Chapter 1

Introduction

Nonlinear phenomena cover all areas of science and engineering, viz. condensed matter physics, plasma physics, optics, biology, chemical science, hydrodynamics etc. Behavior of a nonlinear system is far more complex but interesting than its linear counterpart. For instance, in nonlinear optics, the field comprising the study of response of dielectric media to strong optical field, various astonishing phenomena arise. For sufficiently strong electric field the response of the medium becomes nonlinear and the polarization no longer remains a linear function of the applied electric field, but it also contains higher powers of the applied field. This effect results into generation of new frequencies, also known as the phenomenon of second harmonic generation. Apart from this, in a nonlinear optical medium refractive index becomes intensity dependent, instead of being constant as in the linear medium case, leading to self-focussing of incident light [1]. Similarly, it is now well established that biological processes at the molecular level are regulated by nonlinear waves. An interesting example of nonlinear biological phenomenon is the pattern formation on the bodies of various species. The origin of specific forms of patterns like strips or dots is the result of nonlinear reaction and diffusion process [2]. In the study of magneto hydrodynamics, when the intensity of electromagnetic wave becomes significantly large the nonlinear effects can’t be ignored, which results into formation of Alfven waves, a fundamental mode of magnetized plasma [3].

Depending upon various factors, a nonlinear system can exhibit regular or irregular behavior. The effect of nonlinearity on the physical systems is so prominent that these nonlinear systems need a separate study [4]. Theoretically, the dynamics of these systems are investigated by modelling them through nonlinear
evolution equations (NLEEs) and behavior of the system is studied by the solutions of corresponding NLEE, obtained either analytically or numerically. The presence of nonlinearities make the mathematical analysis of NLEEs very difficult and non-trivial, as Fourier and Laplace transform methods, which are boon to handle any linear system, can’t be applied here. In general, for the nonlinear systems linear superposition principle fails, hence the combination of two independent solutions is no longer a solution. Further, the dynamics of nonlinear systems depend heavily on the initial conditions, which is not the case for linear systems. Due to the above reasons, there is a very little provision of general formalism to solve NLEEs but each problem is new and needs to be studied from a different perspective. One of the formalisms to solve a NLEE is by considering the effect of nonlinearity as a perturbation to the linear system, which is not a correct strategy for the heavily nonlinear system. Despite the complexities arising in both theoretical as well as experimental analysis, nonlinear systems are getting attention because of richness and their applications to real life. The famous and well studied NLEEs are nonlinear Schrödinger equation (NLSE), Kortweg de-Vries (KdV) equation, sine Gordon equations etc. Each of these equations models various physical systems and hence poses a significant role in understanding the behavior of corresponding systems.

This thesis comprises of study of nonlinear wave propagation in three different fields namely nonlinear fiber optics, negative index materials (NIM), and Bose-Einstein condensates (BEC). The reason behind, why these different fields have been compiled in one thesis consistently, is the mathematical analogy between these phenomena. Irrespective of the origin of dispersion and nonlinearity, the basic structure of the equations governing the nonlinear wave propagation in each of these is very much similar to generalized nonlinear Schrödinger equation (GNLSE). The standard form of nonlinear Schrödinger equation (NLSE) is given as

$$i\psi_t + \frac{1}{2}\psi_{xx} + \sigma|\psi|^2\psi = 0, \quad (1.1)$$

where $\psi$ is the field envelope. First term on the L.H.S of Eq. (1.1) represents the evolution in time/space. Second and third terms correspond to diffraction/dispersion and nonlinearity, respectively, of the system. Here, $\sigma = \pm 1$ corresponds to focussing and defocusing nonlinearity, respectively.

Here, we give a brief introduction of all these mediums and discuss how the NLSE arises in each them.
1.1 Nonlinear fiber optics

In the fiber-optic communication system, information is transmitted over a fiber by using a coded sequence of optical pulses, whose width is set by the bit rate of the system. The availability of low-loss silica fibers revolutionized not only the field of optical fiber communications, but also resulted into the beginning of new field of nonlinear fiber optics. This field encompasses myriads of interesting effects and practical applications. In order to understand the nonlinear phenomena in optical fibers, it is necessary to consider the theory of electromagnetic wave propagation in dispersive nonlinear media. Like all electromagnetic phenomena, the propagation of optical fields in fibers is governed by Maxwell’s equations.

In dielectric media and in the absence of free charges or currents Maxwell’s equations are given as

\[
\nabla \times E = \frac{\partial B}{\partial t}, \tag{1.2}
\]

\[
\nabla \times H = \frac{\partial D}{\partial t}, \tag{1.3}
\]

\[
\nabla \cdot D = 0, \tag{1.4}
\]

\[
\nabla \cdot B = 0, \tag{1.5}
\]

where \(E\) and \(H\) are the electric and magnetic field vectors, respectively, and \(D\) and \(B\) are electric and magnetic flux densities which arise in response to \(E\) and \(H\) propagating inside the medium and are related to them through the constitutive relations given as

\[
D = \varepsilon_0 E + P, \tag{1.6}
\]

\[
B = \mu_0 H + M, \tag{1.7}
\]

where \(P\) and \(M\) are the induced electric and magnetic polarization. \(\mu_0, \varepsilon_0\) are permeability and permittivity of free space, respectively. For nonmagnetic material \(M = 0\). Eqs. (1.2-1.5) can be used to obtain the wave equation that describes light propagation in optical fibers. By taking curl of Eq. (1.2) and using Eq. (1.3), (1.6) and (1.7), one can eliminate \(B\) and \(D\) in favor of \(E\) and \(P\) and obtain

\[
\nabla \times \nabla \times E = -\frac{1}{\varepsilon^2} \frac{\partial^2 E}{\partial t^2} - \mu_0 \frac{\partial^2 P}{\partial t^2}, \tag{1.8}
\]
where \( c \) is the speed of light in vacuum and the relation \( \mu_0 \varepsilon_0 = 1/c^2 \) was used. For the wavelength in the range of 0.5 - 2 \( \mu \)m in the optical fibers, nonlinear effects become dominant. Polarization is therefore conveniently expressed as sum of linear and nonlinear terms

\[
P(r, t) = P_L(r, t) + P_{NL}(r, t).
\] (1.9)

In optical fibers all even orders of nonlinearity vanish due to the inversion symmetry in the amorphous silica and hence significant nonlinear contribution is from third order nonlinearity, which is known as Kerr nonlinearity.

Linear \( P_L \) and nonlinear polarization \( P_{NL} \) are related to electric field as [6, 7, 8]

\[
P_L(r, t) = \varepsilon_0 \int_{-\infty}^{t} \chi^{(1)}(r, t') E(r, t') dt'.
\] (1.10)

\[
P_{NL}(r, t) = \varepsilon_0 \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t} dt_2 \int_{-\infty}^{t} dt_3 \chi^{(3)}(t-t_1, t-t_2, t-t_3) E(r, t_1) E(r, t_2) E(r, t_3).
\] (1.11)

Following Eq. (1.4), we have \( \nabla \cdot E = 0 \), now using Eq. (1.9) in Eq. (1.8) later can be rewritten as

\[
\nabla^2 E(r, t) - \frac{1}{c^2} \frac{\partial^2 E(r, t)}{\partial t^2} = \mu_0 \frac{\partial^2 P_L(r, t)}{\partial t^2} + \mu_0 \frac{\partial^2 P_{NL}(r, t)}{\partial t^2}.
\] (1.12)

In slowly varying envelope approximation, electric field \( E(r, t) \) and polarization components \( P_L(r, t), P_{NL}(r, t) \) can be defined as

\[
E(r, t) = \frac{1}{2} \hat{x} (E(r, t) \exp(-i\omega t) + c.c.),
\] (1.13)

\[
P_L(r, t) = \frac{1}{2} \hat{x} (P_L(r, t) \exp(-i\omega t) + c.c.),
\] (1.14)

\[
P_{NL}(r, t) = \frac{1}{2} \hat{x} (P_{NL}(r, t) \exp(-i\omega t) + c.c.),
\] (1.15)

where \( \hat{x} \) is polarization unit vector, and \( E(r, t) \) is slowly varying function of time. \( P_L \) and \( P_{NL} \) can be obtained by substituting Eq. (1.14) in Eq. (1.10) and Eq. (1.15) in Eq. (1.11), respectively. For instantaneous nonlinear response, \( \chi^{(3)}(t_1, t_2, t_3) \) can be approximated as

\[
\chi^{(3)}(t, t_1, t_2, t_3) = \chi^{(3)} \delta(t - t_1) \delta(t - t_2) \delta(t - t_3).
\] (1.16)
Hence, \[ P_{NL}(r,t) = \epsilon_0 \chi^{(3)} E(\mathbf{r},t) E(\mathbf{r},t) E(\mathbf{r},t). \] (1.17)

Using Eqs. (1.13), (1.15) and (1.17), \( P_{NL}(r,t) \) comes out to be

\[ P_{NL}(r,t) = \epsilon_0 \epsilon_{NL} E(\mathbf{r},t). \] (1.18)

where \( \epsilon_{NL} \), the nonlinear dielectric constant, is given as

\[ \epsilon_{NL} = \frac{3}{4} \lambda^{(3)} |E(\mathbf{r},t)|^2. \] (1.19)

Substituting Eq. (1.13) in Eq. (1.12) and using relation (1.15), one can find that the Fourier component of \( E(\mathbf{r},t) \), \( E(\mathbf{r},\omega - \omega_0) \), satisfies Helmholtz equation

\[ \nabla^2 E + \epsilon(\omega) k_0^2 E = 0, \] (1.20)

where \( k_0 = \omega/c \) and

\[ \epsilon(\omega) = 1 + \chi^{(3)} + \epsilon_{NL} \] (1.21)

is the dielectric constant. Since refractive index (\( n \)) and dielectric constant (\( \epsilon \)) are related to each other as \( \epsilon = n^2 \), hence using relation (1.21), refractive index becomes intensity dependent and can be defined as

\[ n = n_0 + n_2 |E|^2, \] (1.22)

where \( n_2 \), the nonlinear-index coefficient is related to third-order susceptibility as

\[ n_2 = \frac{3}{8n} \text{Re}(\chi^{(3)}). \] (1.23)

Eq. (1.22) also signifies that \( \Delta n \), small change in refractive index, is given as

\[ \Delta n = n - n_0 = n_2 |E|^2. \] (1.24)

In order to solve Eq. (1.20) we use method of separation of variable and define

\[ E(\mathbf{r},\omega - \omega_0) = \mathcal{F}(x,y) A(z,\omega - \omega_0) e^{i(kz)}. \] (1.25)
where \( F(x, y) \) is the transverse field distribution. \( A \) is slowly varying envelope, \( \omega_0 \) is a fast carrier frequency and \( \beta_0 \) is the wave number corresponding to the central frequency. \( A \) is normalized such that \( |A|^2 \) represents the optical power. The product of the independent transverse and longitudinal parts leads to two conditional equations:

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) F(x, y) + \epsilon(\omega)k_0^2 F(x, y) = \beta^2 F(x, y) , \quad (1.26)
\]

\[
2i\omega_0 \frac{\partial A}{\partial z} + 2\beta_0(\beta - \beta_0) A(z, \omega) = 0 , \quad (1.27)
\]

where the second derivative of slowly varying envelope has been neglected and the approximation \( \beta^2 - \beta_0^2 = 2\beta_0(\beta - \beta_0) \) has been used. Both the conditions are justified as long as \( \Delta \omega \ll \omega \), where \( \Delta \omega = \omega - \omega_0 \) is the pulse width. Eq. (1.26) is an eigenvalue equation, known as scalar Helmholtz equation, and leads to condition of guided mode and their field distribution \( F(x, y) \) in the fibers. \( \beta \) is the eigenvalue of transverse field distribution. In the absence of nonlinear polarization, solution of Eq. (1.26) is superposition of Bessel and Neumann functions and it can be shown that these can have confined modes only for \( k n_1^2 > \beta^2 > k n_2^2 \), where \( n_1 \) and \( n_2 \) are the refractive indices of core and cladding of fiber, respectively. There may be several \( \beta \) fulfilling the conditions corresponding to multimode, which implies that more than one spatial distribution of field is possible in the fiber. For the small Kerr nonlinearity, the \( \Delta n \) is significantly small, which enables us to solve Eq. (1.26) by first order perturbation method. Replacing \( \epsilon \) by \( n^2 \) in Eq. (1.26), field distribution \( F(x, y) \) and propagation parameter \( \beta \) are found. Then eigen function \( F(x, y) \) are used to calculate the first order correction to the term \( \beta \) due to the term \( \Delta n \). Eigen function \( \tilde{\beta}(\omega) \) becomes

\[
\tilde{\beta}(\omega) = \beta(\omega) + \Delta\beta(\omega)
\]

where

\[
\Delta\beta = \frac{\omega_0^2 n(\omega) \int \int_{-\infty}^{\infty} \Delta n |F(x, y)|^2 dxdy}{c^2 \beta(\omega) \int \int_{-\infty}^{\infty} |F(x, y)|^2 dxdy}.
\]

Unperturbed linear propagation constant is approximated by Taylor expansion around central frequency \( \omega_0 \)

\[
\beta(\omega) = \beta_0 + \beta_1(\omega - \omega_0) + \frac{1}{2} \beta_2(\omega - \omega_0)^2 + \ldots , \quad (1.29)
\]

A similar expression is made for \( \Delta\beta(\omega) \). Cubic and higher terms in the expression
1.1 Nonlinear fiber optics

(1.29) are negligible as long as \( \Delta \omega \ll \omega \). Further, the same condition also implies \( \Delta \beta = \Delta \beta_0 \). This expression is inserted in Eq. (1.27) and a Fourier transformation back to time domain gives the following equation for time dependent slowly varying envelope.

\[
\frac{\partial A}{\partial z} + iA \frac{\partial^2 A}{\partial t^2} = i \Delta \beta_0 A.
\]  

(1.30)

\( \Delta \beta_0 \) term on the R.H.S. contains the effect of nonlinearity. Using \( \beta(\omega) = n(\omega)\omega/c \), and considering that \( F(x, y) \) in Eq. (1.26) does not vary much over the pulse bandwidth, Eq. (1.31) takes the form

\[
\frac{\partial A}{\partial z} + iA \frac{\partial^2 A}{\partial t^2} = i \gamma(\omega_0) |A|^2 A,
\]  

(1.31)

where \( \gamma(\omega) = n_2(\omega_0)\omega_0/c \lambda_{\text{eff}} \). The parameter \( \lambda_{\text{eff}} \) is effective mode area. Eq. (1.31) corresponds to standard NLSE.

Though a large number of nonlinear effects can be successfully explained by Eq. (1.31), but for certain experimental conditions drastic modifications are needed in this equation. For instance, the effect of stimulated inelastic scattering, namely stimulated Raman scattering (SRS) and stimulated Brillouin scattering (SBS), are not included here. If the peak power of the incident pulse crosses a certain threshold level then due to SRS and SBS effects the energy will be transferred to new pulse which is of different wavelength. Another assumption made, while the derivation of this equation is \( \Delta \omega \ll \omega \). For the ultrashort optical pulses, whose width is close to or < 1 ps, the spectral width \( \Delta \omega \) becomes large enough and hence the Raman term can no longer be neglected. Another major simplification while deriving Eq. (1.31) is made by assuming that the polarization of the incident beam is preserved during the propagation. If the system is relaxed from this condition and the coupling between the orthogonal polarization components is considered then the governing equation becomes the coupled NLSE (CNLSE). In the simplest form this is expressed as

\[
i \psi_{1z} + \psi_{1u} + (r_{11}|\psi_1|^2 + r_{12}|\psi_2|^2)\psi_1 = 0,
\]

\[
i \psi_{2z} + \psi_{2u} + (r_{21}|\psi_1|^2 + r_{22}|\psi_2|^2)\psi_2 = 0,
\]  

(1.32)

where \( \psi_1 \) and \( \psi_2 \) are two components of polarization, \( r_{11} \) and \( r_{22} \) are coefficients of self phase modulation (SPM) and \( r_{12} \) and \( r_{21} \) are coefficients of cross phase modulation (XPM). The XPM coefficients, which take care of the coupling between the polarization modes, can also arise between two optical fields of different wavelengths.
but same polarization.

Eq. (1.31) appears in the nonlinear optics in several different contexts. For instance, this equation also describes the continuous wave propagation in a planar waveguide. The transition to the problem of waveguides is accomplished by changing $z \to Z$ and $t \to X$ in (Eq. (1.31)), where $Z$ is the longitudinal and $X$ is the transverse direction of the waveguide. $\lambda_2$ corresponds to the diffraction in the plane of the waveguide. The dimensionality of the underlying equation depends upon the nature of the material under consideration.

### 1.2 Negative index materials

The materials with simultaneous negative values of electric permittivity and magnetic permeability show negative refraction. Such materials do not exist naturally but needed to be constructed artificially. The theoretical idea of negative refraction was given by Veselago in 1967 [9]. In his seminal paper he demonstrated that if a material has negative permeability and permittivity simultaneously, it can still allow the propagation of EM wave through it, but its refractive index is negative. This concept of negative refraction remained an academic interest due to unavailability of naturally occurring negative permeable medium and hence the only possibility was to construct them artificially. Smith et al. [10] were the first one, who showed that composite material constructed in a specific way can have negative effective permeability. It was found that by combining this with the material which has negative electric permittivity, the resulting composite material could still allow the transmission of EM waves in microwave region. After this experimental verification, these metamaterials have been well explored in the linear wave propagation regime. It is found that these materials show interesting properties which challenge the well established concepts in electromagnetism and optics, such as reversal of Snell’s law, modified Doppler effect and obtuse angle for Cherenkov radiations. Pendry has proposed that by exploiting the properties of the negative refraction a perfect lens can be achieved that could focus both propagating and evanescent waves, resulting into overcoming problems with the common lens [11]. Thus far all the properties of NIMs discussed above are for the linear regime of wave propagation, when both effective permittivity and permeability are considered to be independent of the field intensity. In order to create tunable NIMs, whose properties depend on the field intensity, nonlinear features of such NIMs are required.
1.2 Negative index materials

Many theoretical models have been proposed to investigate the nonlinear wave propagation through NIM. First ever attempt in this regard was done by Scalora et al. [12]. They derived a GNLSE, describing the ultrashort pulse propagation through bulk media having magnetic permeability and dielectric susceptibility to be frequency dependent. A (1 + 1) dimensional NLSE with self-steepening term was derived by Wen et al. to study ultrashort pulse propagation with few optical cycles in NIMs [13]. A (3 + 1) dimensional envelope equation was derived for describing pulse propagation in NIM with nonlinear polarization [14]. Many research group have worked in the direction of obtaining CNLSE, in order to take into account both nonlinear polarization and magnetization. Tsironis was the first of them to give a system of CNLSE for the envelopes of electric and magnetic fields in an isotropic and homogeneous NIMs [15]. Next attempt was made by Wen et al. to derive a system of CNLSE modelling few cycle pulse propagation in NIM with nonlinear polarization and magnetization [16]. Besides Kerr nonlinearity, work has been done for the saturable nonlinearity [17] and cubic-quintic nonlinearity in NIM in (1+1) dimension [18].

Electric permittivity $\varepsilon$ and magnetic permeability $\mu$ have dispersive behavior in NIMs. Their frequency dispersion is given by lossy Drude model

$$\varepsilon(\tilde{\omega}) = \varepsilon \left(1 - \frac{1}{\tilde{\omega}(\tilde{\omega} + \gamma_e)}\right),$$

$$\mu(\tilde{\omega}) = \mu \left(1 - \frac{(\omega_{pm}/\omega_{pe})^2}{\tilde{\omega}(\tilde{\omega} + \gamma_m)}\right),$$

where $\tilde{\omega} = \omega/\omega_{pe}$ is the normalized frequency, $\omega_{pe}$ and $\omega_{pm}$ are respective electric and magnetic plasma frequency. For a nonlinear medium embedded in Kerr medium, both the nonlinear electric and magnetic polarizations are given as

$$P_{NL} = \varepsilon_{NL}E = \varepsilon_0\chi^{(3)}_P|E|^2E,$$

$$M_{NL} = \mu_{NL}H = \mu_0\chi^{(3)}_M|H|^2H,$$

where $\chi^{(3)}_P$ and $\chi^{(3)}_M$ are the third order electric and magnetic susceptibility. E and H are the electric and magnetic field. The nonlinear pulse propagation through NIM is symbolized by electric flux density D and magnetic induction B related to electric and magnetic field intensity E and H as $D = \varepsilon \ast E + P_{NL}$, $B = \mu \ast H + M_{NL}$, where $\ast$ is convolution. Using the above expression for D and B and following the
similar procedure as given in Section (1.1), NLSE can be derived governing the wave propagation in NIM [19].

1.3 Bose-Einstein condensates

Bose-Einstein condensates (BEC) is a phenomenon of quantum phase transition, in which a large fraction of Bose gas particles condense into the same quantum state below a critical temperature. This was predicted by Bose and Einstein in 1925, but first experimental realization took place 70 years after the theoretical prediction. It was first visualized in 1995, when different species of alkali metal atoms confined in a magnetic trap condensed to the ground state [20]. The static and dynamical properties of BEC can be described by effective mean field equation, also known as Gross-Pitaevskii (GP) equation, which obliges both the theoretical standpoint as well as experimental relevant conditions [21].

1.3.1 GP mean field model

For sufficiently dilute ultracold atomic gas, consisting of $N$ bosons of mass $m$ confined by external potential $V_{\text{ext}}$, many body Hamiltonian in second quantization can be expressed as [22]

$$
\hat{H} = \int dr \hat{\Psi}^\dagger \hat{H}_0 \hat{\Psi}(r, t) + \frac{1}{2} \int drdr' \hat{\Psi}^\dagger(r, t) \hat{\Psi}^\dagger(r', t) V(r - r') \hat{\Psi}(r', t) \hat{\Psi}(r, t), \quad (1.37)
$$

with $\hat{\Psi}(r, t)$ and $\hat{\Psi}^\dagger(r, t)$ as the boson annihilation and creation field operator, and $V(r - r')$ as two body interaction potential. Single particle operator $\hat{H}_0$ is defined as

$$
\hat{H}_0 = -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}}(r). \quad (1.38)
$$

The basis of mean field approach is Bogoliubov approximation, according to which boson field operator is given as

$$
\hat{\Psi}(r, t) = \Psi(r, t) + \hat{\Psi}'(r, t). \quad (1.39)
$$
Ψ(\(r, t\)) = \(\langle \Psi(r, t) \rangle\) is macroscopic wave function of condensate and \(\Psi'(r, t)\) gives the non-condensate part. Well below the critical temperature, \(\Psi(r, t)\) approaches to zero, which leads to zeroth-order theory of BEC wave function. In the Heisenberg evolution equation
\[
\frac{i\hbar}{\partial t} \frac{\partial}{\partial t} \Psi = [\hat{\Psi}, \hat{H}],
\]  
substituting the Value of \(\hat{H}\) and simplifying, we find that field operator \(\hat{\Psi}(r, t)\) satisfies the following equation
\[
\frac{i\hbar}{\partial t} \frac{\partial}{\partial t} \Psi(r, t) = \left[ \hat{H}_0 + \int d^3r' \hat{V}(r', t) \hat{\Psi}(r', t) \hat{\Psi}(r, t) \right] \Psi(r, t).
\]  
(1.41)

Considering the case of only binary collisions, described by \(s\)-wave scattering length, at low energy among the dilute ultracold gas, the interatomic potential can be replaced with an effective delta-function interaction potential. \(V(r - r') = g\delta(r - r')\), where \(g = 4\pi\hbar^2/m\) is the coupling coefficient. Using this effective interaction potential form and replacing field operator \(\hat{\Psi}\) with the classical field \(\Psi\), Eq. (1.41) yields the GP equation
\[
\frac{i\hbar}{\partial t} \frac{\partial}{\partial t} \Psi(r, t) = \left[ \frac{-\hbar^2}{2m} \nabla^2 + V_{ext}(r) + g|\Psi(r, t)|^2 \right] \Psi(r, t).
\]  
(1.42)

The external potential \(V_{ext}\), in GP model, trap and manipulate the condensate. In the experiments BEC are mainly confined in magnetic trap \(V_{mag}\) or the optical dipole trap \(V_{opt}\). Magnetic traps are harmonic with the form given as
\[
V_{mag} = \frac{1}{2}m(\omega_x^2x^2 + \omega_y^2y^2 + \omega_z^2z^2),
\]  
(1.43)

where the trap frequencies \(\omega_x, \omega_y\) and \(\omega_z\) along the three directions can be different. A famous example of the periodic optical dipole trap is optical lattice [23, 24, 25]. In three-dimensional setting, optical lattice takes the following form
\[
V_{opt}(r) = V_0[\cos^2(k_x x + \phi_x) + \cos^2(k_y y + \phi_y) + \cos^2(k_z z + \phi_z)].
\]  
(1.44)

Apart from these potentials, BEC has also been realized in double well potential (formed by combination of magnetic and optical dipole trap) and linear ramp (gravitation potential) with \(V_{ext} = mgz\).
Lower dimensional GP equations

For the BEC trapped in harmonic potential given by Eq. (1.43), with the assumption $\omega_z << \omega_x = \omega_y = \omega_r$, transverse confinement of condensate becomes significantly tight and condensate becomes effective one-dimensional which is also known as cigar shaped BEC. So the three-dimensional equation (1.42) can be reduced to effective or quasi one-dimensional equation [21]. In order to obtain the equation governing the dynamics of cigar shaped BEC, we consider quasi one-dimensional setting, i.e. $\omega_z << \omega_r$, decomposing the wave function $\Psi$ in a longitudinal and transverse component

$$\Psi(r, t) = \psi(z, t) \Phi(r, t). \quad (1.45)$$

Substituting it into Eq. (1.42) and averaging out the resulting equation in $r$ direction (i.e. multiply by $\Phi^*$ and integrate w.r.t. $r$), we obtain the resulting equation

$$i\hbar \frac{\partial}{\partial t} \psi(z, t) = \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + V(z) + g_{1D} |\psi|^2 \right] \psi(z, t), \quad (1.46)$$

where the effective one-dimensional coupling coefficient is given by

$$g_{1D} = g/2\pi a_r^2 = 2a\hbar \omega_r \quad (1.47)$$

and

$$V(z) = (1/2)m\omega_z^2 z^2. \quad (1.48)$$

The equation structure of (1.46) is similar to that of NLSE with an additional trap.

1.4 Autonomous nonlinear systems and possible excitations

Dynamical systems whose parameters have no explicit dependence on space or time are called autonomous systems. Such systems are modelled by constant coefficient differential equations. In the nonlinear autonomous systems, interplay of nonlinearity and dispersion results into various localized excitations, such as solitons, breathers and rogue waves. Modulation instability (MI), breaking up of a plane wave into pulses, is considered to be one of the reason of these localized excitations.
1.4 Autonomous nonlinear systems and possible excitations

A brief description of MI and the nonlinear excitations are given below:

1.4.1 Modulation instability

Modulation instability (MI) represents a fundamental nonlinear phenomenon. This was first predicted by Benjamin and Feir for deep water waves [26], further it was investigated for electromagnetic waves in cubic nonlinear medium [27]. The MI phenomenon consists of instability of nonlinear plane waves against weak long scale modulation with wave number lower than critical value. Long time evolution results into growth of sidebands and periodic exchange of energy between pump and sidebands during wave propagation. It has been observed in nonlinear optics, plasma physics, condensed matter physics. It results due to the interplay of nonlinearity and dispersion/diffraction. In nonlinear optics, MI is of fundamental importance for formation of both temporal and spatial solitons. Important models for investigating MI of EM wave in nonlinear media are the scalar and vectorial NLSE [28]. Initial stage of instability can be explored with linear stability analysis, which shows exponential increase of side band, long time behavior can be investigated with numerical simulation. This involves truncation of finite number of modes and finding exact periodic solution of NLSE. Various higher order effects such as higher order dispersion, self-steepening (SS) and time delayed Raman effect, which arise in the governing equation during the ultrashort pulse propagation, have found to strongly influence the MI. The third order dispersion contributes none to the MI, whereas forth order dispersion plays a major role in the critical MI frequency and may lead to additional instability regions. SS narrows the MI frequency regions and time delayed Raman nonlinearity alters MI fundamentally. MI is among the major factors limiting the transmission capacity of long haul optical communication system. On the other hand it can also used to generate chains of short optical pulses for high bit rate data transmission [29].

1.4.2 Solitary waves and solitons

Solitary wave

The waves which are of localized shape and continue to travel with constant velocity for a long time without dissipating their energy are referred to as solitary waves.
First description of solitary wave was made by John Scott Russell, who observed it in the Union Canal in Scotland [30].

**Solitons**

Solitons are those solitary waves that retain their individuality under collisions and eventually travel with their original shapes and speeds. This property is interesting in itself, as the interaction though looks similar to the scattering between free particles, but it differs because there is a phase shift between two waves after the collision which signifies the essential role played by the nonlinearity in the system, pictorially depicted in Fig. 1.1. The name soliton was first coined by Zabusky and Kruskal in 1965, to characterize nonlinear solitary waves that do not disperse and preserve their identity during propagation and after a collision. The Greek ending ‘on’ is generally used to describe elementary particles and this word emphasizes the most remarkable feature of these solitary waves. This means that the energy can propagate in the localized form and that the solitary waves emerge from the interaction completely preserved in form and speed with only a phase shift. Now the conditions on the definition of solitons that they emerge without changing their shape and speed have been dropped and less restrictive working definition is used. A classical soliton solution to nonlinear field equation (i) has finite nonzero energy and (ii) is confined to a finite region of space for all time. Under this definition the number of classical field theories which admit soliton solutions has greatly enlarged.

Solitons are mainly categorized in two types

![Figure 1.1: Interaction of solitons.](image-url)
1. **Bright solitons**: These are characterized by localized intensity peak above a otherwise continuous background.

For \( \sigma = +1 \), Eq. (1.1) possesses bright soliton solution which is given as

\[
\psi(\chi, \zeta) = \text{sech}\alpha(\sqrt{2}\chi - \nu \zeta)e^{i(b(\sqrt{2}\chi - \nu \zeta))}, \tag{1.49}
\]

where \( \omega = (b^2 - a^2)/b, a = 1/\sqrt{2} \) and \( \nu = 2b \). A typical intensity profile of bright soliton \( (I_B) \) is shown in Fig. 1.2(a).

2. **Dark solitons**: Dark solitons represent self-trapped beams involving a dark notch in an otherwise bright background. For \( \sigma = -1 \), Eq. (1.1) possesses dark soliton solution which is given as

\[
\psi(\chi, \zeta) = u_0(i \sin \phi + \cos \phi \tanh \Theta)e^{i\Theta}, \tag{1.50}
\]

with \( \Theta = u_0 \cos \phi(\chi - u_0 \zeta \sin \phi) \). Here, \( u_0 \) is the background amplitude and \( \phi \) governs the grayness and speed of the dark soliton. Intensity profile of dark soliton \( (I_D) \) is shown in Fig. 1.2(b).

![Figure 1.2: Intensity profile of (a) bright soliton solution of NLSE taking \( v = 0.3 \) (b) dark soliton solution of NLSE taking \( u_0 = 1 \) and \( \phi = \pi/6 \).](attachment:image.png)

In nonlinear optical fibers such localized pulses, formed due to balance of dispersion and self-phase modulation, are called optical solitons. These represent the examples in which an abstract mathematical concept has produced a large impact on the real world of high technologies. The classical soliton is now being considered as the ideal natural data bit. In BEC, localized waves formed due to balance of two body nonlinear interaction and dispersion are called matter wave solitons. These are considered as concrete application of BEC in future.
1.4.3 Breathers

In contrast to solitons, breathers are internal oscillations and are defined as bound state of nonlinear wave packets. They are considered to be superposition of \( N \)-solitons with a coinciding center. Breathers are spatially localized and temporally oscillating nonlinear waves [31, 32]. The mathematical proof of existence of breathers is given by MacKay and Aubay [33]. Experimental observation of breathers have been made in BEC [34], dispersion managed optical waveguides and fibers [35], Josephson arrays [36]. These kind of solutions have been derived for KdV [37], Gardner equation [38], modified KdV [39] and NLSE. Breather solution of NLSE is known as Akhmediev Breather (AB) [40]. The propagating wave is with an initial constant amplitude and a small periodic perturbation superimposed over it, resulting into time dependent modulation of amplitude. Thus, AB consists of ultrashort pulses, periodic in time and growth-return cycles in space. Mathematical form of AB solution of Eq. (1.1) is given as

\[
\psi(\chi, \zeta) = \left( \frac{\cos \sqrt{2} \chi + i \sqrt{2} \sinh \zeta}{\cos \sqrt{2} \chi - \sqrt{2} \cosh \zeta} \right) e^{i \zeta}. \tag{1.51}
\]

Intensity profile of AB \((I_{AB})\) is plotted in Fig. 1.3.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{IntensityProfile.png}
\caption{Intensity profile of breather solution of NLSE.}
\end{figure}
1.4.4 Rogue waves

These are giant mountainous waves with the amplitude approximately three times than the average wave crest [40]. The first scientific observation of these waves was done in 1994, at the Draupner oil platform in the north sea [41]. Initially the focus of their study was hydrodynamics [42, 43, 44] but now it has been extended to areas of optics [45], capillary waves [46], finance [47] etc.

There are various mechanisms that can generate the extreme waves, but nonlinear mechanisms draw major attention as these waves are very sensitive to initial conditions. Possible nonlinear mechanisms are nonlinear focusing via modulation instability in one-dimensional and two-dimensional crossing, nonlinear spectral instability, focusing with caustic currents and anomalous wind excitation [48, 49].

The defining characteristic of these waves is that they appear from nowhere and disappear without trace [40]. Another feature of these waves is their unusual statistics. They obey L shaped statistics, according to which waves with lower amplitude have more probability of occurrence, but the outliers also occur more frequently in comparison to that allowed by other statistics (Gaussian or Rayleigh statistics) [45].

The most popular model for the study of these waves is NLSE [40]. As it is well confirmed that these waves can not form without continuous supply of energy and hence exact solutions to model rogue waves should contain a finite background. Since the rational solution of NLSE, the Peregrine soliton, lies on a finite background, it models the fundamental rogue wave [50]. A hierarchy of rational solutions and hence of rogue wave can be obtained by the Darboux transformation [51]. Mathematical form of first-order rogue wave solution is given as

\[ \psi(x, \zeta) = \left( 1 - 4 \frac{1 + 2i\chi}{1 + 4\chi^2 + \zeta^2} \right) e^{i\chi}. \]  \hspace{1cm} (1.52)

Second-order rogue wave solutions can be obtained from the first order by the Darboux transformation. The mathematical expression of second-order rogue wave solution is given as

\[ \psi(x, \zeta) = \left( 1 - \frac{K + iH}{D} \right) e^{i\chi}. \]  \hspace{1cm} (1.53)
where

\[ K = \left( \chi^2 + \zeta^2 + \frac{3}{4} \right) \left( \chi^2 + 5\zeta^2 + \frac{3}{4} \right) - \frac{3}{4}, \]
\[ H = \zeta \left( \zeta^2 - 3\chi^2 + 2(\chi^2 + \zeta^2)^2 - \frac{15}{8} \right), \]
\[ D = \frac{1}{3} \left( \chi^2 + \zeta^2 \right)^3 + \frac{1}{4} \left( \chi^2 - 3\zeta^2 \right)^3 + \frac{3}{64} \left( 12\chi^2 + 44\zeta^2 + 1 \right). \]

(1.54)

Intensity profiles of first-order rogue wave \((I_{R1})\) and second-order rogue wave \((I_{R2})\) are plotted in Figs. 1.4(a) and 1.4(b), respectively.

Figure 1.4: Profile of (a) first-order rogue wave solution (b) second-order rogue wave solution of NLSE.

1.5 Non-autonomous nonlinear systems and Self-similar nonlinear waves

The systems we have discussed so far are the homogeneous or autonomous systems. These are the systems in which equation parameters are independent of space or time and hence these variables do not appear explicitly in the equation. But nowadays, due to various inhomogeneities present in the systems and their varying boundary conditions, variable coefficient equations are being studied comparatively more often than their constant coefficient counterparts. These equations with dispersive coefficients are generally not integrable. However, they do possess solitary wave solutions. The study of soliton propagation through density gradient was pioneered by Tappert and Zabusky [52]. This concept was further extended by Chen and Liu [53] to the
solitons in linear inhomogeneous plasma. They found that for NLSE with the linear external potential inverse scattering method can be generalized to have time varying eigenvalues resulting into accelerating solitons. Calegaro and Degaprise [54] introduced the general class of soliton solutions for nonautonomous KdV with varying nonlinearity and dispersion. They showed that the basic properties of solitons, elastic interaction remains preserved, but these solitons move with the variable speed in analogy with particles in external force and names like boomeron and trappon were introduced instead of the usual classical soliton.

In the framework of integrability in GNLSE with varying dispersion, nonlinearity, dissipation or gain, soliton like interactions have been studied [55, 56, 55, 57]. Hasegawa and Kodama studied the behavior of solitons described by NLSE with perturbation and proved the robust nature of soliton [58]. Corney and Bang studied one-dimensional quadratic nonlinear photonic crystal with modulating linear and nonlinear susceptibility [59]. These inhomogeneous nonlinear equations are mathematically handled by invoking self-similar transformation and hence the resulting solutions are known as the self-similar solutions [60, 61, 62]. These have the characteristic property of retaining their shape, but width and amplitude adjust as these waves propagate in the inhomogeneous system. For the homogeneous systems, this type of feature is the result of the presence of internal order or hidden symmetries in the underlying system. These symmetries are exploited in mathematical treatment of the governing equations, resulting into reduction in the dimensionality of the system and hence facilitating the analysis. The self-similar properties have been well studied in many fields of physics such as plasma physics, nuclear physics, hydrodynamics and turbulence. Many natural phenomena showing self-similarity reproduce themselves on different time and space scale. Different forms of self-similar waves have been studied such as parabolic pulses, Hermite Gaussian and hybrid pulse solutions. After the generalization of the inverse scattering technique various integrable inhomogeneous system were discovered. These systems possess sech and tanh shaped self-similar pulses, which are very much consistent with the solitary or soliton solutions given by Hasegawa and Serkin. In these inhomogeneous integrable systems, self-similar waves maintain their identity after interaction resulting into coining of term similaritons. Ponomarenko and Agrawal have studied the behavior of self-similar waves in nonlinear systems having spatial inhomogeneity and gain or loss [60].

Special significance of these results lies in their potential application to the various physical and engineering sciences. For instance, in a real optical-fiber trans-
mission system, the varying dispersion and Kerr nonlinearity are of practical im-
portance, with the consideration of the inhomogeneities resulting from such factors as
the variation in the lattice parameters of the fiber media and fluctuation in the
fiber’s diameter.

1.6 Outline of thesis

Thesis proceeds as follows:

In Chapter 2 we study MI of the continuous wave propagating through NIM,
modelled by coupled cubic-quintic NLSE. We investigate the effects of quintic non-
linearity on the amplitude of gain of MI. The study is done for all the three cases of
MI- temporal MI, spatial MI and spatio-temporal MI. A comparison is made with
the already available results on MI in the absence of quintic term [14]. We find
that quintic nonlinearity helps in achieving MI, in otherwise not possible parameter
regimes. For instance, due to presence of quintic nonlinearity spatial MI can also
occur in the focusing regime, temporal MI becomes possible in the anomalous dis-
ersion for defocusing nonlinearity, while in the case of normal dispersion regime, it
can occur for focusing nonlinearity, and spatiotemporal MI can occur in the focusing
nonlinearity and normal dispersion. Moreover its presence provides extra freedom
to control the amplitude of the MI gain profile.

In Chapter 3, we study the solitary wave solutions for non-polynomial NLSE.
The study is done for two cases: in the first case the nonlinear interaction is of
only non-polynomial form and in the second case cubic nonlinearity is also included
along with the non-polynomial nonlinearity. Chirped dark and bright solitary waves
solutions are obtained in the respective cases. Further, we find that the later case
can be reduced to driven quadratic-cubic NLSE, possessing cnoidal solutions with
plane wave phase, which reduce to bright soliton for a certain parameter.

In the Chapter 4, we present a systematic analytical approach to construct a
family of self-similar waves in inhomogeneous NLSE. We study the effect of tapering
of waveguide or the optical rogue waves propagating through it. This is done by
generalizing the functional form of available tapering profiles, by incorporating Ric-
cati generalization scheme. We found that the intensity of first- and second-order
rogue waves can be increased to a significantly large extent and for the rogue wave
triplets, the distance between the peaks can also be controlled as well. Further in
the same chapter, we study the self-similar wave solutions for variable coefficient
coupled NLSE (ve-CNLSE). Under specific conditions on the self- and cross-phase coupling terms, we find that the equation supports bright-dark, bright, dark similaritons, self-similar breathers and rogue waves as possible localized configurations. For each of these solutions, one of the propagating mode can be made prominent by lowering the strength of its self phase modulation in comparison to the other. We further show that this equation also supports semirational self-similar rogue waves solutions for the equal contribution of self and cross phase modulation.

In Chapter 5 we carry forward the idea of Riccati generalization, discussed in Chapter 4, to control the dynamics of dark and bright similaritons, and first-and second-order self-similar rogue waves in BEC through the modulation of time dependent trapping potential. The analysis is done for the sech² type time-varying quadratic trapping potential for two different choices of linear potential. Further, we show that there exists a class of nonlinearity control function, within the integrability framework of the nonautonomous Gross Pitaevskii equation, for same trapping potential. This variation of the nonlinearity function helps to smoothen out the intensity profile of the solitons trapped in same potential.

Finally in Chapter 6, we summarize the present thesis work, focusing on the main conclusion that we draw from this analysis.
Bibliography


