CHAPTER ONE
1.1 INTRODUCTION:

In Reliability, Engineering, Operations Research and related fields, components of different brands need to be compared with respect to their life times. In Econometrics, different income distributions are to be compared in terms of the corresponding income inequalities and the various random prospects need comparison so that the better one can be chosen. In Biological sciences, the life times (or the remaining life times) of control group of living organisms need to be compared with group receiving drug in order to conclude about the effectiveness of a particular drug. It is also of interest to compare the life times of different breeds of animals in order to decide which breed is to be preferred. In Reliability and Survival Analysis, the remaining life times of a component at different ages need comparison to determine whether the component is ageing with time. The idea of comparison of life times of systems is also utilized in allocating spares to systems in Reliability Theory.

Let us take one very simple example. Suppose we are interested in comparing 100W light bulbs produced by Philips Company and GEC Company. Let $X$ be the life time of a 100W bulb produced by Philips and $Y$ be that of GEC. Then Philips bulb should be preferred to the GEC bulb if $X$ is larger than $Y$ in some sense.
Since, here $X$ and $Y$ are random variables and not simple real numbers, hence comes the role of stochastic orderings. For any two random variables $X$ and $Y$, the simplest way of comparing is by the comparison of their means, i.e. $X$ is said to be larger than $Y$ if

$$E(X) \geq E(Y).$$

However, this type of comparison has two drawbacks. Firstly, such a comparison, based on only two single numbers measuring only centres of the distributions, is often not very informative. Secondly, the means sometimes do not exist. In many practical situations one may have much more information regarding the random variables than just their means. If one is interested in comparing two random variables having same mean, one may think of comparing in terms of dispersions, which can be done by comparing their respective standard deviations. However, such a comparison is, again, based on only two numbers having same type of limitations as that of means. In order to overcome these difficulties, one way or probably the best way of comparing two distributions is through stochastic order relations.
1.2. SOME USEFUL NOTATIONS, DEFINITIONS & THE REVIEW OF LITERATURE

In the literature, several concepts of stochastic order relations have been defined for comparisons of random variables. They are useful for modeling in Reliability and in Economic applications and serve as a mathematical tool to prove important results in applied probability. Examples of applications may be found in Lehmann (1955), Van Zwet (1964), Barlow and Proschan (1975), Ross (1983), Stoyan (1983), Singh (1989), Singh and Jain (1989), Deshpande et al. (1990) and Shaked and Shanthikumar (1994) among others, where stochastic order relations are used in Reliability, Queues and other Stochastic Processes contexts. One such stochastic comparison is based on the respective survival functions. Let the distribution function of $X$ be $F(x) = P(X \leq x)$ and that of $Y$ be $G(x) = P(Y \leq x)$. Write, $F(x) = 1 - F(x)$ and $G(x) = 1 - G(x)$, the respective survival function of $X$ and $Y$.

**DEFINITION 1.2.1:** $X$ is said to be larger than $Y$ in usual ST stochastic ordering, written $X \succeq Y$, if

$$P(X > x) \geq P(Y > x) \quad (1.2.1)$$

for every $x$, i.e. $F(x) \geq G(x)$ for all $x$. This means, at each point $x$, the graph of $F(x)$ lies above the graph of $G(x)$.

In Reliability, apart from the survival function, failure rate (or hazard rate) function plays an important role. Let $f(t)$
be the density function (assumed to exit) corresponding to the survival function \( \bar{F}(t) \). Then the failure rate function \( r_F(t) \) is defined as

\[
    r_F(t) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} P \left[ t \leq X \leq t + \varepsilon \mid X > t \right]
\]

\[
    = \frac{f(t)}{F(t)}, \text{ where defined.} \tag{1.2.2}
\]

Thus, \( r_F(t) \) gives the instantaneous failure rate of \( X \). If we choose \( \varepsilon (> 0) \) very small then, given that the system has survived upto time \( t \), the probability that it will fail before \( t + \varepsilon \) is given approximately by \( \varepsilon . r_F(t) \). If \( r_F(t) \) is nondecreasing in \( t \), we say that \( X \) (or \( F \)) is IFR (Increasing Failure Rate) and if \( r_F(t) \) is nonincreasing in \( t \), we say that \( X \) (or \( F \)) is DFR (Decreasing Failure Rate). Clearly, higher the failure rate, smaller \( X \) should be. This is the intuition behind the failure rate ordering.

**Definition 1.2.2:** Let \( X \) and \( Y \) be two absolutely continuous nonnegative random variables with distribution function \( F \) and \( G \) respectively. \( X \) is said to be larger than \( Y \) in failure rate FR ordering (\( X \preceq Y \)), if

\[
    r_F(x) \preceq r_G(x) \tag{1.2.3}
\]
for all \( x \), or equivalently,

\[
\frac{\bar{F}(x)}{\bar{G}(x)}\text{ is nondecreasing in } x \geq 0. \tag{1.2.4}
\]

Thus, if the variables are not absolutely continuous, by (1.2.4), one can define failure rate ordering. Let \( R_t^X = (X-t|X > t) \) denote the residual life of a random variable \( X \) which has survived up to time \( t \). Then \( R_t^X \Preceq R_t^Y \) for all \( t \geq 0 \) if and only if

\[
P(R_t^X \geq x) \geq P(R_t^Y \geq x) \tag{1.2.5}
\]

for all \( x,t \geq 0 \). If \( X \) and \( Y \) have respective survival function \( \bar{F} \) and \( \bar{G} \), then (1.2.5) can equivalently be written as

\[
\frac{\bar{F}(x+t)}{\bar{F}(t)} \geq \frac{\bar{G}(x+t)}{\bar{G}(t)}
\]

for all \( x,t \geq 0 \), which can equivalently be written as

\[
\frac{\bar{F}(x+t)}{\bar{G}(x+t)} \geq \frac{\bar{F}(t)}{\bar{G}(t)}
\]

for all \( x,t \geq 0 \). This means

\[
\frac{\bar{F}(x)}{\bar{G}(x)}\text{ is nondecreasing in } x \geq 0. \tag{FR}
\]

Hence, by (1.2.4), we have \( X \succeq Y \). Thus, we have the following theorem (cf. Shaked and Shanthikumar (1994)).
THEOREM 1.2.1: $X \succeq Y$ if and only if $R_X^T \succeq R_Y^T$ for all $t \geq 0$.

Since $R_0^X = X$, we see that failure rate ordering is stronger than stochastic ordering. There is another partial ordering called likelihood ratio (LR) ordering which is stronger than failure rate ordering. The formal definition of LR ordering is as follows.

DEFINITION 1.2.3: Let $X$ and $Y$ be two absolutely continuous random variables having density function $f$ and $g$ respectively. $X$ is said to be larger than $Y$ in likelihood ratio ordering, written $X \succeq LR Y$, if

$$\frac{f(x)}{g(x)} \text{ is nondecreasing in } x. \quad (1.2.6)$$

Though, LR ordering does not have any intuitive justification, in many situations it is easy to verify.

Similar to failure rate function, there is another function $\mu_F(x)$, called reversed (or retro, or backward) hazard rate function, defined as

$$\mu_F(x) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \mathbb{P}[x - \epsilon < X \leq x | X \leq x]$$

$$= \frac{f(x)}{F(x)}, \text{ where defined.} \quad (1.2.7)$$

Suppose a system has failed before time $x$. Then for any small positive number $\epsilon$, the probability that it survived up to time $x - \epsilon$...
is given approximately by $e^{jip(x)}$. In the analysis of left censored data, the reversed hazard rate function plays the same role as the hazard rate function plays in the analysis of right censored data (cf. Anderson et al. (1993)). Shaked and Shanthikumar (1994) have defined reversed hazard rate ordering as follows.

**DEFINITION 1.2.4:** For two absolutely continuous nonnegative random variables $X$ and $Y$ having respective reversed hazard rate function $\mu_F$ and $\mu_G$, $X$ is said to be larger than $Y$ in reversed hazard rate RHR ordering, written $X \succcurlyeq Y$ if

$$\mu_F(x) \geq \mu_G(x)$$

(1.2.8)

for all $x \geq 0$. This ordering was discussed, for the first time, to the best of our knowledge, by Keilson and Sumita (1982). They used the terminology "$Y$ is uniformly smaller than $X$ in the negative direction". One can easily see that (1.2.8) is equivalent to

$$\frac{F(x)}{G(x)}$$

is nondecreasing in $x \geq 0$. (1.2.9)

Thus, when the random variables are not absolutely continuous, the reversed hazard rate ordering can be defined in terms of (1.2.9).

One can verify that $X \succcurlyeq Y$ if and only if

$$[X|X \leq t] \geq [Y|Y \leq t]$$
for all $t \geq 0$ (see also Shaked and Shanthikumar, 1994). Thus, reversed hazard rate ordering is stronger than usual stochastic ordering, yet is weaker than LR ordering.

Deshpande et al. (1990) defined another ordering called mean residual life (MR) ordering, which compares the remaining life of two components at different points of time.

**DEFINITION 1.2.5:** For two nonnegative random variables $X$ and $Y$, $X$ is said to be larger than $Y$ in mean residual life (MR) ordering, written $X \succeq_{MR} Y$, if

$$E(R^X_t) \geq E(R^Y_t)$$

for all $t \geq 0$, i.e.

$$E\left[X-t \mid X > t\right] \geq E\left[Y-t \mid Y > t\right] \quad (1.2.10)$$

for all $t \geq 0$. The function $E(X-t \mid X > t)$ is called the mean residual life function or the mean life expectancy at age $t$ and has been shown by Gupta (1975) that the mean residual life function determines the distribution function uniquely. Since stochastic ordering implies expectation ordering, it follows from (1.2.5) that failure rate ordering is stronger than mean residual life ordering. Note that (1.2.10) can equivalently be written as
\[
\frac{\int_0^\infty F(x) \, dx}{F(t)} \geq \frac{\int_0^\infty G(x) \, dx}{G(t)}
\]

for all \( t \geq 0 \). This can equivalently be written as
\[
\frac{d}{dt} \left[ \frac{\int_0^\infty F(x) \, dx}{\int_0^\infty G(x) \, dx} \right] \geq 0.
\]

This means
\[
\frac{\int_0^\infty F(u) \, du}{\int_0^\infty G(u) \, du}
\]

is nondecreasing in \( t \). (1.2.11)

They have also defined harmonic average mean residual life (HAMR) ordering as follows:

**DEFINITION 1.2.6:** \( X \) is said to be larger than \( Y \) in harmonic average mean residual life ordering, written \( X \preceq_{\text{HAMR}} Y \), if
\[
\left[ \frac{1}{X} \int_0^X \frac{du}{e_F(u)} \right]^{-1} \geq \left[ \frac{1}{X} \int_0^X \frac{du}{e_G(u)} \right]^{-1}
\]

for all \( x > 0 \), where \( e_F(t) = E_F(R_t^X) \) and \( e_G(t) = E_G(R_t^Y) \). Note that
\[ \frac{d}{dx} \left[ -\ln \int_x^\infty F(u) \, du \right] = \frac{1}{e_p(x)}. \]

Hence, (1.2.12) can equivalently be written as

\[ \int_0^x d \left[ -\ln \int_u^\infty F(t) \, dt \right] = \int_0^x d \left[ -\ln \int_u^\infty G(t) \, dt \right] \]

for all \( x \geq 0 \), which, in turn, gives

\[ \ln \mu_F - \ln \int_x^\infty F(u) \, du \geq \ln \mu_G - \ln \int_x^\infty G(u) \, du \]

for all \( x \geq 0 \), where \( \mu_F \) and \( \mu_G \) denote the mean corresponding to the distribution function \( F \) and \( G \) respectively. This further reduces to, for all \( x \geq 0 \),

\[ \frac{1}{\mu_F} \int_x^\infty F(u) \, du \geq \frac{1}{\mu_G} \int_x^\infty G(u) \, du. \quad (1.2.13) \]

Now, from (1.2.11), \( X \succeq Y \) implies

\[ \frac{\int_x^\infty F(u) \, du}{\int_x^\infty G(u) \, du} \geq \frac{\int_0^\infty F(u) \, du}{\int_0^\infty G(u) \, du} = \frac{\mu_F}{\mu_G}, \]

which gives (1.2.13). Hence, we see that MR ordering is stronger than HAMR ordering.
Define

\[ l_F(x) = \frac{d}{dx} \left[ -\ln f(x) \right], \quad (1.2.14) \]

\[ r_F(x) = \frac{d}{dx} \left[ -\ln \bar{F}(x) \right] \quad (1.2.15) \]

and

\[ u_F(x) = \frac{d}{dx} \left[ -\ln \int_x^\infty \bar{F}(u)du \right]. \quad (1.2.16) \]

It can be easily checked that \( r_F(x) \) is the usual failure rate function and \( u_F(x) = (e_F(x))^{-1} \). From the previous definitions of partial orderings and the functions defined in (1.2.14)-(1.2.16), Singh (1989) observed the following:

**THEOREM 1.2.2:** For two nonnegative random variables \( X \) and \( Y \) having respective distribution function \( F \) and \( G \),

(i) \( X \preceq Y \) if and only if \( l_F(x) \preceq l_G(x) \) for all \( x \geq 0 \);

(ii) \( X \preceq Y \) if and only if \( r_F(x) \preceq r_G(x) \) for all \( x \geq 0 \);

(iii) \( X \preceq Y \) if and only if \( u_F(x) \preceq u_G(x) \) for all \( x \geq 0 \);

(iv) \( X \preceq Y \) if and only if \( \int_0^x r_F(t)dt \preceq \int_0^x r_G(t)dt \) for all \( x \geq 0 \);
Following is a partial ordering of life distribution which emerges in a natural way if one observes the known partial orderings through the above theorem (cf. Singh, 1989).

**DEFINITION 1.2.7:** X is said to be larger than Y in Weak Likelihood Ratio (WLR) ordering, written $X \preceq Y$, if

$$\int_0^X l_F(t)dt \leq \int_0^X l_G(t)dt$$

(1.2.17)

for all $x \geq 0$. One can easily check that

$$X \preceq Y \text{ if and only if } \frac{f(x)}{g(x)} \geq \frac{f(0)}{g(0)},$$

assuming $\frac{f(0)}{g(0)} \in (0, \infty)$. Again, generalizing the functions defined in (1.2.14)-(1.2.16), one can define

$$v_F(x) = \frac{d}{dx} \left[ -\ln \int_x^\infty \frac{F(t)dt}{u} \right].$$

(1.2.18)

Simple mathematical calculations show that
Generalizing on the same line of Theorem 1.2.2, Singh (1989) defines the following:

**DEFINITION 1.2.8:** For two nonnegative random variables $X$ and $Y$ having respective distribution function $F$ and $G$, $X$ is said to be larger than $Y$ in Variance Residual Life (VR) Ordering, written $\text{VR} \; X \succeq Y$, if

\[ v_F(x) \leq v_G(x) \]

for all $x \geq 0$. Thus, $X \succeq Y$, if and only if

\[
\frac{d}{dx} \left[ -\ln \int_x^\infty F(t) \, dt \right] \leq \frac{d}{dx} \left[ -\ln \int_x^\infty G(t) \, dt \right]
\]

for all $x \geq 0$, which further reduces to

\[
\frac{\int_x^\infty F(t) \, dt}{\int_x^\infty F(t) \, dt} \leq \frac{\int_x^\infty G(t) \, dt}{\int_x^\infty G(t) \, dt}
\]

for all $x \geq 0$. This means

\[
E[(X-x)^2 | X \geq x] = \frac{2e_F(x)}{E[(X-x)^2 | X \geq x]}.
\]
for all $x \leq 0$, i.e.

$$
\frac{\int_0^\infty \int_0^\infty F(t) dt \, du}{\int_0^\infty \int_0^\infty G(t) dt \, du}
$$

is nondecreasing in $x$.

Further, it can be noted that

$$
MR \quad VR
X \preceq Y \Rightarrow X \preceq Y
$$

(refer to Singh, 1989). Thus, we have the following chain of implications:

![Implications Diagram]

Stoyan (1983) defines convex and concave orderings as follows:
DEFINITION 1.2.9: X is said to be larger than Y in convex \text{CX} ordering, written X \succeq Y, if

$$\int_{x}^{\infty} F(t) dt \geq \int_{x}^{\infty} G(t) dt$$ \hspace{1cm} (1.2.20)

for all x \geq 0. Sometimes convex ordering is also known as variability ordering and is written as X \succeq Y (for instance, in Ross, 1983).

DEFINITION 1.2.10: X is said to be larger than Y in concave \text{CV} ordering, written X \preceq Y, if

$$\int_{0}^{X} F(t) dt \geq \int_{0}^{X} G(t) dt$$ \hspace{1cm} (1.2.21)

for all x \geq 0.

Being motivated by the above results and the unified way of formulation of different partial orderings by Singh (1989), Fagiuoli and Pellerey (1993) defined some partial orderings in a general way so that most of the partial orderings discussed earlier become the particular cases of their general orderings. Some of these general orderings were proposed by Averous and Meste (1989), but their presentation was different from one given in Fagiuoli and Pellerey (1993).
For any non-negative absolutely continuous random variable $X$ with density function $f(x)$ and survival function $F(x)$, write

$$T_0(X, x) = f(x)$$

and

$$T_s(X, x) = \frac{\int_0^\infty T_{s-1}(X, t) dt}{\mu_{s-1}(X)}$$

for $s = 1, 2, \ldots$, where

$$\mu_s(X) = \int_0^\infty T_s(X, t) dt,$$

$s = 0, 1, 2, \ldots$ Also define

$$r_s(X, x) = \frac{T_{s-1}(X, x)}{\int_0^\infty T_{s-1}(X, t) dt},$$

Clearly,

$$T_1(X, x) = F(x),$$

$$T_2(X, x) = \frac{1}{\mu_F} \int_x^\infty F(t) dt,$$
the survival function of the equilibrium distribution of $X$, which plays an important role in ageing concepts (cf. Deshpande et al., 1986), whereas $T_2(X,x)$ is the survival function of the equilibrium distribution of the distribution with survival function $T_{s-1}(X,x)$, $s = 2, 3, \ldots$. Further note that
\[
\mu_0(X) = 1, \\
\mu_1(X) = \mu_F,
\]
the mean of $X$ and
\[
r_1(X,x) = r_F(x),
\]
the failure rate function of $X$. The following definition is due to Fagioli and Pellerey (1993).

**DEFINITION 1.2.11:** $X$ is said to be larger than $Y$ in

(a) $s$-FR ordering ($X \preceq Y$) if
\[
\frac{T_s(X,x)}{T_s(Y,x)} \text{ is nondecreasing in } x \geq 0;
\]

(b) $s$-ST ordering ($X \succeq Y$) if
\[
\frac{T_s(X,x)}{T_s(Y,x)} \preceq \frac{T_s(X,0)}{T_s(Y,0)}
\]
for all $x \geq 0$;
(c) $s$-CV ordering $(X \succeq Y)$ if

$$\int_0^\infty \frac{r_s(X,u)}{r_s(X,0)} du \geq \int_0^\infty \frac{r_s(Y,u)}{r_s(Y,0)} du$$

for all $x \geq 0$;

(d) $s$-CX ordering $(X \succeq Y)$ if

$$\int_x^\infty \frac{r_s(X,u)}{r_s(X,0)} du \geq \int_x^\infty \frac{r_s(Y,u)}{r_s(Y,0)} du$$

for all $x \geq 0$;

(e) $s$-SFR (Starting Failure Rate) ordering $(X \succeq Y)$ if

$$r_s(X,0) \leq r_s(Y,0).$$

It has been observed that the following equivalences hold:

0-FR $\iff$ LR; 1-FR $\iff$ FR; 2-FR $\iff$ MR; 3-FR $\iff$ VR;

0-ST $\iff$ WLR; 1-ST $\iff$ ST; 2-ST $\iff$ HAMR; 1-CV $\iff$ CV;

1-CX $\iff$ CX.

All the above mentioned partial orderings have their natural discrete counterparts, e.g. see Shaked and Shanthikumar (1994).
From the existing results and the results proved in Fagiuoli and Pellerey (1993), we have the following chain of implications:

$$0-\text{FR} \iff LR \longrightarrow 1-\text{FR} \iff FR \longrightarrow 2-\text{FR} \iff MR \longrightarrow 3-\text{FR} \iff VR \longrightarrow$$

$$0-\text{ST} \iff WLR \quad 1-\text{ST} \iff ST \quad 2-\text{ST} \iff HAMR \quad 3-\text{ST} \quad \ldots$$

$$0-\text{CX} \quad 1-\text{CX} \iff CX \quad 2-\text{CX} \quad 3-\text{CX} \quad \ldots$$

$$0-\text{CV} \quad 1-\text{CV} \iff CV \quad 2-\text{CV} \quad 3-\text{CV} \quad \ldots$$

$$1-\text{SFR} \quad 2-\text{SFR} \quad 3-\text{SFR}$$

**Fig. 1.2.2**

In connection with different life distributions, there is an interesting aspect of life distributions, known as ageing of distributions. By ageing we mean mathematical specification of degradation of equipment over time. Generally, two kinds of ageing are considered in the literature, viz. positive ageing and negative ageing. Positive ageing means the adverse effect of age on the random residual lifetime of the unit, and the dual concept of negative ageing means beneficial effect of age. Different types of positive and negative ageing have been described in different ways in the literature. It is considered to be axiomatic that any effect of age on the unit which contributes to the reduction of
its residual life time (in some probabilistic sense) is to be taken as the adverse effect and the phenomenon is to be called positive ageing (cf. Deshpande et al. (1986)). The following are few commonly used ageing classes.

**DEFINITION 1.2.12:** A nonnegative random variable having survival function $\overline{F}$ and density $f$ is said to be

(a) Increasing Likelihood Ratio (ILR) if

$$\frac{f(x+t)}{f(t)}$$

is nonincreasing in $t$

for all $x \geq 0$;

(b) Increasing Failure Rate (IFR) if

$$r_F(t) \text{ is nondecreasing in } t; \quad (1.2.26)$$

(c) Increasing Failure Rate in Average (IFRA) if

$$\frac{1}{x} \int_0^x r_F(t) \, dt \text{ is nondecreasing in } x > 0; \quad (1.2.27)$$

(d) Decreasing in Mean Residual Life (DMRL) if

$$E(R^X t) \text{ is nonincreasing in } t;$$

(e) Decreasing in Variance Residual Life (DVRL) if

$$V(X-t|X \geq t) \text{ is nonincreasing in } t;$$
(f) New Better than Used (NBU) if

\[ \overline{F}(x+t) \leq \overline{F}(x) \cdot \overline{F}(t) \]

for all \( x, t \geq 0 \);

(g) New Better than Used in Expectation (NBUE) if

\[ E(R^X_t) \leq E(X). \]

The corresponding negative ageing classes have been defined in the literature with obvious modifications.

Bryson and Siddiqui (1969) and Launer (1984) have shown that the following holds:

\[ \text{ILR} \implies \text{IFR} \implies \text{DMRL} \implies \text{DVRL} \]

\[ \implies \text{IFRA} \implies \text{NBU} \implies \text{NBUE}. \]

Fig. 1.2.3

(1.2.26) can equivalently be written as

\[ \frac{\overline{F}(x+t)}{\overline{F}(t)} \text{ is nonincreasing in } t \]

for all \( x \geq 0 \). It is well known that (see also Barlow and Proschan, 1975) (1.2.27) holds if and only if
\[ \left( \frac{1}{F(x)} \right)^{1/x} \text{ is nonincreasing in } x. \]  

(1.2.28)

Shaked and Shanthikumar (1994) have defined the decreasing reversed hazard rate distributions as under:

**DEFINITION 1.2.13:** A non-negative random variable \( X \) (or its distribution function \( F \)) is said to have decreasing reversed hazard rate if \( \mu_F(x) \) is nonincreasing in \( x \geq 0 \).

The idea of decreasing reversed hazard rate goes back to Barlow et al. (1963), who showed that the reversed hazard rate of \( -X \) is nonincreasing if and only if the hazard rate of \( X \) is nondecreasing. Block et al. (1997) have shown that there does not exist any nonnegative random variable having increasing reversed hazard rate function. They have also shown that the well known ageing distributions exponential, gamma, Weibull all have decreasing reversed hazard rate functions. Sengupta and Nanda (1997) have shown that \( F \) has decreasing reversed hazard rate function if and only if

\[ F^{P+q}(x) = F^P(x)^q \cdot F^q(x)^p \]

for all \( x \geq 0 \) and for all positive integers \( p \) and \( q \). They have also given some other characterization results for the decreasing reversed hazard rate distributions.
Fagiuoli and Pellerey (1993) have defined the generalized ageing criteria in the sense that the ageing properties given in definition 1.2.13 follow as particular cases of their definition.

**DEFINITION 1.2.14:** For $s = 0, 1, 2, \ldots$, $X$ is said to be

(a) $s$-IFR ($s$-DFR) if $r_s(X,x)$ is nondecreasing (nonincreasing) in $x > 0$;

(b) $s$-IFRA ($s$-IFRA) if $\frac{1}{x} \int_0^x r_s(X,t) dt$ is nondecreasing (nonincreasing) in $x > 0$;

(c) $s$-NBU ($s$-NWU) if $T_s(X,x+t) > T_s(X,0)$ for all $x, t \geq 0$.

It can be noted that the following equivalences hold:

0-IFR $\iff$ ILR; 1-IFR $\iff$ IFR; 2-IFR $\iff$ DMRL;
3-IFR $\iff$ DVRL; 1-IFRA $\iff$ IFRA; 1-NBU $\iff$ NBU.

Note that $X$ is $s$-IFR if and only if

$r_s(X,t)$ is nondecreasing in $t \geq 0$.

This means, for all $x, t \geq 0$,

$$\frac{T_{s-1}(X,t)}{\int_0^t T_{s-1}(X,u) du} \leq \frac{T_{s-1}(X,x+t)}{\int_0^{x+t} T_{s-1}(X,u) du}.$$
This, further, can equivalently be written as, for all $x,t \geq 0$,

$$
\left[ x + t \int_{0}^{\infty} \bar{T}_{s-1}(X,u) \, du \right] \bar{T}_{s-1}(X,t) - \left[ \int_{t}^{\infty} \bar{T}_{s-1}(X,u) \, du \right] \bar{T}_{s-1}(X,x+t) \leq 0.
$$

This further gives, for all $x,t \geq 0$,

$$
\frac{d}{dt} \left[ \left( \int_{x+t}^{\infty} \bar{T}_{s-1}(X,u) \, du \right) \left( \int_{t}^{\infty} \bar{T}_{s-1}(X,u) \, du \right) \right] \leq 0,
$$
giving, for all $x,t \geq 0$,

$$
\frac{d}{dt} \left[ \frac{\bar{T}_{s}(X,x+t)}{\bar{T}_{s}(X,t)} \right] \leq 0.
$$

Hence,

$$
\frac{\bar{T}_{s}(X,x+t)}{\bar{T}_{s}(X,t)} \text{ is nonincreasing in } t \quad (1.2.29)
$$

for all $x \geq 0$. This, by taking derivative with respect to $t$ and on using (1.2.23), further reduces to, for all $x,t \geq 0$,

$$
\bar{T}_{s}(X,x+t) \bar{T}_{s-1}(X,t) \leq \bar{T}_{s}(X,t) \bar{T}_{s-1}(X,x+t),
$$
giving, for all $x,t \geq 0$,
This means
\[
\frac{T_s(X,x)}{T_{s-1}(X,x)} \text{ is nonincreasing in } x. \quad (1.2.30)
\]

The following well known definition of Totally Positive (TP) function is taken from Karlin (1968) (see also Barlow and Proschan, 1975).

**DEFINITION 1.2.15:** Let A and B be subsets of the real line. A function \( K(x,y) \) on \( A \times B \) is said to be totally positive of order \( n \) (TP\(_n\)) if, for all \( x_1 < x_2 < \ldots < x_r \) in \( A \) and \( y_1 < y_2 < \ldots < y_r \) in \( B \), \( r = 1, 2, \ldots, n, \)

\[
\begin{vmatrix}
K(x_1, y_1) & K(x_1, y_2) & \cdots & K(x_1, y_r) \\
K(x_2, y_1) & K(x_2, y_2) & \cdots & K(x_2, y_r) \\
\vdots & \vdots & \ddots & \vdots \\
K(x_r, y_1) & K(x_r, y_2) & \cdots & K(x_r, y_r)
\end{vmatrix} \geq 0.
\]

A function which is totally positive of all finite orders, is said to be totally positive (TP).
Following is a basic property of a totally positive function of finite or infinite order, known as its variation diminishing property (cf. Karlin (1968), Barlow and Proschan (1975)).

**THEOREM 1.2.3:** Let \( K(x,y) \) be TP on \( \mathbb{R}^2 \) and \( f \), defined on \( B \), have at most one change of sign (from - to +). Then

\[
g(x) = \int_B K(x,y)f(y)\,dy
\]

has at most one change of sign (from - to +).

Mixtures of distributions arise naturally in a number of reliability situations. For example, suppose a manufacturer produces \( a\% \) of a certain product in one factory and \( (100-a)\% \) in another factory. Due to differences in machines, personnel etc., suppose the life length of a unit produced in first factory has distribution \( F_1 \) and that of a unit produced in second factory has distribution \( F_2 \). After production, units from both the factories flow into a common shipping room, so that outgoing lots consist of a random mixture of the output of the two factories. Now a unit selected at random from the lot would have distribution

\[
F = \frac{a}{100}F_1 + \frac{1-a}{100}F_2,
\]

a mixture of the two underlying distributions (cf. Barlow and Proschan (1975)).
More generally, the distributions being mixed may be infinite in number. For example, suppose an important quality characteristic of the product being manufactured depends on the amount of impurity present in the raw material. Specifically, let the probability distribution of the quality characteristic be $F_\alpha$. Suppose $\alpha$ itself is random with distribution $G(\alpha)$. Then the resulting distribution $F$ of the quality characteristic is given by

$$F(x) = \int_{-\infty}^{\infty} F_\alpha(x) dG(\alpha).$$

Shaked and Shanthikumar (1994) have given the preservation results for LR, FR and ST orderings for mixtures of distributions. In chapter two we have presented the preservation results, under mixture, for s-FR, s-ST, s-CX, s-CV and s-SFR orderings. Thus, the results of FR, ST, MR, VR, HAMR, CX and CV orderings follow as particular cases.

Lynch et al. (1987) showed that for a coherent system of $n$ independent and identically distributed components if the failure rate of one type of component is higher than that of a second type, then certain systems have a higher failure rate if the components of the first type are used rather than components of the second type. Chapter three deals with the similar type of problems with respect to reversed hazard rate ordering. We have established sufficient conditions on structure function of a coherent system of independent and identically distributed (iid)
components to preserve reversed hazard rate ordering and LR ordering. As a particular case, it is shown that the result is true for any k-out-of-n system. A class of structures satisfying a sufficient condition for preservation of reversed hazard rate ordering is also constructed. In this chapter, we also consider the system constructed out of independent but not necessarily identical components. We prove that if each of a set of independent and not necessarily identical components has smaller (larger) reversed hazard rate than the components of second type (of iid components), then some coherent systems have less (more) reversed hazard rate if first set of components are used rather than components of the second type, and as a particular case, it is true for any k-out-of-n system. It has been shown that a coherent system of independent components is a series (parallel) system if and only if the system hazard (reversed hazard) rate is the sum of the individual component hazard (reversed hazard) rates. The preservation of reversed hazard rate ordering under Poisson shock model has also been discussed in this chapter.

Kebir (1994) has characterized, using Laplace transform, the property that two life distributions are ordered in the sense of LR, FR, ST and reversed hazard rate orderings. In chapter four, we give necessary and sufficient conditions under which two life distributions are ordered in the sense of s-FR and s-ST orderings. Hence the condition for LR, FR, ST and HAMR orderings follow as special cases.
Vinogradov (1973) characterized the IFR life distributions with the help of Laplace transform. The DMRL, IFRA, NBU and NBUE life distributions along with their duals have been characterized by Block and Savits (1980). In this chapter we also characterize ageing classes viz. s-IFR, s-IFRA, s-NBU and their duals with the help of Laplace transform. Thus the characterizations for IFR, DMRL, DVRL, IFRA, NBU and their dual classes follow as particular cases.

With the passage of time, a repairable system can be better or worse. A repairable system is said to improve (deteriorate) with time if the interarrival times of failures tend to get larger (smaller) in some sense. Ebrahimi (1989) defined system improvement (deterioration) by comparing the conditional interarrival times of failures. Deshpande and Singh (1995) show that the system improvement (deterioration) defined by Ebrahimi (1989) is actually the comparison of two suitably defined random variables with respect to stochastic ordering. System improvement (deterioration) with respect to LR, FR, ST, MR and VR orderings are defined, using the above concept, by Deshpande and Singh (1995) and Bagai and Jain (1994). In chapter five we define system improvement (deterioration) in s-FR and s-ST sense so that the results of Ebrahimi (1989), Deshpande and Singh (1995) and Bagai and Jain (1994) get extended. From this general system improvement (deterioration), improvement (deterioration) in WLR and HAMR sense follow as particular cases. The system improvement (deterioration)
in s-FR and s-ST sense have been characterized in terms of the ageing aspects of \( X(1) \), the time of the first failure.

Weighted distributions occur in a natural way in specifying probabilities of events as observed and recorded by making adjustments to probabilities of actual occurrence of events taking into account methods of ascertainment. Failure to make such adjustments can lead to wrong conclusions (cf. Rao (1985)). When an investigator collects a sample of observations produced by nature according to a certain model, then the original distribution may not be reproduced due to non-observability of events, partial destruction of observations and sampling with unequal chances to observations (cf. Rao, 1965).

In Reliability, the distribution of the total life time of a component which has survived for \( t_0 \ (> 0) \) units of time, denoted by \( (X|X>t_0) \), is a weighted distribution of the life time \( X \) of the new component. Gupta and Keating (1986) point out that in order to determine the mean life of helicopter the transmissions in operation in a particular aircraft are sometimes studied on a specific date and their total life are measured. Because the transmissions studied are in operation on a particular date and have been operating for some time prior to this date, they constitute a nonrandom sample of all transmissions having longer mean lives than the class from which they are originally drawn.
In Survival Analysis, Schotz and Zelen (1971) point out that the usual estimators of cell kinetic parameters resulting from labelled mitosis experiments are biased because cells with longer DNA synthesis periods have greater probability of being labelled.

So, in all these cases, the original distribution is altered or truncated. Examples of such situations can also be found in observational studies and surveys of research related to forestry, ecology, bio-medicine and reliability. For example, in the distribution of eggs laid by insects, the frequency of zero eggs can't be known. The information about the number of accidents may be partially destroyed or may be only partially ascertained. Another example is the frequency of families with both parents heterozygous for albinism but with no albino children. There is no evidence that the parents are heterozygous unless they have an albino child and the families with such parents and having no albino children get confounded with normal families. The actual frequency of the event 'zero albino children' is thus not ascertainable (refer to Rao (1965, 1985), Patil and Rao (1977, 1978), Mahfoud and Patil (1982) among others).

In chapter six, different partial orderings have been studied in univariate cases in order to see how the original random variable and its weighted version are related with respect to different univariate partial orderings. Since generalized equilibrium distributions can be considered as weighted distributions with some specific weight function, some moment
properties of the generalized equilibrium distributions studied in this chapter, may be of interest.

Multivariate weighted distributions arise in reliability theory in many situations such as missing data, damaged observations, sociological studies and economic data (cf. Patil et al. (1988)). Some results regarding the weighted version of bivariate distributions have been given in Mahfoud and Patil (1982), Patil, Rao and Ratnaparkhi (1986), Arnold and Nagaraja (1991) among others. But the idea of multivariate weighted distributions has not been dealt with in detail in the literature, although Ahmed (1995) has discussed some estimation problems regarding multivariate weighted distributions.

How different multivariate partial orderings of the random vectors are transmitted to their respective weighted versions has been worked out in chapter seven. Here some multivariate dependence properties of the original random vector and its weighted version have been addressed. We have defined the multivariate marginal weighted and conditional weighted densities. If $X = (X_1, Z')'$, where $Z$ has $(p-1)$ components, then it is shown how the multiple correlation coefficient and the regression of $X_1$ on $Z$ can be found out using the multivariate weighted distributions. An example illustrating the method of writing down a multivariate weighted distribution using a given weight function and a joint density of $X_1, X_2, ..., X_p$, is given. In the course of
derivation, multivariate Poisson negative hypergeometric distribution has been derived.

In the Annexure we have given a brief discussion on what we have planned to do further in this line and lastly, we present a bibliography containing relevant references.