \cdot p^{k-1} (1-p)^{n-k}
\]

and

\[
h''(p) = \frac{\Gamma(n+1)}{\Gamma(k) \cdot \Gamma(n-k+1)} \left[ (k-1)p^{k-2}(1-p)^{n-k} - (n-k)p^{k-1}(1-p)^{n-k-1} \right]
\]

\[
= \frac{\Gamma(n+1)}{\Gamma(k) \cdot \Gamma(n-k+1)} \cdot p^{k-2}(1-p)^{n-k} \left[ (k-1) - (n-k) \frac{p}{1-p} \right].
\]

Thus,

\[
\frac{ph''(p)}{h'(p)} = (k-1) - (n-k) \frac{p}{1-p},
\]

which is nonincreasing in \( p \). Hence the result. \( \square \)
REMARK 3.2.3: Several systems other than k-out-of-n systems satisfy the condition that "$p h''(p)/h'(p)$ is nonincreasing in $p$". A simple example is a structure where two components are in parallel and the third one is in series with these two.

Note that the reliability of the system where two components are in parallel and the third one is in series with these two, is given by

$$h(p) = 2p^2 - p^3.$$ 

Hence

$$h'(p) = 4p - 3p^2 \text{ and } h''(p) = 4 - 6p.$$ 

So,

$$\frac{ph''(p)}{h'(p)} = \frac{4-6p}{4-3p},$$

which is nonincreasing in $p$.  

REMARK 3.2.4: The above corollary can also be proved as a particular case of Theorem 4.6 of Boland et al. (1996) given below:

Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ be two sets of independent absolutely continuous random variables where $X_i \stackrel{LR}{=} Y_j$ for all $i, j (i,j = 1,2,\ldots,n)$. Then $X_{(i)} \stackrel{LR}{=} Y_{(i)}$ for all $i = 1,2,\ldots,n$, where $X_{(i)}$ is the $i^{th}$ order statistic of the random sample $X_1, X_2, \ldots, X_n$ and $Y_{(i)}$ is the $i^{th}$ order statistic of the random sample $Y_1, Y_2, \ldots, Y_n$.  


3.3. SOME RESULTS FOR SYSTEMS OF NON-IDENTICAL COMPONENTS

Consider a system of \( n \) independent and not necessarily identical components in which the \( i \)th component has survival function \( F_i(t) = 1 - F_i(t) \), \( i = 1, 2, \ldots, n \). Let \( h(p) = h(p_1, p_2, \ldots, p_n) \) be the system reliability function. Then it is easily seen that the reversed hazard rate function \( \mu_h(F)(t) \) of the system is given by

\[
\mu_h(F)(t) = \sum_{i=1}^{n} \left[ \frac{F_i(t)}{1-h(p)} \frac{\partial h}{\partial p_i} \left( \frac{f_i(t)}{F_i(t)} \right) \right]_{p_i = F_i(t)}.
\]

On the basis of the above expression we can obtain bounds on the system reversed hazard rate as

\[
\left[ \sum_{i=1}^{n} \left(1 - p_i \right) \frac{\partial h}{\partial p_i} \left( \frac{f_i(t)}{F_i(t)} \right) \right] \min \limits_{1 \leq i \leq n} \mu_{F_i}(t) \leq \mu_h(F)(t) \leq \sum_{i=1}^{n} \left(1 - p_i \right) \frac{\partial h}{\partial p_i} \left( \frac{f_i(t)}{F_i(t)} \right) \max \limits_{1 \leq i \leq n} \mu_{F_i}(t).
\]

(3.3.2)
THEOREM 3.3.1: Let \( h(p) \) be the reliability of a coherent system of \( n \) independent and not necessarily identical components. If

\[
\sum_{i=1}^{n} \frac{(1-p_i)^{\frac{\partial h}{\partial p_i}}}{1-h(p)} \text{ is nondecreasing in } p_i, \tag{3.3.3}
\]

for all \( i = 1, 2, \ldots, n \), then

\[
h(F) \preceq h(G) \quad \text{whenever } F_i \preceq G, \quad i = 1, 2, \ldots, n.
\]

Proof: \( \mu_{h(F)}(t) = \sum_{i=1}^{n} \mu_{F_i}(t) (1-F_i(t)) \frac{\partial h}{\partial p_i} \bigg|_{p = F(t)} \)

\[
\preceq \mu_{G}(t) \sum_{i=1}^{n} (1-G_i(t)) \frac{\partial h}{\partial p_i} \bigg|_{p = G(t)}
\]

\[
= \mu_{h(G)}(t).
\]
The first inequality follows from $\bar{F_i} \geq G$ and the second inequality from (3.3.3), together with the fact that $\bar{F_i} \geq G \Rightarrow \bar{F_i}(t) \geq G(t)$ for all $t \geq 0$.

The following well known result can be found in Barlow and Proschan (1975).

**Lemma 3.3.1:** The system reliability satisfies the following identity:

$$h(p) = p_i h(1_i, p) + (1-p_i) h(0_i, p),$$

where $h(a_i, p) = h(p_1, p_2, \ldots, p_{i-1}, a, p_{i+1}, \ldots, p_n)$, $a = 0, 1$, for $i = 1, 2, \ldots, n$.

**Theorem 3.3.2:** Let $h(p)$ be the reliability function of a $k$-out-of-$n$ system of independent and not necessarily identical components. Then

$$\sum_{i=1}^{n} \frac{(1-p_i) \frac{\partial h}{\partial p_i}}{1-h(p)}$$

is nondecreasing in $p_i$ for all $i = 1, 2, \ldots, n$. 

Proof: On using Lemma 3.3.1, we have

\[
\frac{\partial h_i}{\partial p_i} = h(1_i, p) - h(0_i, p) = p \left( \sum_{j \neq i}^n X_j = k-1 \right),
\]

where \( X_j \) is a binary random variable with \( P(X_j = 1) = p_j \) and \( P(X_j = 0) = 1 - p_j \). Thus

\[
\sum_{i=1}^n (1-p_i) \frac{\partial h_i}{\partial p_i} = \sum_{i=1}^n p \left[ X_i = 0, \sum_{j \neq i}^n X_j = k-1 \right] = (n-k+1) \cdot P \left[ \sum_{i=1}^n X_i = k-1 \right],
\]

since each term used in computing \( P \left[ \sum_{i=1}^n X_i = k-1 \right] \) occurs exactly (n-k+1) times in computing \( \sum_{i=1}^n P \left[ X_i = 0, \sum_{j \neq i}^n X_j = k-1 \right] \). Hence, it is enough to show that

\[
\frac{P \left[ \sum_{i=1}^n X_i = k-1 \right]}{P \left[ \sum_{i=1}^n X_i \leq k-1 \right]} \text{ is nondecreasing in } p_i
\]

for \( i = 1, 2, \ldots, n \), or equivalently, that
\[
\frac{P\left[ \sum_{i=1}^{n} X_i \leq k-2 \right]}{P\left[ \sum_{i=1}^{n} X_i \leq k-1 \right]} \text{ is nonincreasing in } p_i
\]

for \( i = 1, 2, \ldots, n \). This is equivalent to showing that, for \( p_i < p'_i \),

\[
0 \leq \begin{vmatrix}
P\left( \sum_{j \neq i} X_j \leq k-2 \mid p_i \right) & P\left( \sum_{j \neq i} X_j \leq k-2 \mid p'_i \right) \\
\vdots & \vdots \\
P\left( \sum_{j \neq i} X_j \leq k-1 \mid p_i \right) & P\left( \sum_{j \neq i} X_j \leq k-1 \mid p'_i \right)
\end{vmatrix}
\]

\[
= \left( p_i q'_i - p'_i q_i \right) \begin{vmatrix}
P\left( \sum_{j \neq i} X_j \leq k-3 \right) & P\left( \sum_{j \neq i} X_j \leq k-2 \right) \\
\vdots & \vdots \\
P\left( \sum_{j \neq i} X_j \leq k-2 \right) & P\left( \sum_{j \neq i} X_j \leq k-1 \right)
\end{vmatrix},
\]

where \( q_i = 1 - p_i \) and \( q'_i = 1 - p'_i \). Since \( p_i q'_i - p'_i q_i < 0 \), it is enough to show that

\[
\begin{vmatrix}
P\left( \sum_{j \neq i} X_j \leq k-2 \right) & P\left( \sum_{j \neq i} X_j \leq k-3 \right) \\
\vdots & \vdots \\
P\left( \sum_{j \neq i} X_j \leq k-1 \right) & P\left( \sum_{j \neq i} X_j \leq k-2 \right)
\end{vmatrix} \leq 0,
\]
which is equivalent to showing that

\[
\begin{vmatrix}
P(\sum_{j \neq i} X_j \leq k-1) & P(\sum_{j \neq i} X_j \leq k-2) \\
\hline 
P(\sum_{j \neq i} X_j \leq k) & P(\sum_{j \neq i} X_j \leq k-1)
\end{vmatrix} \geq 0.
\]

Defining \( Y_j = 1 - X_j \), the above expression reduces to

\[
\begin{vmatrix}
P(\sum_{j \neq i} Y_j \geq n-k) & P(\sum_{j \neq i} Y_j \geq n-k+1) \\
\hline 
P(\sum_{j \neq i} Y_j \geq n-k-1) & P(\sum_{j \neq i} Y_j \geq n-k)
\end{vmatrix} \geq 0. \tag{3.3.4}
\]

It is known that

\[
\begin{vmatrix}
P(\sum_{j \neq i} Y_j \geq k) & P(\sum_{j \neq i} Y_j \geq k+1) \\
\hline 
P(\sum_{j \neq i} Y_j \geq k-1) & P(\sum_{j \neq i} Y_j \geq k)
\end{vmatrix} \geq 0,
\]

(See Karlin (1962) and also Esary and Proschan (1963)). Hence the required result follows by replacing \( k \) by \( n-k \) in (3.3.4).

**REMARK 3.3.1:** It follows from (3.3.2) and Theorem 3.3.2 that for a k-out-of-n system, it is possible to bound the system reversed hazard rate by a pair of bounds which are decreasing if the component reversed hazard rates are decreasing.
REMARK 3.3.2: Several systems other than the k-out-of-n systems satisfy the condition that
\[ \sum_{i=1}^{n} \frac{(1-p_i)^{\delta h}}{1-h(p)} \] is nondecreasing in \( p_i \).

One such example is a structure of independent components where two components are in series and a third component is in parallel with these two.

Let the two components \( T_1 \) and \( T_2 \), having reliability \( p_1 \) and \( p_2 \) be in series and the component \( T_3 \), having reliability \( p_3 \) be in parallel with \( T_1 \) and \( T_2 \). Then the system reliability is given by

\[ h(p) = p_1 p_2 + p_3 - p_1 p_2 p_3. \]

So,
\[ \frac{\partial h}{\partial p_1} = p_2 (1 - p_3), \]
\[ \frac{\partial h}{\partial p_2} = p_1 (1 - p_3) \]
and
\[ \frac{\partial h}{\partial p_3} = 1 - p_1 p_2. \]

Further,
\[ 1 - h(p) = (1 - p_3)(1 - p_1 p_2). \]

Hence
\[ \frac{(1-p_1)^{\delta h}}{1-h(p)} = \frac{p_2 (1-p_1)}{1-p_1 p_2}, \]
This gives
\[
\frac{(1-p_2) \frac{\partial h}{\partial p_2}}{1-h(p)} = \frac{p_1(1-p_2)}{1-p_1 p_2}
\]
and
\[
\frac{(1-p_3) \frac{\partial h}{\partial p_3}}{1-h(p)} = 1.
\]
This gives
\[
\sum_{i=1}^{n} \frac{(1-p_i) \frac{\partial h}{\partial p_i}}{1-h(p)} = 1 + \frac{p_1 + p_2 - 2p_1 p_2}{1-p_1 p_2},
\]
which is nondecreasing in \( p_1 \) and \( p_2 \), and is trivially nondecreasing in \( p_3 \).

As a consequence of Theorem 3.3.1 and Theorem 3.3.2, we have the following corollary.

**COROLLARY 3.3.1:** Let \( T_1 \) be the life time of a \( k \)-out-of-\( n \) system where \( i \)th component has distribution function \( F_i \) and \( T_2 \) be that of another \( k \)-out-of-\( n \) system of iid components which have distribution function \( G \). Let \( F_i \geq G, i = 1, 2, \ldots, n \). Then \( T_1 \leq T_2 \).
THEOREM 3.3.3: \( \mu_{h(F)}(t) = \sum_{i=1}^{k} \mu_{F_i}(t) \) if and only if the system is a parallel system, provided \( \mu_{F_i}(t) > 0 \) for all \( t > 0 \) and for all \( i = 1,2,\ldots,k \).

Proof: From (3.3.1), we have, for any coherent system of \( k \) components

\[ \mu_{h(F)}(t) = \sum_{i=1}^{k} \mu_{F_i}(t) \times \frac{(1-p_i) \frac{\partial h}{\partial p_i}}{1-h(p)} \Bigg|_{p_i=F_i(t)}. \]  

(3.3.5)

If part: Suppose the system is a parallel system. Then

\[ h(p) = 1 - \prod_{i=1}^{k} (1-p_i), \]

which gives

\[ \frac{(1-p_i) \frac{\partial h}{\partial p_i}}{1-h(p)} = 1. \]

Hence the result follows.

Only if part: Given that \( \mu_{h(F)}(t) = \sum_{i=1}^{k} \mu_{F_i}(t) \).  

(3.3.6)

By Lemma 3.3.1,

\[ 1-h(p) = 1 - p_i h(1_i, p) - (1-p_i)h(0_i, p) \]
\[ h(1_i, p) - p_i h(1_i, p) - (1-p_i)h(0_i, p) \]
\[ = (1-p_i) \left[ h(1_i, p) - h(0_i, p) \right] \]
\[ = (1-p_i) \cdot \left( \frac{\partial h}{\partial p_i} \right). \]

Hence, we have, for any \( 0 < p_i < 1 \),

\[ \frac{(1-p_i)\frac{\partial h}{\partial p_i}}{1-h(p)} \leq 1. \] (3.3.7)

From (3.3.5) and (3.3.6), we have

\[ \sum_{i=1}^{k} \mu_{p_i} (t) \cdot \alpha_i(t) = 0, \] (3.3.8)

where

\[ \alpha_i(t) = \left[ 1 - \frac{(1-p_i)\frac{\partial h}{\partial p_i}}{1-h(p)} \right]_{p_i=p_i(t)} \]
\[ \equiv 0. \]

The inequality follows from (3.3.7). Further, since \( \mu_{p_i} (t) > 0 \) for all \( t > 0 \), (3.3.8) gives \( \alpha_i(t) = 0 \) for all \( i = 1, 2, \ldots, k \). This means
for all $i = 1,2,\ldots,k$. This means
\[
\left(1-p_i\right)\frac{\partial h}{\partial p_i}
\left.\right|_{p_i=\bar{F}_i(t)} = 1
\]
for all $i = 1,2,\ldots,k$. Let $F$ be the distribution function of the system. Then, by Lemma 3.3.1, (3.3.9) gives
\[
F(t) = F_i(t) \left[ h(1_i,\bar{F}(t)) - h(0_i,\bar{F}(t)) \right]
\]
for all $i = 1,2,\ldots,k$ and for all $t > 0$. Taking limit as $t$ approaches infinity, we have
\[
h(e_i) = 1
\]
for all $i = 1,2,\ldots,k$, where $e_i$ is the $i^{th}$ unit vector (having unity at the $i^{th}$ place and '0' in all other places). Now, on using (3.3.11) and monotone increasing property of the reliability function $h$, (3.3.10) gives
\[
F(t) = F_1(t) \left[ 1 - h(0,\bar{F}(t)) \right].
\]
\[
= F_1(t) \left[ 1 - \left( \bar{F}_2(t) h(0,1,\bar{F}(t)) + F_2(t) h(0,0,\bar{F}(t)) \right) \right]
\]
\[
= F_1(t) \left[ 1 - \left( \bar{F}_2(t) + F_2(t) h(0,0,\bar{F}(t)) \right) \right]
\]
This, on further simplification, gives

\[ F(t) = \prod_{i=1}^{k} F_i(t), \]

which is the distribution function of a parallel system of \( k \) independent components.

Boland et al. (1994) had observed that for a series system of independent components, the hazard rate function of the system is the sum of the hazard rates of the individual components. The converse also holds as the following theorem shows.

**THEOREM 3.3.4:** Let \( r_h(F)(t) \) be the hazard rate function of a system constructed out of \( k \) independent components with \( i \)th component having hazard rate function \( r_i(t) \), \( i = 1, 2, \ldots k \). Then

\[ r_h(F)(t) = \sum_{i=1}^{k} r_i(t) \]

if and only if the system is a series system, provided \( r_i(t) > 0 \) for all \( t > 0 \) and \( i = 1, 2, \ldots k \).
Proof: It is well known that for any coherent system of $k$ independent components (see also Boland et al. (1994))

$$r_h(F)(t) = \sum_{i=1}^{k} r_i(t) \cdot \frac{\frac{\partial h}{\partial \pi_i} \bigg|_{\pi_i=P_i} h(p)}{h(p)} = \prod_{i=1}^{k} r_i(t) . \quad (3.3.12)$$

If part: Suppose the system is a series system. Then

$$h(p) = \prod_{i=1}^{k} p_i ,$$

which gives

$$\frac{p_i \frac{\partial h}{\partial \pi_i} \bigg|_{\pi_i=P_i}}{h(p)} = 1 .$$

Hence the result follows.

Only if part: Given that $r_h(F)(t) = \sum_{i=1}^{k} r_i(t)$.

(3.3.13)

By Lemma 3.3.1,

$$h(p) = p_i h(l_i, p) + (1-p_i) h(0_i, p)$$

$$= p_i \left[ h(l_i, p) - h(0_i, p) \right]$$

$$= p_i \left( \frac{\partial h}{\partial \pi_i} \right) .$$

Hence, we have, for any $0 < p_i < 1$,
From (3.3.12) and (3.3.13), we have

\[
\sum_{i=1}^{k} r_i(t) \cdot \beta_i(t) = 0, \tag{3.3.15}
\]

where

\[
\beta_i(t) = \left[ 1 - \frac{\frac{\partial h}{\partial p_i}}{h(p)} \bigg|_{p_i=F_i(t)} \right] = 0.
\]

The inequality follows from (3.3.14). Further, since \( r_i(t) \) > 0 for all \( t > 0 \), (3.3.15) gives \( \beta_i(t) = 0 \) for all \( i = 1, 2, \ldots, k \). This means

\[
\left. \frac{\partial h}{\partial p_i} \right|_{p_i=F_i(t)} = 1
\]

for all \( i = 1, 2, \ldots, k \). This means

\[
\left[ \frac{\partial h}{\partial p_i} \right]_{p_i=F_i(t)} = h(p) \bigg|_{p_i=F_i(t)} \tag{3.3.16}
\]

for all \( i = 1, 2, \ldots, k \). Let \( F \) be the survival function of the system. Then, by Lemma 3.3.1, (3.3.16) gives
\( F(t) = F_i(t) \left[ h(1,i,F(t)) - h(0,i,F(t)) \right] \) \hspace{1cm} (3.3.17)

for all \( i = 1,2,\ldots,k \) and for all \( t > 0 \). Taking limit as \( t \) approaches zero, we have

\[ h(0,i,1) = 0, \] \hspace{1cm} (3.3.18)

for all \( i = 1,2,\ldots,k \), where \( h(0,i,1) \) is the reliability function \( h \) evaluated at \( p_i = 0 \) and \( p_j = 1 \) for all \( j \neq i \). Now, on using (3.3.18) and monotone increasing property of the reliability function \( h \), (3.3.17) gives

\[ F(t) = F_1(t).h(1,F(t)) \]

\[ = F_1(t) \left[ F_2(t)h(1,1,F(t)) + F_2(t)h(1,0,F(t)) \right] \]

\[ = F_1(t).F_2(t).h(1,1,F(t)). \]

This, on further simplification, gives

\[ F(t) = \prod_{i=1}^{k} F_i(t), \]

which is the survival function of a series system of \( k \) independent components.
3.4. CLOSURE OF REVERSED HAZARD RATE ORDERING UNDER POISSON SHOCK MODEL

In this section we establish the closure of reversed hazard rate ordering under Poisson shock model. Let $X$ and $Y$ be two discrete random variables having mass function $p_k = P(X = k)$ and $q_k = P(Y = k)$. Write $P(k) = P(X \leq k)$ and $Q(k) = P(Y \leq k)$. We define the reversed hazard rate ordering for two discrete random variables as follows:

**DEFINITION 3.4.1**: $X$ is said to be larger than $Y$ in discrete reversed hazard rate ordering ($X \triangleright Y$, or $P(k) \triangleright Q(k)$) if

$$\frac{p_k}{P(k)} \geq \frac{q_k}{Q(k)}$$

for all $k = 0, 1, 2, ...$

Suppose that a device is subject to a sequence of shocks occurring randomly as events in a Poisson process with constant intensity $\lambda$. Suppose further that the probability that the device survives $k$ ($\geq 0$) shocks is $\bar{F}(k)$, where $1 = \bar{F}(0) \geq \bar{F}(1) \geq \bar{F}(2) \geq ...$. Denoting the system life by $T$, the distribution function of the device is given by
\[ F(t) = P(T \leq t) \]
\[ = \sum_{k=0}^{\infty} \left[ \exp(-\lambda t) \frac{(\lambda t)^k}{k!} \right] . P(k), \]

where \( P(k) = 1 - \bar{P}(k) \). Consider another device having life time \( S \), which is also subject to shocks occurring randomly as events in a Poisson process with same intensity \( \lambda \) and the device has the probability \( \bar{Q}(k) \) of surviving the first \( k \) shocks, where \( 1 = \bar{Q}(0) \geq \bar{Q}(1) \geq \bar{Q}(2) \geq \ldots \). Then the distribution function of the device is given by

\[ G(t) = P(S \leq t) \]
\[ = \sum_{k=0}^{\infty} \left[ \exp(-\lambda t) \frac{(\lambda t)^k}{k!} \right] . Q(k), \]

where \( Q(k) = 1 - \bar{Q}(k) \).

The following theorem shows that if \( P(k) \) and \( Q(k) \) are ordered with respect to the discrete reversed hazard rate ordering, then the system life time \( T \) and \( S \) are ordered with respect to the reversed hazard rate ordering.

**THEOREM 3.4.1:** If \( P(k) \geq Q(k) \) for all \( k = 1,2,\ldots \), then \( T \geq S \).
Proof: Given that

\[ \frac{P_k}{P(k)} \geq \frac{q_k}{Q(k)} \]

for all \( k = 0,1,2,\ldots \). This can equivalently be written as

\[ \frac{P(k)}{Q(k)} \]

is nondecreasing in \( k = 0,1,2,\ldots \).

This means, for some real \( c (> 0) \), \( P(k) - c.Q(k) \) has atmost one change of sign in \( k \) and if one such change does occur, it occurs from - to +. Again, since \( \left[ \exp(-\lambda t) \frac{(\lambda t)^k}{k!} \right] \) is TP2 in \((k,t)\), for all \( k \in (0,1,2,\ldots) \) and \( t \in (0,\infty) \) (cf. Karlin, 1968), on using variation diminishing property, we have

\[ \sum_{k=0}^{\infty} \exp(-\lambda t) \frac{(\lambda t)^k}{k!} \left[ P(k) - c.Q(k) \right] \]

has atmost one change of sign in \( t \) and if one such change does occur, it occurs from - to +. This gives

\[ \frac{\sum_{k=0}^{\infty} \exp(-\lambda t) \frac{(\lambda t)^k}{k!} P(k)}{\sum_{k=0}^{\infty} \exp(-\lambda t) \frac{(\lambda t)^k}{k!} Q(k)} \]

is nondecreasing in \( t \), or equivalently,

\[ \frac{F(t)}{G(t)} \]

is nondecreasing in \( t \).

Hence the result.