Chapter 5

On limits of sequences of elements algebraic over a complete field

5.1 Origin of the problem

Let \( v \) be a non-trivial real (rank one) valuation of a field \( K \) with respect to which it is complete and \( \bar{v} \) be the unique extension of \( v \) to a fixed algebraic closure \( \bar{K} \) of \( K \). In 1917, Ostrowski proved that the valued field \((\bar{K}, \bar{v})\) is complete if and only if \( \bar{K} \) is a finite extension of \( K \) (see [Os1] or [Bou2, p.467-468]). In 1927, Artin and Schreier gave a characterization of such fields by proving that the algebraic closure \( \bar{K} \) of a field \( K \) is a finite extension of \( K \) if and only if either \( K \) is algebraically closed or \( K \) is a real closed field and \( \bar{K} = K(\sqrt{-1}) \) (cf. [Ar-Sc], [Jac, §11.7, Theorem 11.14]). Hence for many interesting complete valued fields \((K, v)\) like the completions of algebraic number fields, the limit of a Cauchy sequence of elements of \((\bar{K}, \bar{v})\) does not always belong to \( \bar{K} \). This gives rise to the following natural question.

Is it possible to characterize those Cauchy sequences \( \{b_n\} \) of elements of \( \bar{K} \) whose limit is not in \( \bar{K} \)?

In this chapter, we give a characterization of such sequences (modulo addition by null sequences) which will be stated after introducing some notations in the next
section (see Theorem 5.2.2). It is also proved that when a Cauchy sequence \( \{b_n\} \) of elements of \( K \) is such that the sequence \( \{[K(b_n) : K]\} \) of degrees of the extensions \( K(b_n)/K \) does not tend to infinity as \( n \) approaches infinity, then \( \{b_n\} \) has a limit in \( K \).

5.2 Definitions, notations and statements of the main results

Let \( K \) be a complete valued field with respect to a real valuation \( v \) and \((K, \bar{v})\) be as above. For any \( a \) belonging to \( K \), \( \text{deg} a \) will denote the degree of the extension \( K(a)/K \). When \( a \) belongs to \( K \setminus K \), let \( M(a, K) \) denote the subset of real numbers given by

\[
M(a, K) = \{v(a - 0) \mid \beta \in \overline{K}, [K(\beta) : K] < [K(a) : K]\}
\]

and \( \delta_K(a) \) the supremum of \( M(a, K) \).

In 2001, Aghigh and Khanduja [Ag-Khl] proved that \( M(\alpha, K) \) has a maximum element for each \( a \) in \( K \setminus K \), if and only if every simple algebraic extension of \((K, v)\) is defectless\(^1\). When \((K, v)\) has the above property, then to each \( a \in K \setminus K \), one can associate an element \( a_1 \) (not necessarily unique) belonging to \( \overline{K} \) of smallest degree over \( K \) such that \( \bar{v}(a - a_1) = \delta_K(a) \); such a pair \((a, a_1)\) is called a distinguished pair. In other words, a pair \((a, a_1)\) of elements of \( \overline{K} \) is called a distinguished pair (more precisely a \((K, v)\)-distinguished pair) if the following three conditions are satisfied:

(i) \( \bar{v}(a - a_1) = \delta_K(a) \),

(ii) \( \text{deg} a > \text{deg} a_1 \),

(iii) if \( \beta \) belonging to \( \overline{K} \) has degree strictly less than that of \( a_1 \), then \( \bar{v}(a - \beta) < \bar{v}(a - a_1) \).

A sequence \( \{a_n\} \) of elements of \( \overline{K} \) will be called an inverted distinguished sequence (more precisely \((K, v)\)-inverted distinguished sequence) if \( a_0 \in K \), \((a_{i+1}, a_i)\)

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\(^1\)Indeed this result has been proved in [Ag-Khl] when \((K, v)\) is a henselian valued field of arbitrary rank.
is a distinguished pair for each \( i \geq 0 \) and \( v(\alpha_{n+1} - \alpha_n) = \delta_K(\alpha_{n+1}) \to \infty \) as \( n \to \infty \) (see §5.4 for examples).

Observe that every \((K, v)\)-inverted distinguished sequence is a Cauchy sequence of elements of \( \overline{K} \) with respect to \( \bar{v} \).

We shall denote the prolongation of \( \bar{v} \) to the completion \( \overline{K}^e \) of \( \overline{K} \) with respect to \( \bar{v} \) by \( \bar{v} \). With the above notation, we prove:

**THEOREM 5.2.1.** Let \((K, v)\) be a complete rank one valued field and \( \bar{v} \) be the unique prolongation of \( v \) to a fixed algebraic closure \( \overline{K} \) of \( K \). Let \( \{b_n\} \) be a Cauchy sequence of elements of \( \overline{K} \) with respect to \( \bar{v} \). Then \( \{b_n\} \) converges to an element of \( \overline{K} \) with respect to \( \bar{v} \) provided the sequence \( \{\deg b_n\} \) does not tend to infinity as \( n \to \infty \).

**THEOREM 5.2.2.** Let \((K, v)\) and \((\overline{K}, \bar{v})\) be as above and \( \overline{K}^e \) be the completion of \( \overline{K} \) with respect to \( \bar{v} \). Assume that \( K(\alpha) \) is a defectless extension of \((K, v)\) for every \( \alpha \) in \( \overline{K} \). Then an element \( t \) of \( \overline{K}^e \) is transcendental over \( K \) if and only if there exists an inverted distinguished sequence of elements of \( \overline{K} \) which converges to \( t \).

As an application of the above theorem, we shall prove:

**COROLLARY 5.2.3.** Let \((K, v)\) be a complete discrete valued field with value group \( \mathbb{Z} \). Let \( p \) be a prime number different from the characteristic of \( K \). Let \( s \) be a positive integer such that \( \frac{v(p)}{p - 1} \leq s \). If \( \pi \) belonging to \( K \) has \( v \)-valuation 1, then the series \( \sum_{n=1}^{\infty} (\pi^n \cdot \pi^{-1/p^n}) \) converges to an element which is transcendental over \( K \).

The following corollary is an immediate consequence of the above result.

**COROLLARY 5.2.4.** Let \( p \) be a prime number. The series \( \sum_{n=1}^{\infty} (p^n \cdot p^{-1/p^n}) \) converges to an element which is transcendental over the field of \( p \)-adic numbers.
The result stated below follows quickly from Theorem 5.2.2 and Example 5.4.2.

**COROLLARY 5.2.5.** Let $p_1, p_2, \ldots$ be a sequence of distinct prime numbers and $\mathbb{Q}$ be the field of rational numbers. Then $\sum_{i=1}^{\infty} \sqrt{p_i} t^i$ is transcendental over the field $\mathbb{Q}(t)$ of Laurent series in an indeterminate $t$.

It may be pointed out that Theorem 5.2.2 generalizes Propositions 2.1, 2.2 of [A-P-Z4] which are proved in the particular case when $K$ is the field of $p$-adic numbers. Our proof is more rigorous as it takes care of non-discrete valuations also.

### 5.3 Proof of Theorem 5.2.1

We prove a lemma which will be used in the proof of Theorem 5.2.1. It is also of independent interest.

**LEMMA 5.3.1.** Let $v$ be a henselian valuation of arbitrary rank of a field $K$ and $(\overline{K}, \overline{v})$ be as above. Let $f(x)$ and $g(x)$ belonging to $K[x]$ be two monic, irreducible, separable polynomials of same degree $m$. Let $\alpha, \beta$ be any roots of $f(x)$ and $g(x)$ respectively. Then the roots $\alpha_1, \ldots, \alpha_m$ of $f(x)$ and $\beta_1, \ldots, \beta_m$ of $g(x)$ can be arranged such that $v(\alpha_i - \beta_i) \geq v(\alpha - \beta)$ for $1 \leq i \leq m$.

**Proof.** Let $N$ denote the smallest normal extension of $K$ containing $\alpha_1, \ldots, \alpha_m$ and $\beta_1, \ldots, \beta_m$. Set $\delta = v(\alpha - \beta)$ and $H = \{ \sigma \in \text{Gal}(N/K) \mid v(\sigma(\alpha) - \alpha) \geq \delta \}$. Keeping in mind that $v(\alpha - \beta) = \delta$, we see that

$$H = \{ \sigma \in \text{Gal}(N/K) \mid v(\alpha - \sigma(\alpha)) \geq \delta \} = \{ \sigma \in \text{Gal}(N/K) \mid v(\beta - \sigma(\beta)) \geq \delta \}. \quad (5.1)$$

Clearly $H$ is a subgroup of $\text{Gal}(N/K)$ and $H \supseteq \text{Gal}(N/K(\alpha)) \cup \text{Gal}(N/K(\beta))$. Thus if $M$ denotes the fixed field of $H$, then $M \subseteq K(\alpha) \cap K(\beta)$. Let $r, s$ denote respectively the degrees of $M/K$ and $K(\alpha)/M$. Let $\{\sigma_1, \ldots, \sigma_r\}$ be the set of $K$-automorphisms of $N$ which are distinct on $M$. Let $\{\tau_1, \ldots, \tau_s\}$ be a system
of coset representatives of $\text{Gal}(N/M)/\text{Gal}(N/K(a))$, i.e., $\tau_1, \ldots, \tau_s$ are distinct on $K(\alpha)$ but identity on $M$. Let $\{\tau'_1, \ldots, \tau'_r\}$ be a system of coset representatives of $\text{Gal}(N/M)/\text{Gal}(N/K(\beta))$. We claim that the sets $\{\sigma_i \circ \tau_j(\alpha) \mid 1 \leq i \leq r, 1 \leq j \leq s\}$ and $\{\sigma_i \circ \tau'_j(\beta) \mid 1 \leq i \leq r, 1 \leq j \leq s\}$ constitute respectively all the $K$-conjugates of $\alpha$ and $\beta$. Keeping in mind that $[K(\alpha) : K] = [K(\beta) : K] = rs$, the claim is proved once we show that when $(i, j) \neq (k, l)$, then

$$\sigma_i \circ \tau_j(\alpha) \neq \sigma_k \circ \tau_l(\alpha) \quad (5.2)$$

and

$$\sigma_i \circ \tau'_j(\beta) \neq \sigma_k \circ \tau'_l(\beta). \quad (5.3)$$

To prove (5.2), note that if $i = k$ and $\sigma_i \circ \tau_j(\alpha) = \sigma_k \circ \tau_l(\alpha)$, then $\tau_j(\alpha) = \tau_l(\alpha)$ which is possible only if $j = l$. When $i \neq k$, then the equality $\sigma_i \circ \tau_j(\alpha) = \sigma_k \circ \tau_l(\alpha)$ implies $\sigma^{-1}_k \circ \sigma_i \circ \tau_j \circ \sigma^{-1}_l(\alpha) = \alpha$ which implies $\sigma^{-1}_k \circ \sigma_i \circ \tau_l \circ \sigma^{-1}_l$ is identity on $K(\alpha)$ and thus identity on $M$. As $\tau_j$, $\tau_l$ are identity on $M$, this leads to $\sigma^{-1}_k \circ \sigma_i$ being identity on $M$ which is possible only if $i = k$. This completes the verification of (5.2).

Similarly we can verify (5.3) and hence the claim. The lemma follows from the claim once we show that

$$v(\sigma_i \circ \tau_j(\alpha) - \sigma_i \circ \tau'_j(\beta)) \geq \delta, \quad 1 \leq i \leq r, \quad 1 \leq j \leq s.$$ 

Keeping in view that $(K, v)$ is henselian and that $\tau_j$, $\tau'_j$ belong to the group $H = \text{Gal}(N/M)$ given by (5.1), the above inequality follows because

$$v(\sigma_i \circ \tau_j(\alpha) - \sigma_i \circ \tau'_j(\beta)) = v(\tau_j(\alpha) - \tau'_j(\beta))$$

$$= v(\tau_j(\alpha) - \alpha + \alpha - \beta + \beta - \tau'_j(\beta))$$

$$\geq \min \{v(\tau_j(\alpha) - \alpha), v(\alpha - \beta), v(\beta - \tau'_j(\beta))\}$$

$$= \delta.$$
COROLLARY 5.3.2. Let \( v \) be a real, henselian valuation of a field \( K \). Let \( \{ \beta_j \} \) be a Cauchy sequence of elements of \( K \) with respect to \( \tilde{v} \) such that each \( \beta_j \) is separable of same degree over \( K \). Let \( f_j(x) = x^h + a_{ij}x^{h-1} + \ldots + a_{hj} \) be the minimal polynomial of \( \beta_j \) over \( K \). Then the sequences \( \{ a_{ij} \} \) are Cauchy with respect to \( v \) for \( 1 \leq i \leq h \).

Proof. If the sequence \( \{ \beta_j \} \) converges to 0, then \( \tilde{v}(\beta_j) \) tends to infinity as \( j \to \infty \) and hence \( v(a_{ij}) \to \infty \) as \( j \to \infty \) for \( 1 \leq i \leq h \) so that \( \{ a_{ij} \} \to 0 \).

Suppose that \( \{ \beta_j \} \) does not converge to 0. We first show that there exists \( j_0 \) such that for all \( j \geq j_0 \), \( \tilde{v}(\beta_j) = \tilde{v}(\beta_{j_0}) \). The assumption \( \{ \beta_j \} \) does not tend to 0 implies that there exists a real number \( I > 0 \) such that for any given \( j \), there exists \( k > j \) satisfying

\[
\tilde{v}(\beta_k) < I.
\]

Since \( \{ \beta_j \} \) is a Cauchy sequence, there exists \( j_0 \) such that

\[
\tilde{v}(\beta_k - \beta_j) \geq I \text{ for all } j \geq j_0, \ k \geq j_0.
\]  \hfill (5.4)

Given \( j_0 \), there exists \( k_0 > j_0 \) with

\[
\tilde{v}(\beta_{k_0}) < I.
\]

The above inequality together with (5.4) shows that for \( j \geq j_0 \), we have

\[
\tilde{v}(\beta_j) = \min\{\tilde{v}(\beta_j - \beta_{k_0}), \tilde{v}(\beta_{k_0})\} = \tilde{v}(\beta_{k_0}) = \mu \text{ (say).}
\]

Keeping in mind the relations between the roots and coefficients of a polynomial and applying Lemma 5.3.1 to the polynomials \( f_j(x) \), \( f_{j+1}(x) \) (with roots \( \beta_j \), \( \beta_{j+1} \) respectively), we quickly deduce that for \( j \geq j_0 \),

\[
\tilde{v}(a_{ij} - a_{ij+1}) \geq (i-1)\mu + \tilde{v}(\beta_j - \beta_{j+1})
\]

which shows that \( \{ a_{ij} \} \) is a Cauchy sequence as \( \{ \beta_j \} \) is so.
Proof of Theorem 5.2.1. Let $b$ denote the limit of $\{b_n\}$ in the completion of $(\overline{K}, \nu)$. Clearly we need to prove the theorem when $b \neq 0$. It is given that $\{\deg b_n\}$ does not tend to infinity as $n$ approaches infinity. If necessary on replacing $\{b_n\}$ by a subsequence, we may suppose that $b_n$ has same degree and same separable degree over $K$ for all $n \geq 1$. In case $\text{char } K = p > 0$, on replacing $\{b_n\}$ by $\{b_n^k\}$ for a suitable $k > 0$, we may assume that each $b_n$ is separable over $K$ and $[K(b_n) : K]$ is same for all $n$. Let $f_n(x) = x^h + a_{1n}x^{h-1} + \ldots + a_{hn}$ denote the minimal polynomial of $b_n$ over $K$. Since $(\overline{K}, \nu)$ is complete and $\{b_n\}$ is a Cauchy sequence, $\{a_{in}\}_n$ are Cauchy sequences for $1 \leq i \leq h$ by virtue of Corollary 5.3.2. Denote by $a_i$ the limit of the sequence $\{a_{in}\}_n$ and by $f(x)$ the polynomial $x^h + a_1x^{h-1} + \ldots + a_h$. We are going to prove that

\[ \lim_{n \to \infty} \nu(f_n(b)) = \infty. \tag{5.5} \]

As \( \lim_{n \to \infty} f_n(b) = f(b) \), the above equation (5.5) leads to $f(b) = 0$ which will prove that $b$ is algebraic over $K$ as desired.

To verify (5.5), write the Taylor series expansion of $f_n(b)$ as

\[ f_n(b) = (b - b_n) \frac{f_n'(b_n)}{1!} + (b - b_n)\frac{f_n''(b_n)}{2!} + \ldots + (b - b_n)^h \frac{f_n^{(h)}(b_n)}{h!}. \tag{5.6} \]

Keeping in view that $b \neq 0$ and arguing as in the proof of Corollary 5.3.2, we see that $\nu(b_n) = \nu(b)$ for sufficiently large values of $n$. As the coefficients of $f_n(x)$ converge to the respective coefficients of $f(x)$, so whenever $a_i \neq 0$, then $\nu(a_{in}) = \nu(a_i)$ for sufficiently large $n$. Thus we conclude that there exists a real number $C$ and an integer $n_0$ such that for all $n \geq n_0$ and for $1 \leq k \leq h$, we have

\[ \nu \left( \frac{f_n^{(k)}(b_n)}{k!} \right) \geq C, \quad \nu(b - b_n) > 0. \]

The expansion (5.6) of $f_n(b)$ together with the two inequalities mentioned above implies that $\nu(f_n(b)) \geq C + \nu(b - b_n)$ for all $n \geq n_0$. Since $\nu(b - b_n)$ tends to infinity as $n$ approaches infinity, it now follows that $\lim_{n \to \infty} \nu(f_n(b)) = \infty$. This proves (5.5)
and completes the proof of the theorem.

5.4 Some examples and preliminary results

We first give some examples of inverted distinguished sequences.

EXAMPLE 5.4.1. Let $K$ be the field of $p$-adic numbers with the valuation $v$ characterized by $v(p) = 1$ and $v$ be the prolongation of $v$ to the algebraic closure $\overline{K}$ of $K$. Let $\{\alpha_n\}$ be a sequence of elements of $\overline{K}$ defined by $\alpha_0 = p$,

$$
\alpha_n = p + p(p^{-1/p}) + \cdots + p^n(p^{-1/p^n}), \quad n \geq 1.
$$

Then $\{\alpha_n\}$ is a $(K, v)$-inverted distinguished sequence. For this it is enough to verify that $\deg \alpha_{n-1} < \deg \alpha_n$,

$$
\delta_K(\alpha_n) = v(\alpha_n - \alpha_{n-1}) = n - \frac{1}{p^n}, \quad n \geq 1, \quad (5.7)
$$

and that

whenever $\beta \in \overline{K}$, $v(\alpha_n - \beta) = n - \frac{1}{p^n}$, then $\deg \beta \geq \deg \alpha_{n-1}$. \quad (5.8)

We first calculate Krasner’s constant $\omega_K(\alpha_n)$, defined for any $\alpha$ belonging to $\overline{K}\setminus K$ by

$$
\omega_K(\alpha) = \max\{v(\alpha - \alpha') | \alpha' \neq \alpha \text{ runs over $K$-conjugates of $\alpha$}\}.
$$

Let $\zeta_r$, $r \geq 1$ denote a primitive $p^r$-th root of unity. Keeping in view that

$$
\prod_{i=1}^{p^r-1} (1 - \zeta_r^i) = p^r \quad \text{and} \quad v(1 - \zeta_r) = v(1 - \zeta_r) \quad \text{when} \quad p \text{ does not divide} \ j,
$$

we conclude that

$$
\bar{v}(1 - \zeta_r) \leq \frac{r}{p^r - p^r-1} \leq \frac{1}{p - 1};
$$

in fact $\bar{v}(1 - \zeta_r) = \frac{1}{p - 1}$. It now follows that if $\sigma$ is any automorphism of $\overline{K}/K$.
which does not fix $p^{-1/p'}$, then for $0 < r < r'$, we have

$$\bar{v}(\sigma(p'^r.p^{-1/p'}) - p'^r.p^{-1/p'}) \leq r - \frac{1}{p'} + 1$$
$$\leq r' - \frac{1}{p'}$$
$$< r' - \frac{1}{p'}$$
$$\leq \bar{v}(\sigma(p'^r.p^{-1/p'}) - p'^r.p^{-1/p'}).$$

Consequently it follows that

$$\omega_K(\alpha_n) = \bar{v}((1 - \zeta_1)p^n.p^{-1/p^n}) = \frac{1}{p-1} + n - \frac{1}{p^n}, \ n \geq 1. \quad (5.9)$$

Applying (5.9) for $n - 1$, we see that

$$\omega_K(\alpha_{n-1}) = \frac{1}{p-1} + (n - 1) - \frac{1}{p^{n-1}} < n - \frac{1}{p^n} = \bar{v}(\alpha_n - \alpha_{n-1}).$$

The above inequality, by virtue of Krasner's Lemma, implies that $K(\alpha_{n-1}) \subseteq K(\alpha_n)$. Hence $K(\alpha_n) = K(p^{1/p^n})$ is an extension of degree $p^n$ of $K$. In particular, $\deg \alpha_{n-1} < \deg \alpha_n$.

To verify (5.7), it is clearly enough to show that whenever $\gamma \in \overline{K}$ is such that

$$\bar{v}(\alpha_n - \gamma) > n - (1/p^n)$$

then $\deg \gamma \geq \deg \alpha_n$. On writing $\alpha_n$ as $\alpha_{n-1} + p^n(p^{-1/p^n})$ and keeping in view that $\bar{v}(\alpha_n - \gamma) > n - (1/p^n)$, we have

$$\bar{v}(\alpha_{n-1} - \gamma) = n - \frac{1}{p^n}.$$ 

Applying (5.9) for $n - 1$, it follows from the above equality that $\bar{v}(\alpha_{n-1} - \gamma) > \omega_K(\alpha_{n-1})$, which by virtue of Krasner's Lemma gives $K(\alpha_{n-1}) \subseteq K(\gamma)$. As $\bar{v}(\alpha_{n-1} - \gamma) = n - \frac{1}{p^n}$ and $\alpha_{n-1} - \gamma \in K(\gamma)$, it follows that the index of ramification of $K(\gamma)/K$ (with respect to $v$) is $\geq p^n$. Consequently $\deg \gamma \geq p^n = \deg \alpha_n$ which proves (5.7).
To verify (5.8), note that \( v(\alpha_n - \beta) = n - \frac{1}{p^n} \) together with \( v(\alpha_n - \alpha_{n-1}) = n - \frac{1}{p^n} \) implies that \( v(\beta - \alpha_{n-1}) \geq n - \frac{1}{p^n} > \omega_K(\alpha_{n-1}) \). Applying Krasner’s Lemma, we see that \( K(\alpha_{n-1}) \subseteq K(\beta) \) which in turn implies that \( \deg \beta \geq \deg \alpha_{n-1} \) as desired.

**EXAMPLE 5.4.2.** Let \( \mathbb{Q} \) be the field of rational numbers and \( p_1, p_2, \ldots \) be a sequence of distinct rational primes. Let \( K = \mathbb{Q}(t) \) be the field of Laurent series in an indeterminate \( t \) and \( v \) denote the \( t \)-adic valuation on \( K \) with \( v(t) = 1 \). Then the sequence \( \{\alpha_n\} \) defined by \( \alpha_0 = t, \alpha_n = t + t^2 + \sqrt{p_1} + t^3 + \sqrt{p_2} + \ldots + t^n \sqrt{p_n} \) is an inverted distinguished sequence with respect to \( v \). To prove this assertion, we first show that \( \bar{K}(\alpha_n) = K(t^{1/p_1}, \ldots, t^{1/p_n}) \). As usual, \( \bar{v} \) will denote the unique prolongation of \( v \) to the algebraic closure \( \bar{K} \) of \( K \).

Keeping in mind that the \( \bar{v} \)-residue \( \left( \frac{\alpha_n - t}{t} \right) \) of \( \left( \frac{\alpha_n - t}{t} \right) \) satisfies \( \left( \frac{\alpha_n - t}{t} \right)^2 = p_1 \), we see that the polynomial \( x^2 - p_1 \) has a simple zero in the residue field of the valuation obtained by restricting \( \bar{v} \) to \( K(\alpha_n) \). Applying Hensel’s Lemma [Rib, Chapter 5, Theorem 1], we conclude that the polynomial \( x^2 - p_1 \) has a zero in \( K(\alpha_n) \), i.e., \( \sqrt{p_1} \in K(\alpha_n) \). So \( K(\alpha_n) = K(\sqrt{p_1}, t^{1/p_1} + \sqrt{p_2} + \ldots + t^{n-1} \sqrt{p_n}) \). Repeating the above argument with \( t + t^2 \sqrt{p_2} + \ldots + t^{n-1} \sqrt{p_n} \), we see that \( \sqrt{p_2} \in K(\alpha_n) \). Continuing this process, we conclude that \( K(\alpha_n) = K(\sqrt{p_1}, \ldots, \sqrt{p_n}) \) is an extension of degree \( 2^n \) over \( K \).

Keeping in mind that any automorphism \( \sigma \) of \( \bar{K}/K \) maps \( \sqrt{p_i} \) to \( \pm \sqrt{p_i} \), it is clear that

\[
\omega_K(\alpha_n) = n.
\]

As the characteristic of the residue field of \( \bar{v} \) is zero, \( K(\alpha_n)/K \) is a tame extension. It now follows from Theorem 2.2.A and the above equality that

\[
\delta_K(\alpha_n) = \omega_K(\alpha_n) = n.
\]

Thus the sequence \( \{\alpha_n\} \) will be proved to be an inverted distinguished sequence once we prove that \( (\alpha_n, \alpha_{n-1}) \) is a distinguished pair for \( n \geq 1 \). Note that \( \bar{v}(\alpha_n - \alpha_{n-1}) = \bar{v}(\sqrt{p_n} t^n) = n = \delta_K(\alpha_n) \). So it only remains to be shown that

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whenever $\beta \in \overline{K}$ satisfies $\nu(\alpha_n - \beta) = \delta_K(\alpha_n) = n$, then $\deg \beta \geq \deg \alpha_{n-1}$. Now $\nu(\alpha_n - \beta) = \nu(\alpha_n - \alpha_{n-1}) = n$ together with the formula for $\omega_K(\alpha_n)$ implies that

$$\nu(\beta - \alpha_{n-1}) \geq n > \omega_K(\alpha_{n-1})$$

which by Krasner’s Lemma shows that $K(\alpha_{n-1}) \subseteq K(\beta)$. Therefore $\deg \beta \geq \deg \alpha_{n-1}$ as desired.

The following lemmas will be used in the proof of Theorem 5.2.2. The first three lemmas proved below are already known (cf. [Ag-Kh2]). For reader’s convenience, these are proved here.

**Lemma 5.4.3.** Let $(K, \nu)$ be a complete valued field and $(\overline{K}, \overline{\nu})$ be as above. If $(\alpha, \alpha_1)$ and $(\alpha_1, \alpha_2)$ are two $(K, \nu)$-distinguished pairs of elements of $K$, then $\overline{\nu}(\alpha - \alpha_2) = \delta_K(\alpha_1) < \delta_K(\alpha)$.

**Proof.** Since $\deg \alpha_2 < \deg \alpha_1$ and $(\alpha, \alpha_1)$ is a distinguished pair, it follows from the definition of a distinguished pair that $\overline{\nu}(\alpha - \alpha_2) < \overline{\nu}(\alpha - \alpha_1)$. Consequently by the strong triangle law, we have

$$\delta_K(\alpha_1) = \overline{\nu}(\alpha_1 - \alpha_2) = \min\{\overline{\nu}(\alpha_1 - \alpha), \overline{\nu}(\alpha - \alpha_2)\} = \overline{\nu}(\alpha - \alpha_2),$$

as desired.

**Lemma 5.4.4.** Let $(K, \nu)$ and $(\overline{K}, \overline{\nu})$ be as in the foregoing lemma. If $(\alpha, \alpha_1)$, $(\alpha, \beta_1)$ and $(\beta_1, \beta_2)$ are $(K, \nu)$-distinguished pairs of elements of $K$, then so is $(\alpha_1, \beta_2)$.

**Proof.** Observe that by virtue of the hypothesis, $\deg \alpha_1 = \deg \beta_1$ and

$$\overline{\nu}(\alpha_1 - \beta_1) \geq \min\{\overline{\nu}(\alpha_1 - \alpha), \overline{\nu}(\alpha - \beta_1)\} = \delta_K(\alpha). \quad (5.10)$$
Applying Lemma 5.4.3 to the distinguished pairs \((\alpha, \beta_1)\) and \((\beta_1, \beta_2)\), we see that
\[
\bar{v}(\alpha - \beta_2) = \bar{v}(\beta_1 - \beta_2) < \delta_K(\alpha).
\] (5.11)

It follows from (5.10), (5.11) and the strong triangle law that
\[
\bar{v}(\alpha_1 - \beta_2) = \min\{\bar{v}(\alpha_1 - \beta_1), \bar{v}(\beta_1 - \beta_2)\} = \bar{v}(\beta_1 - \beta_2).
\] (5.12)

If \(\gamma\) is in \(\overline{K}\) and \(\deg \gamma < \deg \alpha_1 = \deg \beta_1\), then keeping in view Lemma 5.4.3, we see that \(\bar{v}(\beta_1 - \gamma) \leq \delta_K(\beta_1) < \delta_K(\alpha)\). Therefore (5.10) together with the strong triangle law shows that
\[
\bar{v}(\alpha_1 - \gamma) = \min\{\bar{v}(\alpha_1 - \beta_1), \bar{v}(\beta_1 - \gamma)\} = \bar{v}(\beta_1 - \gamma).
\] (5.13)

If \(\gamma\) is as above, then \(\bar{v}(\beta_1 - \gamma) \leq \bar{v}(\beta_1 - \beta_2)\) with strict inequality if \(\deg \gamma < \deg \beta_2\). Therefore keeping in view (5.12) and (5.13), we conclude that
\[
\bar{v}(\alpha_1 - \gamma) = \min\{\bar{v}(\alpha_1 - \beta_1), \bar{v}(\beta_1 - \gamma)\} = \bar{v}(\alpha_1 - \beta_2)
\]
for all \(\gamma\) in \(\overline{K}\) with \(\deg \gamma < \deg \alpha_1\). In fact the above inequality is strict if \(\deg \gamma < \deg \beta_2\). This proves that \((\alpha_1, \beta_2)\) is a distinguished pair.

\textbf{DEFINITION.} Let \((K, v)\) and \((\overline{K}, \overline{v})\) as above. Let \(\alpha\) be an element of \(\overline{K}\). A chain \(\alpha = a_0, \ldots, a_r\) will be referred to as a saturated distinguished chain for \(\alpha\) (with respect to \((K, v)\)) if \((a_i, a_{i+1})\) is a \((K, v)\)-distinguished pair for \(0 \leq i < r - 1\) and \(a_r \in K\).

\textbf{LEMMA 5.4.5.} Let \((K, v)\) and \((\overline{K}, \overline{v})\) as in Lemma 5.4.3. If \(\alpha = \alpha_0, \alpha_1, \ldots, \alpha_r\) and \(\alpha = \beta_0, \beta_1, \ldots, \beta_s\) are two saturated distinguished chains for an element \(\alpha\) belonging to \(\overline{K}\), then \(r = s\).

\textit{Proof.} We apply induction on \(\deg \alpha\). When \(\deg \alpha = 2\), then clearly \(r = s = 1\). Assume that \(\alpha \in \overline{K}\) has degree greater than 2 and the result is true for chains of
those elements of \( \overline{K} \) which have degree less than the degree of \( \alpha \). As the lemma is trivially true when \( r = s = 1 \), we may assume, if necessary after renaming that \( s \geq 2 \). By Lemma 5.4.4, \((\alpha_1, \beta_2)\) is a distinguished pair. Hence \( \alpha_1, \alpha_2, \ldots, \alpha_r \) and \( \alpha_1, \beta_2, \ldots, \beta_s \) are two saturated distinguished chains for \( \alpha_1 \). The lemma now follows by induction.

**Lemma 5.4.6.** Let \((K,v)\) and \((\overline{K},\overline{v})\) be as above. Let \( \alpha = \alpha_0, \alpha_1, \ldots, \alpha_r \) be a saturated distinguished chain (with respect to \((K,v)\)) for an element \( \alpha \) of \( \overline{K}\setminus K \). Then

\[
\sup\{\overline{v}(\alpha - a) \mid a \in K\} = \overline{v}(\alpha - \alpha_r).
\]

**Proof.** We prove the result by induction on the length \( r \) of a saturated distinguished chain for \( \alpha \). If \( r = 1 \), then \((\alpha, \alpha_1)\) is a distinguished pair and \( \alpha_1 \in K \). The desired equality is obvious in the present case, because by definition

\[
\overline{v}(\alpha - \alpha_1) = \sup\{\overline{v}(\alpha - \beta) \mid \beta \in \overline{K}, \ deg \beta < deg \alpha\}.
\]

Suppose that the result is true for elements of \( \overline{K}\setminus K \) having saturated distinguished chain of length \( r - 1 \). Let \( \alpha \) be an element of \( \overline{K} \) with a saturated distinguished chain \( \alpha = \alpha_0, \ldots, \alpha_r \). Keeping in mind that \( deg \alpha > 1 \) and Lemma 5.4.3, we have for any \( a \in K \),

\[
\overline{v}(\alpha_1 - a) \leq \delta_K(\alpha_1) < \delta_K(\alpha);
\]

it now follows from the strong triangle law that

\[
\overline{v}(\alpha - a) = \min\{\overline{v}(\alpha - \alpha_1), \overline{v}(\alpha_1 - a)\} = \overline{v}(\alpha_1 - a). \tag{5.14}
\]

It is immediate from (5.14) and the induction hypothesis that

\[
\sup\{\overline{v}(\alpha - a) \mid a \in K\} = \sup\{\overline{v}(\alpha_1 - a) \mid a \in K\} = \overline{v}(\alpha_1 - \alpha_r).
\]

As \( \alpha_r \in K \), we have \( \overline{v}(\alpha_1 - \alpha_r) = \overline{v}(\alpha - \alpha_r) \) by (5.14); this together with the above equation gives the desired equality.
As in the previous chapters for \( L \subseteq K, R(L) \) will denote the residue field of the valuation obtained by restricting \( \overline{v} \) to \( L \).

**Lemma 5.4.7.** Let \( (K,v) \) and \( (K,\overline{v}) \) be as in Lemma 5.4.5 and \( \{\beta_i\} \) be a sequence of elements of \( \overline{K} \) having the same degree. Let \( \eta \) be an element of \( \overline{K}\backslash K \) such that \( \overline{v}(\eta - \beta_i) > \delta_K(\beta_i) \) for each \( i \). Suppose that the sequence \( \{\overline{v}(\eta - \beta_i)\} \) is strictly monotonically increasing with limit strictly greater than \( \delta_K(\eta) \). Then there exists \( j \) such that \( \overline{v}(K(\eta)) = \overline{v}(K(\beta_j)) \) and \( R(K(\eta)) = R(K(\beta_j)) \) for all \( i \geq j \).

**Proof.** Since \( \overline{v}(\eta - \beta_i) > \delta_K(\beta_i) \) for \( i \geq 1 \), on applying Generalized Fundamental Principle (stated in §2.2), we obtain

\[
\overline{v}(K(\beta_i)) \subseteq \overline{v}(K(\eta)), \ R(K(\beta_i)) \subseteq R(K(\eta)). \quad (5.15)
\]

Keeping in view that the sequence \( \{\overline{v}(\eta - \beta_i)\} \) is strictly monotonically increasing, we see that

\[
\overline{v}(\beta_i - \beta_{i+1}) = \min\{\overline{v}(\eta - \beta_i), \overline{v}(\eta - \beta_{i+1})\} = \overline{v}(\eta - \beta_i) > \delta_K(\beta_i).
\]

By virtue of Generalized Fundamental Principle, the above inequality implies that

\[
\overline{v}(K(\beta_i)) \subseteq \overline{v}(K(\beta_{i+1})), \ R(K(\beta_i)) \subseteq R(K(\beta_{i+1})). \quad (5.16)
\]

As all the extensions \( K(\beta_i)/K \) are of the same degree, it is clear from (5.16) that there exists a positive integer \( j \) such that for all \( i \geq j \),

\[
\overline{v}(K(\beta_i)) = \overline{v}(K(\beta_{i+1})), \ R(K(\beta_i)) = R(K(\beta_{i+1})).
\]

The above equations together with (5.15) and (5.16) give

\[
\overline{v}(K(\beta_i)) = \bigcup_{i \geq 1} \overline{v}(K(\beta_i)) \subseteq \overline{v}(K(\eta)), \ R(K(\beta_i)) = \bigcup_{i \geq 1} R(K(\beta_i)) \subseteq R(K(\eta)).
\]
Therefore the lemma follows immediately once we show that

$$
\nu(K(\eta)) \subseteq \bigcup_{i \geq 1} \nu(K(\beta_i)), \quad R(K(\eta)) \subseteq \bigcup_{i \geq 1} R(K(\beta_i)).
$$

To prove (5.17), it is clearly enough to show that if $F(\eta)$ is any element of $K[\eta]/K$, $F(x)$ being a polynomial over $K$ of degree less than $\deg \eta$, then there exists a natural number $k$ such that

$$
\nu(F(\eta)) > 0.
$$

Write $F(x) = c \prod_{l=1}^{m} (x - \gamma_l)$. Since $\deg \gamma_l \leq \deg F(x) < \deg \eta$, we have

$$
\nu(\eta - \gamma_l) \leq \delta_K(\eta) \text{ for } 1 \leq l \leq m.
$$

By hypothesis $\lim_{l \to \infty} \nu(\eta - \beta_l) > \delta_K(\eta)$. So there exists a number $k$ such that

$$
\nu(\eta - \beta_k) > \delta_K(\eta).
$$

It now follows from (5.19), (5.20) and the strong triangle law that for $1 \leq l \leq m$,

$$
\nu(\beta_k - \gamma_l) = \min \{ \nu(\beta_k - \eta), \nu(\eta - \gamma_l) \} = \nu(\eta - \gamma_l) \leq \delta_K(\eta).
$$

On writing $\frac{F(\eta)}{F(\beta_k)}$ as $\prod_{l=1}^{m} \left( 1 + \frac{\eta - \beta_k}{\beta_k - \gamma_l} \right)$ and using (5.20) and (5.21), we obtain (5.18).
5.5 A characterization of transcendental elements of $\overline{K}^c$

In this section, we prove Theorem 5.2.2 which is proved with the motivation of giving a characterization of all those Cauchy sequences $\{b_n\}$ of elements of $\overline{K}$ whose limit is not in $\overline{K}$.

**Proof of Theorem 5.2.2.** Let $\{\alpha_n\}_{n > 0}$ be an inverted distinguished sequence of elements of $\overline{K}$, converging to an element $t$ of $\overline{K}^c$. It will be shown that $t$ is transcendental over $K$. Applying Lemma 5.4.3 to the distinguished pairs $(\alpha_{i+1}, \alpha_i)$ and $(\alpha_{i+2}, \alpha_{i+1})$, we see that

$$v(\alpha_{i+1} - \alpha_i) < v(\alpha_{i+2} - \alpha_{i+1}), \ i \geq 0. \quad (5.22)$$

We first prove that for all $j \geq 0$

$$v(t - \alpha_j) = \delta_K(\alpha_{j+1}). \quad (5.23)$$

Fix any $j \geq 0$. Since $\lim_{n \to \infty} v(t - \alpha_n) = \infty$, we can choose an integer $m > j$ such that

$$v(t - \alpha_m) > \delta_K(\alpha_{j+1}). \quad (5.24)$$

It follows from (5.22), (5.24) and the strong triangle law that

$$v(t - \alpha_j) = v(t - \alpha_m + \alpha_m - \alpha_{m-1} + \ldots + \alpha_{j+1} - \alpha_j)$$

$$= v(\alpha_{j+1} - \alpha_j) = \delta_K(\alpha_{j+1})$$

which proves (5.23).

Suppose, to the contrary, that $t$ is algebraic over $K$. Choose an integer $j$ such
that \( \deg \alpha_j > \deg t \). Consequently by the definition of \( \delta_K(\alpha_j) \), we have

\[
\bar{v}(t - \alpha_j) \leq \delta_K(\alpha_j). \tag{5.25}
\]

But (5.23) together with (5.22) implies that \( \bar{v}(t - \alpha_j) = \delta_K(\alpha_{j+1}) > \delta_K(\alpha_j) \) which contradicts (5.25). This contradiction proves that \( t \) is transcendental over \( K \).

Conversely suppose that an element \( t \) of \( \overline{K}^\times \) is transcendental over \( K \). We shall define an inverted distinguished sequence \( \{\alpha_n\} \) of elements of \( K \) having \( t \) as the limit. For this we shall first inductively define pairs \( (\alpha_j, \delta_j) \) belonging to \( K \times K \) satisfying the following properties (\( P_1 \)) and (\( P_2 \)) for \( i \geq 0 \) and then verify that \( \{\alpha_n\} \)

is an inverted distinguished sequence with limit \( t \):

\begin{align*}
(\text{\(P_1\)}) & \quad \max\{\bar{v}(t - \beta) \mid \beta \in K, \ \deg \beta \leq \deg \alpha_n\} = \delta_n; \\
(\text{\(P_2\)}) & \quad \max\{\bar{v}(t - \beta) \mid \beta \in K, \ \deg \beta < \deg \alpha_n\} = \delta_{n-1}.
\end{align*}

We first construct the pair \( (\alpha_0, \delta_0) \). The set \( M = \{\bar{v}(t - a) \mid a \in K\} \) is bounded above, for otherwise one can choose a sequence \( \{a_n\} \) of elements of \( K \) such that \( \bar{v}(a_n - t) \to \infty \) as \( n \to \infty \). This is impossible as \( K \) is complete and \( t \notin K \). Let \( \delta_0 \)

denote the supremum of \( M \). We now prove that \( \delta_0 \in M \). Since \( K \) is dense in \( \overline{K}^\times \), there exists \( \xi \in \overline{K} \) such that \( \bar{v}(t - \xi) > \delta_0 \). Note that \( \xi \notin K \). It follows by virtue of the strong triangle law that

\[
\bar{v}(t - a) = \bar{v}(\xi - a) \quad \text{for each} \ a \in K.
\]

In particular

\[
\sup\{\bar{v}(t - a) \mid a \in K\} = \sup\{\bar{v}(\xi - a) \mid a \in K\}. \tag{5.26}
\]

By hypothesis, every simple algebraic extension of \((K, \bar{v})\) is defectless. Therefore \( \xi \) has a saturated distinguished chain, say \( \xi = \xi_0, \xi_1, \ldots, \xi_r \) (cf. [Ag-Kh2, Theorem 1.2], [Agh, Theorem 2.1.2]). Using (5.26) and Lemma 5.4.6, we see that

\[
\delta_0 = \sup\{\bar{v}(\xi - a) \mid a \in K\} = \bar{v}(\xi - \xi_r).
\]

Denote \( \xi_r \) by \( \alpha_0 \); this defines the pair \( (\alpha_0, \delta_0) \). As induction hypothesis, suppose
that the pairs \((\alpha_0, \delta_0), \ldots, (\alpha_{n-1}, \delta_{n-1})\) have been defined satisfying the properties \((P_1)\) and \((P_2)\). Let \(A_n\) be the subset of \(\overline{K}\) defined by

\[ A_n = \{ \alpha \in \overline{K} \mid \deg \alpha > \deg \alpha_{n-1}, \overline{v}(t - \alpha) > \delta_{n-1} \}. \]

Let \(h\) denote the minimum of the set \(\{ \deg \alpha \mid \alpha \in A_n \}\) and \(B_n\) the subset of \(A_n\) given by

\[ B_n = \{ \beta \in \overline{K} \mid h = \deg \beta > \deg \alpha_{n-1}, \overline{v}(t - \beta) > \delta_{n-1} \}. \] (5.27)

Define a subset \(S_n\) of \(\mathbb{R}\) given by

\[ S_n = \{ \overline{v}(t - \beta) \mid \beta \in B_n \}. \] (5.28)

We claim that the set \(S_n\) has a maximum element which will be denoted by \(\delta_n\) and can be written as \(\overline{v}(t - \alpha_n)\), \(\alpha_n \in B_n\); this would define the pair \((\alpha_n, \delta_n)\). It will be shown later that the pair \((\alpha_n, \delta_n)\) satisfies the properties \((P_1)\) and \((P_2)\). Suppose that our above claim is false. Two cases arise:

**Case I.** \(S_n\) is not bounded above.

**Case II.** \(S_n\) is bounded above but has no maximum element.

We first consider Case I. In this case for any natural number \(i\), we can choose \(\beta_i \in B_n\) such that \(\overline{v}(t - \beta_i) \geq i\). So \(t = \lim \beta_i\) must belong to \(\overline{K}\) by virtue of Theorem 5.2.1, contrary to the hypothesis. This contradiction disposes of Case I.

We now consider Case II. Recall that \(\delta_n\) is the supremum of the set \(S_n\) defined by (5.28). Choose a sequence \(\{\beta_i\}\) of elements of \(B_n\) such that \(\overline{v}(t - \beta_i) < \overline{v}(t - \beta_{i+1})\) for all \(i\) and \(\lim_{i \to \infty} \overline{v}(t - \beta_i) = \delta_n\). Since \(\overline{K}\) is dense in \(\overline{R}\), there exists an element \(\alpha \in \overline{K}\) such that \(\overline{v}(t - \alpha) \geq \delta_n\). Let \(\eta\) be an element of \(\overline{K}\) of smallest degree satisfying \(\overline{v}(t - \eta) \geq \delta_n\). Note that \(\deg \eta > h\) by virtue of the assumption that \(\delta_n \not\in S_n\). We
shall obtain the desired contradiction by showing that $K(\eta)/K$ is not a defectless extension. This will be accomplished by proving that there exists a member $\beta_j$ of the sequence $\{\beta_i\}$ such that

$$v(K(\eta)) = v(K(\beta_j)), \quad R(K(\eta)) = R(K(\beta_j)).$$

(5.29)

As $\deg \eta > h = \deg \beta_j$, (5.29) quickly yields that the extension $K(\eta)/K$ is not defectless, which is contrary to the hypothesis. This contradiction proves that Case II also does not arise, thereby proving our claim that $S_n$ has a maximum element.

We now prove (5.29). By virtue of Lemma 5.4.7, (5.29) is immediately proved once it is shown that for each $i > 1$, we have

$$v(\eta - \beta_i) = v(t - \beta_i) > \delta_K(\beta_i)$$

(5.30)

and

$$\delta_n > \delta_K(\eta).$$

(5.31)

We now verify (5.30). For any $\gamma \in \overline{K}$ with $\deg \gamma < h$, by virtue of the choice of $h$ and the definition of $B_n$ given by (5.27), we have

$$v(t - \gamma) \leq \delta_{n-1}, \quad v(\beta_i - t) > \delta_{n-1}$$

for all $i > 1$; consequently

$$v(\beta_i - \gamma) = \min\{v(\beta_i - t), v(t - \gamma)\} = v(t - \gamma) \leq \delta_{n-1}.$$ 

Since the above inequality holds for all $\gamma \in \overline{K}$ with $\deg \gamma < h$, we conclude that

$$\delta_K(\beta_i) \leq \delta_{n-1}, \quad i \geq 1.$$ 

(5.32)

Recall that by choice of $\eta$, $v(\eta - t) \geq \delta_n$. Since $\delta_{n-1} < v(t - \beta_i) < \delta_n$ for each $i$, it
now follows from the strong triangle law and (5.32) that
\[ \bar{v}(\eta - \beta_i) = \bar{v}(t - \beta_i) > \delta_{n-1} \geq \delta_K(\beta_i) \]
which proves (5.30).

As regards (5.31), note that for any \( \alpha \in \overline{K} \) with \( \deg \alpha < \deg \eta \), we have \( \bar{v}(t - \alpha) < \delta_n \) which gives
\[ \bar{v}(\eta - \alpha) = \min \{ \bar{v}(\eta - t), \bar{v}(t - \alpha) \} = \bar{v}(t - \alpha) < \delta_n. \]

Recall that every simple algebraic extension of \((K, v)\) is defectless. Consequently by virtue of [Ag-Khl, Theorem 1.1] and the above inequality, we have
\[ \delta_K(\eta) = \max \{ \bar{v}(\eta - \alpha) \mid \alpha \in \overline{K}, \ \deg \alpha < \deg \eta \} < \delta_n \]
which proves (5.31). This completes the proof of our claim that \( S_n \) contains its supremum \( \delta_n \) which can be written as \( v(t - \alpha_n) \) with \( \alpha_n \) in \( B_n \).

Thus we have inductively proved the existence of pairs \((\alpha_n, \delta_n)\) with \( \alpha_n \) belonging to \( B_n \) (given by (5.27)) such that \( \bar{v}(t - \alpha_n) = \delta_n \). The choice of \( \alpha_0, \alpha_1, \ldots, \alpha_n \) shows that
\[ \max \{ \bar{v}(t - \beta) \mid \beta \in \overline{K}, \ \deg \beta \leq \deg \alpha_n \} = \delta_n \quad (5.33) \]
and
\[ \max \{ \bar{v}(t - \beta) \mid \beta \in \overline{K}, \ \deg \beta < \deg \alpha_n \} = \delta_{n-1} \quad (5.34) \]
and thus we have constructed a sequence \((\alpha_i, \delta_i)\) in \( \overline{K} \times \mathbb{R} \) satisfying properties \((P_1)\) and \((P_2)\).

The sequence \( \{\alpha_n\} \) will be an inverted distinguished sequence with limit \( t \) once we prove the following statements (a) and (b).

(a) \((\alpha_n, \alpha_{n-1})\) is a distinguished pair;
(b) \( \bar{v}(\alpha_n - \alpha_{n-1}) = \delta_{n-1} \to \infty \) as \( n \to \infty \).
We first verify (b). Since \( v(t - \alpha_n) = \delta_n > \delta_{n-1} = v(t - \alpha_{n-1}) \), we have
\[
\bar{v}(\alpha_n - \alpha_{n-1}) = \min\{\bar{v}(\alpha_n - t), \bar{v}(t - \alpha_{n-1})\} = \delta_{n-1}.
\]
To show that \( \lim_{n \to \infty} \delta_n = \infty \), we prove that if \( M \) is any given positive real number, then there exists an integer \( j \) such that \( \delta_j \geq M \). As \( \overline{K} \) is dense in \( \overline{K}^c \), we can choose \( \beta \in \overline{K} \) such that \( \bar{v}(t - \beta) \geq M \). Since \{deg \( \alpha_i \)\} is a strictly increasing sequence, we can choose \( j \) such that \( \deg \beta \leq \deg \alpha_j \). It follows from (5.33) that \( \delta_j \geq \bar{v}(t - \beta) \geq M \) as desired.

To prove (a), note that if \( \gamma \in \overline{K} \) is such that \( \deg \gamma < \deg \alpha_n \), then by virtue of (5.34), we have \( \bar{v}(t - \gamma) \leq \delta_{n-1} \). Therefore
\[
\bar{v}(\alpha_n - \gamma) = \min\{\bar{v}(\alpha_n - t), \bar{v}(t - \gamma)\} = \bar{v}(t - \gamma) \leq \delta_{n-1}.
\]
(5.35)

If \( \gamma \in \overline{K} \) is such that \( \deg \gamma < \deg \alpha_{n-1} \), then (5.34) implies that
\[
\bar{v}(\alpha_n - \gamma) = \min\{\bar{v}(\alpha_n - t), \bar{v}(t - \gamma)\} = \bar{v}(t - \gamma) < \delta_{n-1}.
\]
(5.36)

Assertion (a) quickly follows from (5.35), (5.36) and the above assertion (b). This completes the proof of the theorem.

5.6 Proof of Corollary 5.2.3

Set \( \alpha_0 = 1, \alpha_n = \pi^{s-1/p} + \pi^{s-1/p^2} + ... + \pi^{ns-1/p^n} \). In view of Theorem 5.2.2, it is enough to prove that the sequence \{\alpha_n\} is an inverted distinguished sequence with respect to \( v \). We establish this by showing that \([K(\alpha_n) : K] = p^n\) for all \( n \geq 1 \),
\[
\delta_{K}(\alpha_n) = \bar{v}(\alpha_n - \alpha_{n-1}) = ns - \frac{1}{p^n}
\]
(5.37)
and that \( \alpha_{n-1} \) belonging to \( \overline{K} \) is of least degree which satisfies (5.37).

Note that \( K(\alpha_n) \subseteq K(\pi^{1/p^n}) \). So \([K(\alpha_n) : K]\) is a divisor of \( p^n \). As \( \text{char } K \neq p \), \( K(\alpha_n)/K \) is a separable extension. We first calculate Krasner's constant \( \omega_K(\alpha_n) \).
Let $\zeta_r$, $r \geq 1$ denote a primitive $p^r$-th root of unity. Keeping in mind the hypothesis $\frac{v(p)}{p-1} \leq s$ and arguing exactly as in Example 5.4.1, we see that

$$v(1 - \zeta_r) \leq \frac{rv(p)}{p^r - p^{r-1}} \leq \frac{v(p)}{p-1} \leq s;$$

in fact $v(1 - \zeta_1) = \frac{v(p)}{p-1}$. Consequently for any automorphism $\sigma$ of $K/K$, which does not fix $\pi^{-1/p^r}$, we have for $0 < r < r'$ with $r, r'$ in $\mathbb{Z}$ the inequality

$$v(\sigma(\pi^{rs} \pi^{-1/p^r}) - \pi^{rs} \pi^{-1/p^r'}) \leq rs - \frac{1}{p^r} + s \leq r's - \frac{1}{p^r'} < r's - \frac{1}{p^r'} \leq v(\sigma(\pi^{rs} \pi^{-1/p^r'}) - \pi^{rs} \pi^{-1/p^r'}).$$

It now follows that

$$\omega_K(\alpha_n) = v((1 - \zeta_1)\pi^{ns} \pi^{-1/p^n}) = \frac{v(p)}{p-1} + ns - \frac{1}{p^{n-1}} n \geq 1. \quad (5.38)$$

Applying formula (5.38) for $n = 1$ and keeping in mind the inequality $\frac{v(p)}{p-1} \leq s$, we obtain

$$\omega_K(\alpha_{n-1}) = \frac{v(p)}{p-1} + (n-1)s - \frac{1}{p^{n-1}} < ns - \frac{1}{p^n} = v(\alpha_n - \alpha_{n-1}).$$

Since each $\alpha_i$ is separable over $K$, the above inequality, together with Krasner’s Lemma implies that $K(\alpha_{n-1}) \subseteq K(\alpha_n)$. This proves that $K(\alpha_n) = K(\pi^{1/p^n})$ is an extension of degree $p^n$ of $K$. Since $v(\alpha_n - \alpha_{n-1}) = ns - \frac{1}{p^n}$, and $\deg \alpha_{n-1} < \deg \alpha_n$, (5.37) is proved as soon as we show that whenever $\beta \in K$ is such that

$$v(\alpha_n - \beta) > ns - \frac{1}{p^n} \quad (5.39)$$
then $\deg \beta \geq p^n$. Rewriting (5.39) as

$$\tilde{v}(\alpha_{n-1} + \pi^n \pi^{-1/p^n} - \beta) > n s - \frac{1}{p^n}$$

and using strong triangle law, we have

$$\tilde{v}(\alpha_{n-1} - \beta) = n s - \frac{1}{p^n}.$$  (5.40)

Using formula (5.38) for $n - 1$, we conclude from (5.40) that

$$\tilde{v}(\alpha_{n-1} - \beta) > \omega_K(\alpha_{n-1}).$$

The above inequality, in view of Krasner's Lemma, shows that $K(\alpha_{n-1}) \subseteq K(\beta)$. Therefore $\alpha_{n-1} - \beta \in K(\beta)$. As the value group of $v$ is $\mathbb{Z}$, it now follows from (5.40) that the index of ramification of $K(\beta)/K$ (with respect to $v$) is not less than $p^n$. So $\deg \beta \geq p^n$. This proves (5.37). Arguing exactly as in the concluding lines of Example 5.4.2, one can easily check that if $\beta \in \overline{K}$ is such that $\tilde{v}(\alpha_n - \beta) = \delta_K(\alpha_n)$, then $\deg \beta \geq \deg \alpha_{n-1}$. So $(\alpha_n, \alpha_{n-1})$ is a distinguished pair. This completes the proof of the corollary.