Chapter 2

IDEAL THEORY OF SO-RINGS

\$2.1. **Ideals**

In this section we show that the lattice of ideals of a class of so-rings is a distributive lattice.

First we recall the notion of an ideal in so-rings introduced by G.V.S. Acharyulu[3].

Definition 2.1.1[3]. Let R be a so-ring. A subset N of R is said to be an *ideal* of R if the following are satisfied

(I1). if $(x_i : i \in I)$ is summable family in R and $x_i \in N$ for every $i \in I$

then $\Sigma x_i \in N$

(I2). if $x \in R$ and $y \in N \ni x \leq y$ then $x \in N$,

(I3). if $x \in N$ and $r \in R$ then $xr, rx \in N$.

Definition 2.1.2. Let R be a so-ring and let S be a subset of R. Then the intersection of all ideals of R containing S is called the *ideal generated by* S and is denoted by $< S >$.

Example 2.1.3. Consider the so-ring $R = [0, 1]$, the unit interval of real numbers with $\Sigma(x_i : i \in I)$ defined as $\sup\{x_i \mid i \in I\}$ and $x \cdot y$ defined as $\inf\{x, y\}$. Let $S = \{x, y\}$ where $x, y \in R = [0, 1]$. Then the ideal generated by $S = \{x, y\}$ is $< \{x, y\} >$ $= \bigcap\{I \mid I \text{ is an ideal of } [0, 1] \ \& \ \{x, y\} \subseteq I\} = \bigcap\{[0, x_i] \mid \{x, y\} \subseteq [0, x_i]\} = [0, t]$ where $t = \sup\{x, y\}$.

Definition 2.1.4. A so-ring is said to be *complete* if every family in it is summable.

Theorem 2.1.5. Let R be a complete so-ring, then for any S contained in R, $< S >= \{x \in R \mid x \leq \Sigma r_i x_i r_i', x_i \in S, r_i, r_i' \in R\}$.

Proof. Let $T = \{x \in R \mid x \leq \Sigma r_i x_i r_i', x_i \in S, r_i, r_i' \in R\}$.

First we prove T is an ideal of R:

Let $(x_i : i \in I)$ be a summable family in R and $x_i \in T \ \forall i \in I$.

Then $x_i \leq \Sigma_j r_{i_j} x_{i_j} r'_{i_j}, x_{i_j} \in S, r_{i_j}, r'_{i_j} \in R$.

$\Rightarrow \exists\ h_i \in R \ni \Sigma_j r_{i_j} x_{i_j} r'_{i_j} = x_i + h_i \ \forall i \in I$.

$\Rightarrow \exists\ \Sigma_i h_i \in R \ni \Sigma_i \Sigma_j r_{i_j} x_{i_j} r'_{i_j} = \Sigma_i x_i + \Sigma_i h_i$.

$\Rightarrow \Sigma_i x_i \leq \Sigma_i \Sigma_j r_{i_j} x_{i_j} r'_{i_j}$ and hence $\Sigma_i x_i \in T$.

Let $x \in T$ and $y \in R \ni y \leq x$. Then $x \leq \Sigma r_i x_i r'_i, x_i \in S, r_i, r'_i \in R$.

$\Rightarrow y \leq \Sigma r_i x_i r'_i, x_i \in S, r_i, r'_i \in R$ and hence $y \in T$.

Let $x \in T$ and $r \in R$. Then $x \leq \Sigma r_i x_i r'_i, x_i \in S, r_i, r'_i \in R$ and $r \in R$.

$\Rightarrow xr \leq \Sigma r_i x_i (r'_i r)$, $rx \leq \Sigma (r r_i) x_i r'_i$ and hence $xr, rx \in T$.

Hence T is an ideal of R.

Since, for any $x \in s$, $x \leq 1x1$, we have $S \subseteq T$.

To prove T is smallest, let N be an ideal of $S \ni S \subseteq N$.

Now for any $x \in T$, $x \leq \Sigma r_i x_i r'_i$, $x_i \in S, r_i, r'_i \in R$.

Since $S \subseteq N$, each $x_i \in N$. Then $\Sigma r_i x_i r'_i \in N$.

$\Rightarrow x \in N$ and hence $T \subseteq N$.

Hence T is the smallest ideal containing S. □

The following is an example of a complete so-ring in which $< \{x, y\} >$ is the set mentioned in theorem 2.1.5.

Example 2.1.6. As observed in the example 2.1.3, in the complete so-ring $R = [0, 1]$, $< \{x, y\} >= [0, t]$ where $t = \sup\{x, y\}$. Now

$T = \{a \in R \mid a \leq \sup_i\{\ \inf\{r_i, x_i, r'_i\}\}, r_i, r'_i \in [0, 1], x_i \in \{x, y\}\}$

$= \{a \in R \mid a \leq \sup\{x, y\}\} = [0, t]$ where $t = \sup\{x, y\}$.

Remark 2.1.7. If R is a partial semiring and S is a subset of R then

$$< S >= \{x \in R \mid x = \Sigma_i r_i x_i r'_i, x_i \in S, r_i, r'_i \in R\}.$$

Remark 2.1.8. Let $\{N_\alpha \mid \alpha \in \Delta\}$ be a family of ideals of a complete so-ring R. Then $< \bigcup N_\alpha >$ is the smallest ideal containing each N_α.

Following example illustrates that the set union of two ideals in a so-ring need not be an ideal.

Example 2.1.9. Consider the so-ring $R = \{0, a, b, c, d, 1\}$ with Σ on R defined by

$$\Sigma x_i = \begin{cases} x_j,\ if\ x_i = 0\ \forall i \neq j,\ for\ some\ j, \\ d,\ if\ x_j = a,\ x_k = b\ or\ x_j = b,\ x_k = c\ for\ some\ j,\ k,\ x_i = 0\ \forall i \neq j, k, \\ undefined,\ otherwise. \end{cases}$$

and '$\cdot$' defined by

$$x \cdot y = \begin{cases} 0,\ if\ x \neq 1,\ y \neq 1, \\ x,\ if\ y = 1, \\ y,\ if\ x = 1. \end{cases}$$

Then the ideals of R are $\{0\}$, $\{0, a\}$, $\{0, b\}$, $\{0, c\}$, $\{0, a, b, c, d\}$, R. Now $\{0, a\} \cup \{0, b\} = \{0, a, b\}$ is not an ideal of R, since $a + b = d$ which is not in $\{0, a, b\}$.

Remark 2.1.10. Let $\{I_\alpha \mid \alpha \in \Delta\}$ be a family of ideals of a complete so-ring R. Then we denote $< \bigcup\limits_\alpha I_\alpha >$ as $\bigvee\limits_\alpha I_\alpha$ and thus $\bigvee\limits_\alpha I_\alpha = \{x \in R \mid x \leq \Sigma_\alpha x_\alpha, x_\alpha \in I_\alpha, \alpha \in \Delta\}$. In fact $I_1 \bigvee I_2 = \{x \in R \mid x \leq x_1 + x_2, x_1 \in I_1, x_2 \in I_2\}$ is the smallest ideal of R containing I_1 and I_2.

Acharyulu, G.V.S.[3] defined $NP = \{x \in R \mid x = \Sigma_i a_i b_i$ for some $a_i \in N, b_i \in P\}$ for partial ideals N, P of a partial semiring R. For any ideals N, P we define NP in a so-ring as follows

Definition 2.1.11. Let N and P be ideals of a so-ring R. Then

$NP = \{x \in R \mid x \leq \Sigma a_i b_i \text{ for some } a_i \in N, b_i \in P\}$.

Theorem 2.1.12. If N and P are ideals of a complete so-ring R, then NP is an ideal of R. Moreover $NP \subset N \cap P$.

Proof. Note that $NP = \{x \in R \mid x \leq \Sigma_i a_i b_i \text{ for some } a_i \in N, b_i \in P\}$.

First we prove that NP is an ideal of R:

Let $(x_i : i \in I)$ be a summable family in NP.

Then $x_i \leq \Sigma_\alpha a_{i_\alpha} b_{i_\alpha}$, $a_{i_\alpha} \in N$, $b_{i_\alpha} \in P$, $i \in I$, $i_\alpha \in \Delta$.

$\Rightarrow \exists\ h_i \in R \ni \Sigma_\alpha a_{i_\alpha} b_{i_\alpha} = x_i + h_i\ \forall i \in I$.

$\Rightarrow \exists\ \Sigma_i h_i \in R \ni \Sigma_{i\in I}\Sigma_{\alpha\in\Delta} a_{i_\alpha} b_{i_\alpha} = \Sigma_{i\in I} x_i + \Sigma_{i\in I} h_i$.

$\Rightarrow \Sigma_i x_i \leq \Sigma_{i\in I}\Sigma_{\alpha\in\Delta} a_{i_\alpha} b_{i_\alpha}$.

Hence $\Sigma_i x_i \in NP$.

Let $x \in R \ni x \leq y$ where $y \in NP$. Then $x \leq y$ and $y \leq \Sigma_i a_i b_i$, $a_i \in N$, $b_i \in P$.

$\Rightarrow x \leq \Sigma_i a_i b_i$, $a_i \in N$, $b_i \in P$. Hence $x \in NP$.

Let $x \in NP$ and $r \in R$. Then $x \leq \Sigma_i a_i b_i$, $a_i \in N$, $b_i \in P$ and $r \in R$.

$\Rightarrow rx \leq \Sigma_i (ra_i) b_i$, $ra_i \in N$, $b_i \in P$ and $xr \leq \Sigma_i a_i (b_i r)$, $a_i \in N$, $b_i r \in P$.

Hence $rx, xr \in NP$. Therefore NP is an ideal of R.

Now for any $x \in NP$, $x \leq \Sigma_i a_i b_i$, $a_i \in N$, $b_i \in P$.

Since N, P are ideals of R, $a_i b_i \in N \cap P$. $\Rightarrow x \in N \cap P$.

Hence $NP \subset N \cap P$. □

The following is an example of a so-ring R in which $N \cap P$ need not be contained in NP.

Example 2.1.13. Consider the so-ring $R = \{0, u, v, x, y, 1\}$ with Σ defined on R by

$$\Sigma x_i = \begin{cases} x_j, \ if \ x_i = 0 \ \forall i \neq j, \ for \ some \ j, \\ undefined, \ otherwise. \end{cases}$$

and '$\cdot$' defined by the following table:

.	0	u	v	x	y	1
0	0	0	0	0	0	0
u	0	u	0	0	0	u
v	0	0	v	0	0	v
x	0	0	0	0	0	x
y	0	0	0	0	0	y
1	0	u	v	x	y	1

Then for ideals $\{0, x, y\}$, $\{0, u, x\}$ of R, $\{0, x, y\} \cap \{0, u, x\} = \{0, x\}$ whereas $\{0, x, y\}\{0, u, x\} = \{0\}$.

The following theorems can be proved easily.

Theorem 2.1.14. Ideal(R), the set of all ideals of a complete so-ring R forms a complete so-ring with the operations partial sum as '$\bigvee$' and multiplication as '$\cdot$'.

Theorem 2.1.15. Ideal(R), the set of all ideals of a complete so-ring R forms a complete lattice with supremum as '$\bigvee$' and infimum as 'intersection'.

Definition 2.1.16. A so-ring R is said to be *Noetherian* if and only if it satisfies the ascending chain condition on ideals of R.

The so-rings $pfn(D, D)$, $Mfn(D, D)$, $Mset(D, D)$, $\mathbb{R}$ and $\mathbb{N}$ are all Noetherian where as the following is an example of a so-ring which is not Noetherian.

Example 2.1.17. Consider the so-ring $R = [0, 1]$ as in example 0.6.2. Since $[0, 0.1] \subset [0, 0.2] \subset ...$ is a nonterminating ascending chain of ideals of R, R is not Noetherian.

Theorem 2.1.18. The following conditions on a so-ring R are equivalent

(1) R is Noetherian

(2) any nonempty collection of ideals of R has a maximal element

(3) every ideal of R is finitely generated.

Proof. (1)$\Rightarrow$(2): Suppose R is Noetherian.

Let $\mathcal{C} = \{I_i \mid I_i \text{ is an ideal of } R,\ i \in I\}$ be a nonempty family of ideals of R.

Since $\mathcal{C}$ is nonempty, $\exists$ an ideal in $\mathcal{C}$, let it be I_1.

If I_1 is maximal then we get (2).

Suppose I_1 is not maximal, $\exists$ an ideal I_2 in $\mathcal{C} \ni I_1 \subset I_2$.

If I_2 is maximal then we get (2).

Suppose I_2 is not maximal, $\exists$ an ideal I_3 in $\mathcal{C} \ni I_1 \subset I_2 \subset I_3$.

Continuing this process, we get either a maximal element in $\mathcal{C}$ or we get an ascending chain of ideals $I_1 \subset I_2 \subset ...$ of R.

Since R is Noetherian, $\exists\ I_k \in \mathcal{C} \ni I_k = I_{k+1} = ...$

Hence $\mathcal{C}$ has a maximal element.

(2)$\Rightarrow$(3): Suppose any nonempty collection of ideals of R has a maximal element and let I be an ideal of R.

Let $\mathcal{C}$ be the family of all finitely generated ideals of R contained in I.

Since (0) is a finitely generated ideal of R contained in I, $\mathcal{C} \neq \emptyset$.

Then by assumption, $\mathcal{C}$ has a maximal element, let it be $H =< \{a_1, ..., a_n\} >$.

Let $b \in I$ and let $H_b =< \{a_1, ..., a_n, b\} >$.

Since $H \subseteq H_b$ and H is maximal element, we get $H_b = H$. $\Rightarrow b \in H\ \forall b \in I$.

Hence $I = H =< \{a_1, ..., a_n\} >$, a finitely generated ideal of R.

(3)$\Rightarrow$(1): Suppose every ideal of R is finitely generated.

Let $I_1 \subseteq I_2 \subseteq ...$ be an ascending chain of left ideals of R.

Then $I = \bigcup_{i \in I} I_i$ is an ideal of R.

$\Rightarrow I$ is finitely generated. i.e., $I =< \{a_1, ..., a_n\} >$.

$\Rightarrow \exists\, n \in \mathbb{N} \ni I \subseteq I_n \subseteq I$ and $I_n = I_{n+1} = ...$

Hence R is Noetherian. $\square$

Definition 2.1.19. A nonempty subset A of a partial semiring R is said to be *subtractive* if it satisfies the condition:

for any $a, b \in R$, $a + b \in A$ and $b \in A$ implies that $a \in A$.

Definition 2.1.20. A nonempty subset A of a partial semiring R is said to be *strong* if it satisfies the condition:

for any $a, b \in R$, $a + b \in A$ implies that $a, b \in A$.

Example 2.1.21. Consider the partial semiring $R = \{0, u, v, x, y, 1\}$ with Σ defined on R by

$$\Sigma x_i = \begin{cases} x_j,\ if\ x_i = 0\ \forall i \neq j,\ for\ some\ j, \\ 1,\ if\ x_j = u,\ x_k = v\ for\ some\ j, k\ and\ x_i = 0\ \forall\ i \neq j, k \\ undefined,\ otherwise. \end{cases}$$

and '·' defined by the following table:

.	0	u	v	x	y	1
0	0	0	0	0	0	0
u	0	u	0	x	0	u
v	0	0	v	0	y	v
x	0	0	x	0	u	x
y	0	y	0	v	0	y
1	0	u	v	x	y	1

Then the set $\{0, u, v, 1\}$ is strong subset and hence subtractive subset. And the set $\{x, 1\}$ is subtractive subset of R but not strong, whereas $\{u, 1\}$ is not subtractive subset of R.

Remark 2.1.22. In a so-ring R, every ideal is strong and hence subtractive.

Proof. Let I be an ideal of a so-ring R and let $a, b \in R \ni a + b \in I$.

Since $a \leq a + b$ and $b \leq a + b$, we have $a, b \in I$.

Hence I is strong ideal. □

Definition 2.1.23. A partial semiring R is said to be *left austere* if and only if R has no nonzero subtractive left partial ideals.

Since every ideal is subtractive in a so-ring R, a so-ring R is left austere if it has no nonzero left ideals.

$pfn(D, D)$, $Mfn(D, D)$ and $Mset(D, D)$ are all left austere so-rings, whereas the so-ring defined as in example 2.1.13 is not left austere.

Definition 2.1.24. A partial semiring R is said to be *entire* if it has no nonzero divisors. i.e., for any $a, b \in R$, $ab = 0$ then $a = 0$ or $b = 0$.

Theorem 2.1.25. If R is a left austere so-ring then R is entire.

Proof. Suppose $\exists\ 0 \neq a,\ 0 \neq b \in R \ni ab = 0$ and take $I' = \{r \in R \mid rb = 0\}$.

First we prove that I' is a left ideal of R.

Let $(x_i : i \in I)$ be a summable family in $R \ni x_i \in I'\ \forall i \in I$.

Then $x_i b = 0\ \forall i \in I.\ \Rightarrow (\Sigma_i x_i)b = 0$.

Hence $\Sigma_i x_i \in I'$.

Let $x \in R$ and $y \in I' \ni x \leq y$.

Then $x \in R$ and $yb = 0 \ni x \leq y.\ \Rightarrow xb \leq yb = 0.\ \Rightarrow xb = 0$.

Hence $x \in I'$.

Let $r \in R$ and $x \in I'$.

Then $r \in R$ and $xb = 0.\ \Rightarrow (rx)b = r(xb) = 0$.

Hence $rx \in I'$.

Moreover $0 \neq a \in I'$.

Hence I' is a nonzero left ideal of R.

Since R is left austere, $I' = R$. $\Rightarrow 1 \in I'$ so that $1 \cdot b = 0$, a contradiction.

Hence R is entire. $\square$

In general, the converse of the above theorem is not true. This is illustrated by the following example.

Example 2.1.26. Consider the so-ring $R = [0, 1]$ as defined in the example 0.7.2. For any nonzero $x \in R$, $[0, x]$ is nonzero ideal of R and hence R is not left austere. Since $a \cdot b = \inf\{a, b\} = 0$ implies $a = 0$ or $b = 0$, $R = [0, 1]$ is an entire so-ring.

Remark 2.1.27. Let R be a complete so-ring and $a \in R$. Then the smallest ideal generated by $\{a\}$ is $< \{a\} >= \{x \in R \mid x \leq \Sigma r_i a s_i,\ r_i, s_i \in R\}$. We call $< \{a\} >$ as the *principal ideal generated by* a and we denote it by (a).

Lemma 2.1.28. The join of any two principal ideals in a complete so-ring R is a principal ideal.

Proof. Let $(a), (b)$ be two principal ideals of R.

We prove that $(a) \bigvee (b) = (a + b)$.

So let $x \in (a) \bigvee (b)$, then $x \leq \Sigma_i r_i a s_i + \Sigma_j r'_j b s'_j \leq (\Sigma_{i,j} r_i + r'_j)(a + b)(\Sigma_{i,j} s_i + s'_j)$ for some $r_i, r'_j, s_i, s'_j \in R$ and hence $x \in (a + b)$.

Conversely if $x \in (a + b)$, then $x \leq \Sigma_i r_i (a + b) s_i \leq \Sigma_i r_i a s_i + \Sigma_i r_i b s_i$

$\Rightarrow x \in (a) \bigvee (b)$.

Hence the lemma. $\square$

Remark 2.1.29. In any complete so-ring R, $(a_1) \bigvee (a_2) \bigvee ... \bigvee (a_n) = (a_1 + a_2 + ... + a_n)$.

Proof. We prove this by mathematical induction on n.

Clearly the result is true for $n = 1$ and from the above lemma the result is true for $n = 2$.

Suppose $(a_1) \bigvee (a_2) \bigvee ... \bigvee (a_n) = (a_1 + a_2 + ... + a_n)$.

Then $(a_1) \bigvee (a_2) \bigvee ... \bigvee (a_n) \bigvee (a_{n+1}) = (a_1 + a_2 + ... + a_n) \bigvee (a_{n+1})$

$= (a_1 + a_2 + ... + a_n + a_{n+1})$ (by lemma 2.1.28).

Hence the remark. $\square$

Definition 2.1.30. Let I,J be two ideals of a so-ring R. Then define $I + J = \{x + y \mid x \in I, y \in J\}$.

Definition 2.1.31. A so-ring R is said to have decomposition property if and only if for any $a, b, c \in R$, $a \leq b + c$ implies $a = b_1 + c_1$ where $0 \leq b_1 \leq b$, $0 \leq c_1 \leq c$.

Theorem 2.1.32. Let I, J be ideals of a so-ring R satisfying the decomposition property. Then $I + J$ is an ideal of R. In fact $I \bigvee J = I + J$.

Proof. First we prove that $I + J$ is an ideal of R.

Let $(x_i : i \in \Delta)$ be a summable family in R and $x_i \in I + J \ \forall i \in \Delta$.

Then $x_i = y_i + z_i$, $y_i \in I$, $z_i \in J \ \forall i \in \Delta$.

$\Rightarrow \Sigma x_i = \Sigma y_i + \Sigma z_i$, $\Sigma y_i \in I$, $\Sigma z_i \in J$. and hence $\Sigma x_i \in I + J$.

Let $x \in R$, $y \in I + J \ni x \leq y$. Then $x \leq y = p + q$ for some $p \in I, q \in J$.

Then by decomposition property $\exists\ 0 \leq p_1 \leq p, 0 \leq q_1 \leq q \ni x = p_1 + q_1$.

$\Rightarrow x = p_1 + q_1$ for some $p_1 \in I, q_1 \in J$ and hence $x \in I + J$.

Let $r \in R$ and $x \in I + J$. Then $r \in R$ and $x = y + z$, $y \in I$, $z \in J$.

$\Rightarrow rx = ry + rz$, $ry \in I$, $rz \in J$ and $xr = yr + zr$, $yr \in I$, $zr \in J$.

$\Rightarrow rx,\ xr \in I + J$.

Hence $I + J$ is an ideal of R.

Now we prove that $I + J = I \bigvee J$.

Clearly $I \subseteq I + J$ and $J \subseteq I + J$.

Let K be any ideal of $R \ni I \subseteq K$ and $J \subseteq K$.

Then $I + J \subseteq K$ and hence $I + J$ is the smallest ideal of R containing I and J.

Hence the theorem. $\square$

The following is an example of a so-ring R in which the decomposition property fails and $I + J$ is not an ideal for ideals I, J.

Example 2.1.33. Let $(R, \Sigma, \cdot)$be the so-ring as in example 2.1.9. For $c, d \in R$, $c < d = a + b$ and $\exists$ no $0 \leq x \leq a, 0 \leq y \leq b \ni c = x + y$. Thus the decomposition property fails. And for ideals $\{0, a\}$, $\{0, b\}$ of R, $\{0, a\} + \{0, b\} = \{0, a, b, d\}$ which is not an ideal since $c < d$ and $c \notin \{0, a, b, d\}$.

Theorem 2.1.34. The lattice of ideals of a so-ring R satisfying the decomposition property is a distributive lattice.

Proof. Let I, J, K be ideals of R.

Clearly $(I \bigwedge J) \bigvee (I \bigwedge K) \subseteq I \bigwedge (J \bigvee K)$.

Let $x \in I \bigwedge (J \bigvee K)$. $\Rightarrow x \in I, x = y + z$ for some $y \in J$, $z \in K$.

Since $y \leq y + z$, $z \leq y + z$, we have $y, z \in I$.

$\Rightarrow x = y + z$ for $y \in I \bigwedge J, z \in I \bigwedge K$.

$\Rightarrow I \bigwedge (J \bigvee K) \subseteq (I \bigwedge J) \bigvee (I \bigwedge K)$.

Hence the theorem. $\square$

Following is an example of a so-ring in which the decomposition property fails and the lattice of ideals is not distributive.

Example 2.1.35. Let $(R, \Sigma, \cdot)$ be the so-ring as in example 2.1.9. Consider $I = \{0, a\}$, $J = \{0, b\}$, $K = \{0, c\}$ then $I \bigwedge (J \bigvee K) = \{0, a\}$ whereas $(I \bigwedge J) \bigvee (I \bigwedge K) = \{0\}$.

Lemma 2.1.36. For any ideals I, J, K of a complete so-ring R, $I(J \bigvee K) = IJ \bigvee IK$.

Proof. Let $x \in I(J \bigvee K)$.

Then $x \leq \Sigma y_i z_i$ where $y_i \in I$ and $z_i \leq z_i' + z_i''$, $z_i' \in J$, $z_i'' \in K$.

$\Rightarrow x \leq \Sigma y_i(z_i' + z_i'') = \Sigma y_i z_i' + \Sigma y_i z_i'' \in IJ \bigvee IK$.

Conversely let $x \in IJ \bigvee IK$.

Then $x \leq \Sigma y_i y_i' + \Sigma z_j z_j'$ where $y_i, z_j \in I$, $y_i', z_j' \in K$.

$\Rightarrow x \leq \Sigma(y_i + z_i)(y_i' + z_i')$ where $y_i + z_i \in I$, $y_i' + z_i' \in J \bigvee K$.

$\Rightarrow x \in I(J \bigvee K)$.

Hence the lemma. □

Following the notion of 'a well inside b' in a semiring introduced by Golan[16], we have

Definition 2.1.37. Let a, b be elements of a partial semiring R. Then we say that a *is well inside* b, written as $a \lhd b$, if and only if $\exists\ c \in R \ni ac = ca = 0,\ \&\ c + b = 1$.

Remark 2.1.38. In any so-ring R, $0 \lhd 0$ and $a \lhd 1$ for $a \in R$.

Proof. Since $\exists\ 1 \in R \ni 0 \cdot 1 = 1 \cdot 0 = 0$ and $1 + 0 = 1$, we have $0 \lhd 0$.

Since $\exists\ 0 \in R \ni a \cdot 0 = o \cdot a = 0$ and $0 + 1 = 1$, we have $a \lhd 1\ \forall a \in R$. □

Remark 2.1.39. If a, b are elements of a partial semiring R satisfying $a \lhd b$ then $ab = a = ba$.

Proof. Since $a \lhd b$, $\exists\ c \in R \ni ac = ca = 0$ and $c + b = 1$.

$\Rightarrow a(c + b) = a$ and $(c + b)a = a$. Hence $ab = a = ba$. □

Theorem 2.1.40. If R is a commutative complete so-ring, then the set S of all elements I of ideal(R) satisfying the condition that $a \in I$ implies $a \lhd b$ for some $b \in I$ is a subso-ring of ideal(R).

Proof. Let $S = \{I \in ideal(R) \mid a \in I \text{ implies } a \lhd b \text{ for some } b \in I\}$.

To prove S is a subso-ring of R, it is enough to prove S is closed with respect to the operations partial sum '$\bigvee$', multiplication '$\cdot$' of the so-ring ideal(R).

Let $(I_\alpha : \alpha \in \Delta)$ be a family of ideals in S.

Then $\bigvee I_\alpha$ is an ideal of R.

Let $a \in \bigvee I_\alpha$. Then $a \leq \Sigma_\alpha a_\alpha$ for $a_\alpha \in I_\alpha$.

$\Rightarrow \exists\ b_\alpha \in I_\alpha \ni a_\alpha \lhd b_\alpha \forall \alpha \in \Delta$.

$\Rightarrow \exists\ c_\alpha \in R \ni a_\alpha c_\alpha = 0, c_\alpha + b_\alpha = 1\ \forall \alpha \in \Delta$.

Take $d = b_{\alpha_1} b_{\alpha_2} ... + c_{\alpha_1} b_{\alpha_2} ... + ... + c_{\alpha_1} c_{\alpha_2} ... b_{\alpha_n} ... + ...$ and $c = c_{\alpha_1} c_{\alpha_2} ... c_{\alpha_n} .....$

Then $ac = 0$, $c + d = 1$ and $d \in \bigvee I_\alpha$ and hence $a \lhd d$ for some $d \in \bigvee I_\alpha$.

Hence $\bigvee I_\alpha \in S$.

Let I, H be ideals of S, then IH is an ideal of R.

Let $a \in IH$. Then $a \in I \cap H$. $\Rightarrow \exists\ b \in I,\ b' \in H \ni a \lhd b, a \lhd b'$.

$\Rightarrow \exists\ c,\ c' \in R \ni ac = 0, c + b = 1$ and $ac' = 0, c' + b' = 1$.

Take $d = bc' + cb' + cc'$. Then $ad = 0, bb' + d = 1$ for $bb' \in IH$.

$\Rightarrow a \lhd bb'$ for $bb' \in IH$ and hence $IH \in S$.

Hence S is a sub so-ring of ideal(R). □

Definition 2.1.41. An element a in a so-ring R is said to be *complemented* if and only if $a \lhd a$. We denote the set of all complemented elements of R by comp(R). i.e., comp(R) $= \{a \in R \mid a \lhd a\}$.

Definition 2.1.42. We denote the set of all elements of a so-ring R which commutes with every element of R by $C(R)$. i.e., $C(R) = \{a \in R \mid ab = ba\ \forall b \in R\}$.

Theorem 2.1.43. An ideal I of a complete so-ring R is complemented in ideal(R) if $I = (a)$ for some $1 \neq a \in comp(R) \cap C(R)$.

Proof. Suppose $I = (a)$ for some $1 \neq a \in comp(R) \cap C(R)$.

Then $\exists\ b \in R \ni a + b = 1, ab = 0 = ba$ and $ar = ra\ \forall r \in R$.

Take $H = (b)$, the ideal generated by b.

Since $1 = a + b \in (a) \bigvee (b) = I \bigvee H$, $I \bigvee H = R$.

To prove $IH = 0$, take $x \in I \cap H$. Then $x \leq \Sigma_i r_i a s_i, x \leq \Sigma_j r'_j b s'_j$ for $r_i, r'_j, s_i, s'_j \in R$.

$\Rightarrow xb \leq (\Sigma_i r_i a s_i)b = \Sigma_i r_i s_i(ab) = 0$ and $xa \leq (\Sigma_j r'_j b s'_j)a = \Sigma_j r'_j(ba)s'_j = 0$.

$\Rightarrow xa = 0, xb = 0$.

Since $1 = a + b$, we have $x = xa + xb = 0$, hence $I \cap H = 0$ and hence $IH = 0$.

Hence the theorem. □

Definition 2.1.44. An ideal D of a so-ring R is said to be *dense* if it satisfies the condition: for any $r \in R$, $rD = 0$ implies $r = 0$.

Example 2.1.45. Let $[0, 1]$ be the unit interval of real numbers. For any family $(x_i : i \in I)$ in $[0, 1]$, define $\Sigma(x_i : i \in I)$ as the $\sup\{x_i \mid i \in I\}$ and for any x, y in $[0, 1]$, $x \cdot y$ as the usual product. Then $[0, 1]$ is a so-ring and for any $x \in [0, 1]$, $[0, x]$ is a dense ideal of $[0, 1]$.

Theorem 2.1.46. For a complete so-ring R, we have

(i). R is dense

(ii). if D is dense and $D \subset D'$ then D' is dense

(iii). if D and D' are dense, so are DD' and $D \bigcap D'$.

Proof. (i). Let $r \in R \ni rR = 0$. Then $r = r1 = 0$. Hence R is dense.

(ii). Suppose D is dense and $D \subset D'$.

Let $r \in R \ni rD' = 0$. For any $x \in rD$, $x \leq \Sigma r d_i$, $d_i \in D$.

$\Rightarrow x \leq \Sigma r d_i$ for $d_i \in D \subset D'$.

$\Rightarrow x \in rD' = 0$ and hence $rD = 0$. Since D is dense, $r = 0$.

Hence D' is a dense ideal of R.

(iii). Suppose D and D' are dense ideals of R.

Let $r \in R \ni r(DD') = 0$. Then $(rD)D' = 0$. Since D' is dense, $rD = 0$.

Since D is dense, $r = 0$.

Hence DD' is a dense ideal of R.

Since $DD' \subset D \bigcap D'$, $D \bigcap D'$ is a also dense ideal of R. □

Definition 2.1.47. Let A and B be any two subsets of a partial semiring R. Then we define $(A : B) = \{r \in R \mid rB \subset A\}$.

Remark 2.1.48. If A, B are ideals of a complete so-ring R then $(A : B)$ is also an ideal of R.

Proof. Let $(x_i : i \in I)$ be a summable family in R and $x_i \in (A : B)$ $\forall i \in I$.

Then $x_i B \subset A$ $\forall i \in I$. $\Rightarrow \bigvee_{i \in I}(x_i B) \subset A$.

$\Rightarrow (\Sigma x_i)B \subset A$ and hence $\Sigma x_i \in (A : B)$.

Let $x \in R$ and $y \in (A : B) \ni x \leq y$.

Then $yB \subset A$ and $x \leq y$. For any $t \in xB$, $t \leq \Sigma x b_i$, $b_i \in B$.

$\Rightarrow t \leq \Sigma y b_i$, $b_i \in B$. $\Rightarrow t \in yB$.

$\Rightarrow xB \subset yB \subset A$ and hence $x \in (A : B)$.

Let $r \in R$ and $x \in (A : B)$. Then $r \in R$ and $xB \subset A$.

$\Rightarrow (rx)B = r(xB) \subset rA \subset A$ and hence $rx \in (A : B)$.

Now $(xr)B = x(rB) \subseteq xB \subseteq A$ and hence $xr \in (A : B)$.

Hence $(A : B)$ is an ideal of R. □

Remark 2.1.49. It may be noted that the above remark is true for partial semirings and thus $(A : B)$ is a partial ideal of R if A and B are partial ideals of R.

Theorem 2.1.50. If A, B, C are ideals of a complete so-ring R then $(AB)C = A(BC)$. Moreover $AB \subset C \iff A \subset (C : B)$.

Proof. First we prove that $(AB)C = A(BC)$:

Let $x \in (AB)C$. Then $x \leq \Sigma_i x_i c_i$, $x_i \in AB$, $c_i \in C$, $i \in I$.

Since $x_i \in AB$, $\forall i \in I$, $x_i \leq \Sigma_j a_{i_j} b_{i_j}$ where $a_{i_j} \in A$, $b_{i_j} \in B$, $j \in J$.

$\Rightarrow x \leq \Sigma_i[\Sigma_j a_{i_j} b_{i_j}]c_i = \Sigma_i \Sigma_j a_{i_j}(b_{i_j} c_i) \in A(BC)$.

In the similar way, we can prove if $x \in A(BC)$ then $x \in (AB)C$.

Hence $(AB)C = A(BC)$.

Now we prove that $AB \subset C \Leftrightarrow A \subset (C : B)$:

Suppose $AB \subset C$. Then for any $a \in A$, $aB \subset AB \subset C$.

$\Rightarrow a \in (C : B)$ and hence $A \subset (C : B)$.

Conversely suppose that $A \subset (C : B)$.

Let $x \in AB$. Then $x \leq \Sigma a_i b_i$, $a_i \in A$, $b_i \in B$, $i \in I$.

Since $a_i \in A$, $a_i \in (C : B)$ $\forall i \in I$. $\Rightarrow a_i B \subset C$ $\forall i \in I$.

$\Rightarrow a_i b_i \in C$ $\forall i \in I$. $\Rightarrow x \leq \Sigma a_i b_i \in C$ and hence $AB \subset C$.

Hence the theorem. $\square$

Corollary 2.1.51. If $A, B, C, A_i (i \in I)$ are ideals of a complete so-ring R then

(i). $((A : B) : C) = (A : CB)$

(ii). $(\bigvee A_i)B = \bigvee(A_i B)$

(iii). $AR = A = RA$

(iv). $(A : R) = A, (A : A) = R$.

Proof. (i). $X \subseteq ((A : B) : C) \Leftrightarrow XC \subseteq (A : B) \Leftrightarrow (XC)B \subseteq A$

$\Leftrightarrow X(CB) \subseteq A \Leftrightarrow X \subseteq (A : CB)$.

(ii). $(\bigvee A_i)B \subseteq X \Leftrightarrow \bigvee A_i \subseteq (X : B) \Leftrightarrow A_i \subseteq (X : B)$ $\forall i \in I$

$\Leftrightarrow A_i B \subseteq X$ $\forall i \in I \Leftrightarrow \bigvee(A_i B) \subseteq X$.

(iii). Since A is an ideal of R, $AR = A = RA$.

(iv). $X \subseteq (A : R) \Leftrightarrow XR \subseteq A \Leftrightarrow X \subseteq A$.

For any $X \subseteq R$, $XA \subseteq A$ and hence $X \subseteq (A : A)$. Hence $(A : A) = R$.

Hence the corollary. □

Definition 2.1.52. Let A be any subset of a partial semiring R. Then the *left annihilator of* A, denoted by A^*, defined as $(0 : A)$. i.e., $A^* = \{r \in R \mid rA = 0\}$.

Theorem 2.1.53. Let $K,\ K_1,\ K_2$ be ideals of a complete so-ring R. Then

(i). K is dense if and only if $K^* = 0$

(ii). $(K_1 \bigvee K_2)^* = K_1^* \bigcap K_2^*$.

Proof. (i). K is dense ideal $\Leftrightarrow$ for each nonzero $r \in R$, $rK \neq 0 \Leftrightarrow K^* = 0$.

(ii). $x \in (K_1 \bigvee K_2)^* \Leftrightarrow x(K_1 \bigvee K_2) = 0 \Leftrightarrow xK_1 \bigvee xK_2 = 0 \Leftrightarrow xK_1 = 0$

and $xK_2 = 0 \Leftrightarrow x \in K_1^* \bigcap K_2^*$.

Hence the theorem. □

Theorem 2.1.54. In any commutative so-ring R, we have the following

(i). $K \subset J \Rightarrow J^* \subset K^*$

(ii). $K \subset K^{**}$

(iii). $K^{***} = K^*$.

Proof. (i). Suppose $K \subset J$. Then for any $x \in J^*$, $xJ = 0$. $\Rightarrow xK \subset xJ = 0$.

$\Rightarrow x \in K^*$ and hence $J^* \subset K^*$.

(ii). For any $x \in K$, $yx = 0\ \forall y \in K^*$. $\Rightarrow K^*x = 0$. $\Rightarrow xK^* = 0$ and hence $x \in K^{**}$.

(iii). By (i) and (ii), we get $K^{***} = K^*$.

Hence the theorem. □

Theorem 2.1.55. Let I be an ideal I of a commutative complete so-ring R. Then the following are equivalent

(i). $H \bigvee I^* = (HI : I)$

(ii). $HI = KI$ implies $H \bigvee I^* = K \bigvee I^*$ for all ideals H and K of R.

Proof. (i)⇒(ii): Suppose $H \bigvee I^* = (HI : I)$ and $HI = KI$ for any ideals $H,\ K$ of R.

Then $H \bigvee I^* = (HI : I) = (KI : I) = K \bigvee I^*$.

(ii)⇒(i): Suppose $HI = KI$ implies $H \bigvee I^* = K \bigvee I^*$ for any ideals $H,\ K$ of R.

Then for any $a \in (HI : I)$, $aI \subseteq HI$. $\Rightarrow aI \bigvee HI = HI$. $\Rightarrow (\{a\} \bigvee H)I = HI$.

$\Rightarrow (\{a\} \bigvee H) \bigvee I^* = H \bigvee I^*$. $\Rightarrow a \in H \bigvee I^*$. Hence $(HI : I) \subseteq H \bigvee I^*$.

Now for any $a \in H \bigvee I^*$, $a \leq x_1 + x_2$ for some $x_1 \in I^*$, $x_2 \in H$.

$\Rightarrow aI \subseteq (x_1 + x_2)I$ for $x_1 I = 0$ and $x_2 \in H$.

$\Rightarrow aI \subseteq x_1 I \bigvee x_2 I = x_2 I \subseteq HI$. $\Rightarrow a \in (HI : I)$. Hence $H \bigvee I^* \subseteq (HI : I)$.

Hence the theorem. □

Now we define the notion of a maximal ideal in so-ring as follows.

Definition 2.1.56. A proper ideal I of a so-ring R is said to be *maximal* if and only if it is not properly contained in any other proper ideal of R.

Example 2.1.57. Consider the so-ring R as in example 2.1.13. The ideal $\{0, u, v, x, y\}$ is the only maximal ideal of R.

Theorem 2.1.58. Every proper ideal of a so-ring R is contained in some maximal ideal of R.

Proof. Let I be a proper ideal of R.

Suppose I is not maximal. Then $\exists$ a proper ideal J of $R \ni I \subset J$.

Take $\mathcal{C} = \{K \in \text{ideal}(R) \mid I \subset K\}$.

Since $J \in \mathcal{C}$, $\mathcal{C} \neq \emptyset$.

Moreover $(\mathcal{C}, \subseteq)$ is a partially ordered set.

For any simply ordered family $\{K_i \mid i \in I\}$ in $\mathcal{C}$, $K^* = \bigcup K_i$ is an upper bound in $\mathcal{C}$.

Then by Zorn's lemma, $\mathcal{C}$ has a maximal element.

Hence the theorem. □

Theorem 2.1.59. A proper ideal I of a complete so-ring R is maximal if and only if for any element $x \in R$, $x \notin I$, $I \bigvee (x) = R$.

Proof. Suppose I is maximal ideal and let $x \in R \ni x \notin I$.

Then $I \subset I \bigvee (x) \subseteq R$. Since I is maximal, $I \bigvee (x) = R$.

Conversely suppose that $I \bigvee (x) = R$ for $x \in R$, $x \notin I$.

Let J be an ideal of $R \ni I \subset J \subseteq R$. Then $\exists\ x \in J \subseteq R \ni x \notin I$.

$\Rightarrow I \bigvee (x) = R. \Rightarrow R = I \bigvee (x) \subseteq J.$

Hence $J = R$. Hence the theorem. □

$2.2. **Prime & semiprime ideals**

In this section we prove that the ideal lattice of a class of so-rings is isomorphic to the lattice of all open sets of the stone space.

We begin with the following.

Definition 2.2.1. A proper ideal P of a so-ring R is said to *prime* if and only if for any ideals A, B of R, $AB \subset P$ implies $A \subset P$ or $B \subset P$.

Example 2.2.2. Consider the so-ring $R = [0, 1]$ as in example 0.6.2. $[0, x]$, $[0, y]$, $[0, z]$ are ideals of $R \ni [0, x] \cdot [0, y] \subseteq [0, z]$ where $x, y, z \in [0, 1]$. Then $[0,\inf\{x, y\}] \subseteq [0, z]$. Since R is a chain, either $x \leq y$ or $y \leq x$. This implies either $[0, x] \subseteq [0, z]$ or $[0, y] \subseteq [0, z]$. Hence every ideal $[0, z]$ of R is a prime ideal of R.

Theorem 2.2.3. If P is a proper ideal of a complete so-ring R then the following are equivalent

(i). P is prime

(ii). $\{arb \mid r \in R\} \subseteq P \Leftrightarrow a \in P$ or $b \in P$.

Proof. (i)$\Rightarrow$(ii): Suppose P is prime and take $P' = \{arb \mid r \in R\}$.

If $a \in P$ or $b \in P$ then clearly $P' \subseteq P$.

Suppose $P' \subseteq P$ and take $A = (a), B = (b)$.

Let $x \in AB$. Then $x \leq \Sigma_i a_i b_i$ for $a_i \in (a), b_i \in (b)$.

$\Rightarrow \forall i \in I$, $a_i \leq \Sigma_j r_{i_j} a s_{i_j}$ and $b_i \leq \Sigma_k r'_{i_k} b s'_{i_k}$, r_{i_j}, s_{i_j}, r'_{i_k}, $s'_{i_k} \in R$.

$\Rightarrow x \leq \Sigma_i(\Sigma_j r_{i_j} a s_{i_j})(\Sigma_k r'_{i_k} b s'_{i_k})$.

$\Rightarrow x \leq \Sigma_i \Sigma_j \Sigma_k [r_{i_j}(a s_{i_j} r'_{i_k} b) s'_{i_k}]$.

Since $P' \subseteq P$ and P is an ideal of R, we have $x \in P$.

$\Rightarrow AB \subseteq P$. $\Rightarrow A = (a) \subseteq P$ or $B = (b) \subseteq P$.

Hence $a \in P$ or $b \in P$.

(ii)$\Rightarrow$(i): Suppose $P' = \{arb \mid r \in R\} \subseteq P \Leftrightarrow a \in P$ or $b \in P$.

Let A, B be ideals of R such that $AB \subseteq P$ and suppose $A \nsubseteq P$

Then $\exists\, x \in A \ni x \notin P$.

For any $y \in B$, $ry \in B\ \forall r \in R$. $\Rightarrow \{xry \mid r \in R\} \subseteq AB \subseteq P$.

$\Rightarrow x \in P$ or $y \in P$. $\Rightarrow y \in P$ and hence $B \subseteq P$.

Hence P is prime ideal. □

The following results can be obtained directly from theorem 2.2.3.

Corollary 2.2.4. If P is a prime ideal of a complete so-ring R and $a, b \in R$ then the following are equivalent

(i). if $ab \in P$ then $a \in P$ or $b \in P$

(ii). if $ab \in P$ then $ba \in P$.

Corollary 2.2.5. An ideal P of a commutative complete so-ring R is prime if and only if for any $a, b \in R$, $ab \in P$ implies $a \in P$ or $b \in P$.

Definition 2.2.6. A so-ring R is said to be *prime* if and only if (0) is a prime ideal.

$pfn(D, D)$, $Mfn(D, D)$ and $Mset(D, D)$ are prime so-rings for any nonempty set D. It may be noted that the so-ring R considered in example 2.1.13 is not a prime so-ring.

Corollary 2.2.7. A so-ring R is prime if and only if $1 \neq 0$ and for each pair of nonzero elements a, b in R there exists r in R such that $arb \neq 0$.

Definition 2.2.8. A nonempty subset A of a so-ring R is said be an *m-system* if and only if for any $a, b \in A$, $\exists\ r \in R \ni arb \in A$.

Example 2.2.9. Consider the so-ring as in example 2.1.13. Then the set $\{0, u, v\}$ is an m-system of R.

Corollary 2.2.10. A proper ideal P of a complete so-ring R is prime if and only if $R \setminus P$ is an m-system.

Proof. An ideal P of R is prime

$\Leftrightarrow$ ($\{arb \mid r \in R\} \subseteq P$ then $a \in P$ or $b \in P$) (by theorem 2.2.3).

$\Leftrightarrow$ ($a \notin P$ and $b \notin P$ then $\{arb \mid r \in R\} \nsubseteq P$)

$\Leftrightarrow$ for every $a,\ b \in R \setminus P$, $\exists\ r \in R \ni arb \in R \setminus P$

$\Leftrightarrow$ $R \setminus P$ is an m-system. □

Theorem 2.2.11. If A is an m-system of elements of a complete so-ring R and if I is an ideal of R which is maximal among all those ideals of R disjoint from A then P is prime.

Proof. Take $\mathcal{C} = \{J \in \text{ideal}(R) \mid J \bigcap A = \emptyset\}$.

Then $I \in \mathcal{C}$ and I is maximal element in $\mathcal{C}$.

Let H, K be ideals of $R \ni HK \subseteq I$ and suppose $H \nsubseteq I$ and $K \nsubseteq I$.

$\Rightarrow I \subset H \bigvee I$ and $I \subset K \bigvee I$.

By the maximality of I, $(H \bigvee I) \bigcap A \neq \emptyset$ and $(K \bigvee I) \bigcap A \neq \emptyset$.

$\Rightarrow \exists\, x, y \in A \ni x \in H \bigvee I$ and $y \in K \bigvee I$.

Since A is an m-system, $\exists\, r \in R \ni xry \in A$, $x \leq x_1 + x_2$ and $y \leq y_1 + y_2$

for some $x_1 \in H, y_1 \in K, x_2, y_2 \in I$.

Then $xry \leq (x_1 + x_2)r(y_1 + y_2) = x_1ry_1 + x_1ry_2 + x_2ry_1 + x_2ry_2 \in HK \bigvee I = I$.

$\Rightarrow I \bigcap A \neq \emptyset$, a contradiction.

Hence I is a prime ideal of R. □

Remark 2.2.12. Any maximal ideal of a complete so-ring R is prime.

Proof. Let M be a maximal ideal of R and $a,\ b \in R \ni I' = \{arb \mid r \in R\} \subseteq M$.

Suppose $a \notin M$. $\Rightarrow M \subset M \bigvee (a)$. $\Rightarrow M \bigvee (a) = R$.

$\Rightarrow 1 \in M \bigvee (a)$. $\Rightarrow 1 \leq \Sigma r_i a s_i + m$ for some $r_i,\ s_i \in R$, $m \in M$.

$\Rightarrow b \leq (\Sigma r_i a s_i + m)b = \Sigma r_i(a s_i b) + mb \in M$ and hence $b \in M$.

Thus $\{arb \mid r \in R\} \subseteq M$ if and only if $a \in M$ or $b \in M$.

Hence M is maximal ideal of R. □

The following is an example of a so-ring R in which a prime ideal is not a maximal ideal.

Example 2.2.13. Consider the so-ring $R = [0, 1]$ as in example 0.6.2. Then for any $x \in R$, the ideal $[0, x]$ is prime but not maximal.

Theorem 2.2.14. Every prime ideal I of a complete so-ring R contains a minimal prime ideal.

Proof. Take $\mathcal{C} = \{P \in \text{ideal}(R) \mid P \text{ is prime and } P \subseteq I\}$.

Then $I \in \mathcal{C}$ and hence $(\mathcal{C}, \subseteq)$ is a nonempty partially ordered set.

Let $\{H_i \mid i \in \Delta\}$ be a descending chain of prime ideals of R contained in I.

Then $H = \bigcap_{i \in \Delta} H_i$ is an ideal of R such that $H \subseteq I$.

To prove H is prime, let $a,\ b \in R \ni \{arb \mid r \in R\} \subseteq H$ and suppose $a \notin H$.

$\Rightarrow a \notin H_k$ for some $k \in \Delta$.

Since $a \notin H_k$, $\{arb \mid r \in R\} \subseteq H_k$ and H_k is prime, we have $b \in H_k$.

Now $\forall i \leq k$, $H_k \subseteq H_i$ and hence $b \in H_i$ $\forall i \leq k$, $i \in \Delta$.

Now $\forall i > k$, $H_i \subseteq H_k$ and hence $a \notin H_i$.

Since $\{arb \mid r \in R\} \subseteq H_i$, H_i is prime and $a \notin H_i$, we have $b \in H_i$ $\forall i > k$, $i \in \Delta$.

$\Rightarrow b \in H_i$ $\forall i \in \Delta$ and hence $b \in H = \bigcap_{i \in \Delta} H_i$.

Hence H is a prime ideal of R.

Thus $H \in \mathcal{C}$ and H is a lower bound of $\{H_i \mid i \in \Delta\}$ in $\mathcal{C}$.

Then by Zorn's lemma, $\mathcal{C}$ has a minimal element.

Hence the theorem. □

Theorem 2.2.15. If I is an ideal of a complete so-ring R and if H is minimal among those ideals of R properly containing I then $K = \{r \in R \mid rH \subseteq I\}$ is a prime ideal of R.

Proof. By remark 2.1.48, $K = \{r \in R \mid rH \subseteq I\} = (I : H)$ is an ideal of R.

Let $A,\ B$ be ideals of $R \ni AB \subseteq K$ and suppose $B \nsubseteq K$.

$\Rightarrow (AB)H \subseteq I$ and $\exists\ x \in B \ni x \notin K$.

Suppose $BH \subseteq I$. Then $B \subseteq K$, a contradiction.

Hence $BH \nsubseteq I$. $\Rightarrow I \subset I \bigvee BH \subseteq H$.

By the minimality of H, $I \bigvee BH = H$. $\Rightarrow AI \bigvee A(BH) = AH$.

$\Rightarrow AH \subseteq I \bigvee I \subseteq I$. $\Rightarrow A \subseteq K$.

Hence K is a prime ideal of R. □

Lemma 2.2.16. Let I be an ideal of a complete so-ring R and $a \in R \ni a \notin I$. Then there exists a prime ideal P of R such that $I \subset P$ and $a \notin P$.

Proof. Take $\mathcal{C} = \{J \in \text{ideal}(R) \mid I \subset J \ \& \ a \notin J\}$.

Then by Zorn's lemma, $\mathcal{C}$ has a maximal element, let it be P.

Now we prove that P is prime:

Let $H,\ K$ be ideals of $R \ni HK \subseteq P$ and suppose $H \nsubseteq P$ & $K \nsubseteq P$.

Then $P \subset H \bigvee P$ and $P \subset K \bigvee P$. $\Rightarrow a \in H \bigvee P$ and $a \in K \bigvee P$.

$\Rightarrow a \in (H \bigvee P)(K \bigvee P) \subseteq HK \bigvee P = P$, a contradiction.

Hence P is a prime ideal of R. □

Definition 2.2.17. Let R be a so-ring. For $a \in R$, let $r(a) = \{P \in spec(R) \mid a \notin P\}$. The topological space $\tau(R)$ defined on $spec(R)$ (the set of all prime ideals of R) such that the sets of the form $\{r(a) \mid a \in R\}$ is a subbase for open sets is called the *stone space* of R.

Lemma 2.2.18. Let $r(I) = \{P \in spec(R) \mid I \not\subset P\}$. Then

(i). $r(I) \bigcap r(J) = r(I \bigcap J)$ for any ideals I, J of R

(ii). $r(\bigvee I_\alpha) = \bigcup r(I_\alpha)$ for any family $\{I_\alpha \mid \alpha \in \Delta\}$ of ideals of R

(iii). $r(a) = r((a))$ for any $a \in R$

(iv). $a \in I$ if and only if $r(a) \subset r(I)$.

Proof. (i). Let $P \in r(I) \bigcap r(J)$. Then $I \not\subset P$ and $J \not\subset P$.

$\Rightarrow IJ \not\subset P$ (Since P is prime). $\Rightarrow I \bigcap J \not\subset P$ and hence $p \in r(I \bigcap J)$.

Now let $P \in r(I \bigcap J)$. $\Rightarrow I \bigcap J \not\subset P$.

$\Rightarrow I \not\subset P$ and $J \not\subset P$ and hence $P \in r(I) \bigcap r(J)$.

Therefore $r(I) \bigcap r(J) = r(I \bigcap J)$.

(ii). $P \in r(\bigvee I_\alpha) \Leftrightarrow \bigvee I_\alpha \not\subset P \Leftrightarrow I_\alpha \not\subset P$ for some $\alpha \in \Delta \Leftrightarrow P \in \bigcup r(I_\alpha)$.

Hence $r(\bigvee I_\alpha) = \bigcup r(I_\alpha)$.

(iii). $P \in r(a) \Leftrightarrow a \notin P \Leftrightarrow (a) \not\subset P \Leftrightarrow P \in r((a))$.

Hence $r(a) = r((a))$ for any $a \in R$.

(iv). Suppose $a \in I$ and let $P \in r(a)$. Then $a \in I$ and $a \notin P$.

$\Rightarrow I \not\subset P$. $\Rightarrow P \in r(I)$ and hence $r(a) \subset r(I)$.

Suppose $r(a) \subset r(I)$ and $a \notin I$.

Then by lemma 2.2.16, $\exists$ a prime ideal $P \ni I \subset P$ and $a \notin P$.

$\Rightarrow P \in r(a) \subset r(I)$ and hence $I \not\subset P$, a contradiction.

Hence $a \in I$. Hence the lemma. □

Theorem 2.2.19. The ideal lattice of a complete so-ring R is isomorphic to the lattice of all open sets of the stone space of R.

Proof. Let I be an ideal of R.

Then by lemma 2.2.18, $r(I) = r(\bigvee_{a \in I}(a)) = \bigcup_{a \in I} r((a)) = \bigcup_{a \in I} r(a)$.

Since each $r(a)$ is an open set of the stone space of R, $r(I)$ is an open set of the stone space of R.

Let C be any open set of the stone space of R, then $C = \bigcup F_\alpha$ where each $F_\alpha = r(a_1^\alpha) \cap ... \cap r(a_n^\alpha)$ for some $a_1^\alpha, ..., a_n^\alpha$.

Note that $F_\alpha = r(a_1^\alpha) \cap ... \cap r(a_n^\alpha) = r((a_1^\alpha)) \cap ... \cap r((a_n^\alpha)) = r((a_1^\alpha) \cap ... \cap (a_n^\alpha))$ (by lemma 2.2.18).

Hence $C = \bigcup r(I_\alpha) = r(\bigvee I_\alpha)$ where $I_\alpha = (a_1^\alpha) \cap ... \cap (a_n^\alpha)$.

Hence every open set can be represented as $r(I)$ for some ideal I of R.

Since $r(I) = r(J) \Leftrightarrow I = J$, it follows that every open set can be uniquely represented as $r(I)$ for some ideal I of R.

Hence the mapping $I \mapsto r(I)$ is an isomorphism from the lattice of ideals of R to the lattice of all open sets of the stone space of R. $\square$

Now we study semiprime ideals in so-rings.

Definition 2.2.20. A proper ideal I of a so-ring R is said to be *semiprime* if and only if for any ideal H of R, $H^2 \subseteq I$ implies $H \subseteq I$.

Example 2.2.21. Let $(R, \Sigma, \cdot)$ be the so-ring as in example 0.6.2. For any $x \in R$, every ideal $[0, x]$ is semiprime.

Clearly every prime ideal is semiprime. The following is an example of so-ring R in which a semiprime ideal is not a prime ideal.

Example 2.2.22. Let $(R, \Sigma, \cdot)$ be the so-ring as in example 2.1.13. For the ideals $\{0, u\}$, $\{0, v\}$ and $\{0, x, y\}$ of R, $\{0, u\}\{0, v\} = \{0\} \subset \{0, x, y\}$ and $\{0, u\} \nsubseteq \{0, x, y\}$, $\{0, v\} \nsubseteq \{0, x, y\}$. Hence $\{0, x, y\}$ is not prime. However the ideal $\{0, x, y\}$ is semiprime.

Theorem 2.2.23. If I is an ideal of a complete so-ring R then the following are equivalent

(i). I is semiprime

(ii). $\{ara \mid r \in R\} \subseteq I \Leftrightarrow a \in I$.

Proof. (i)$\Rightarrow$(ii): Suppose I is semiprime and take $P' = \{ara \mid r \in R\}$.

If $a \in I$ then clearly $P' \subseteq I$.

Now suppose $P' \subseteq I$ and take $A = (a)$.

Let $x \in A^2$. Then $x \leq \Sigma_i a_i b_i$ for $a_i, b_i \in (a)$.

$\Rightarrow a_i \leq \Sigma_j r_{i_j} a s_{i_j}$, $b_i \leq \Sigma_k r'_{i_k} a s'_{i_k}$ $\forall i \in I$, $r_{i_j}, s_{i_j}, r'_{i_k}, s'_{i_k} \in R$.

$\Rightarrow x \leq \Sigma_i (\Sigma_j r_{i_j} a s_{i_j})(\Sigma_k r'_{i_k} a s'_{i_k})$.

$\Rightarrow x \leq \Sigma_i \Sigma_j \Sigma_k [r_{i_j}(a s_{i_j} r'_{i_k} a) s'_{i_k}]$.

Since $P' \subseteq I$ and I is ideal, we have $x \in I$. $\Rightarrow A^2 \subseteq I$.

$\Rightarrow A = (a) \subseteq I$ and hence $a \in I$.

Therefore $P' \subseteq I \Leftrightarrow a \in I$.

(ii)$\Rightarrow$(i): Suppose $P' = \{ara \mid r \in R\} \subseteq I \Leftrightarrow a \in I$.

Let A be an ideal of R such that $A^2 \subseteq I$ and let $a \in A$.

Then $\{ara \mid r \in R\} \subseteq A^2 \subseteq I$. $\Rightarrow a \in I$ and hence $A \subseteq I$.

Hence I is semiprime. □

Definition 2.2.24. A nonempty subset A of a so-ring R is a *p-system* if and only if for any $a \in A \; \exists \; r \in R \ni ara \in A$.

Clearly every m-system is a p-system. The following is an example of a so-ring R in which a p-system is not an m-system.

Example 2.2.25. Let $(R, \Sigma, \cdot)$ be the so-ring as in example 2.1.13. Then the subset $\{u,\ v\}$ of R is a p-system. But it is not an m-system since for $u,\ v \in \{u,\ v\}$ and for any $r \in R$, $urv = 0 \notin \{u,\ v\}$.

Corollary 2.2.26. A proper ideal I of a complete so-ring R is semiprime if and only if $R \setminus I$ is a p-system.

Proof. An ideal P of R is semiprime

$\Leftrightarrow$ ($\{ara \mid r \in R\} \subseteq P$ then $a \in P$) (by theorem 2.2.23)

$\Leftrightarrow$ ($a \notin P$ then $\{ara \mid r \in R\} \nsubseteq P$)

$\Leftrightarrow$ for any $a \in R \setminus P \; \exists \; r \in R \ni ara \in R \setminus P$

$\Leftrightarrow$ $R \setminus P$ is a p-system. □

Theorem 2.2.27. A nonempty subset A of a complete so-ring R is a p-system if and only if it is the union of m-systems.

Proof. Suppose $A = \bigcup_{i \in \Delta} B_i$ where each B_i is an m-system.

Since every m-system is p-system and union of p-systems is again a p-system, A is a p-system.

Conversely suppose that A is a p-system and let $a_0 \in A$.

$\Rightarrow \exists\, r_0 \in R \ni a_1 = a_0 r_0 a_0 \in A$.

$\Rightarrow \exists\, r_1 \in R \ni a_2 = a_1 r_1 a_1 = (a_0 r_0 a_0) r_1 (a_0 r_0 a_0) \in A$.

Continuing this process, we get a subset $B = \{a_0, a_1, ...\}$ of A, which forms an m-system.

Hence A is the union of m-systems. □

We denote the set $\{H \in spec(R) \mid I \subset H\}$ by $V(I)$ and $\bigcap V(I)$ by $\sqrt{I}$. We call $\sqrt{0}$ as the lower nil radical of R.

Theorem 2.2.28. A proper ideal I of a complete so-ring R is semiprime if and only if $I = \sqrt{I}$.

Proof. Suppose I is a semiprime ideal of R and take $A = R \setminus I$.

By corollary 2.2.26, A is a p-system.

By theorem 2.2.27, $A = \bigcup_{i \in \Delta} B_i$, B_i is an m-system, $i \in \Delta$.

Take $\mathcal{C} = \{J \in \text{ideal}(R) \mid J \bigcap B_i = \emptyset \text{ and } I \subseteq J\}$. Clearly $I \in \mathcal{C}$.

Then $(\mathcal{C}, \subseteq)$ is a nonempty poset in which every simply ordered family has an upper bound.

By Zorn's lemma, $\mathcal{C}$ has a maximal element. Let it be K_i.

i.e., $K_i \bigcap B_i = \emptyset$ and $I \subseteq K_i$ $\forall i \in \Delta$.

By theorem 2.2.11, each K_i is prime.

$\Rightarrow I \subseteq \bigcap K_i \subseteq \bigcap (R \setminus B_i) = I.$

$\Rightarrow I = \bigcap \{K_i \mid K_i \text{ is prime ideal of } R \ \ni \ I \subseteq K_i\} = \bigcap V(I).$

Hence $I = \sqrt{I}$.

Conversely suppose that $I = \sqrt{I} = \bigcap \{H \mid H \text{ is prime, } I \subseteq H\}$.

$\Rightarrow R \setminus I = R \setminus \bigcap \{H \in \text{spec}(R) \mid I \subseteq H\} = \bigcup \{R \setminus H \mid H \in \text{Spec}(R),\ I \subseteq H\}.$

Since H is prime, $R \setminus H$ is m-system.

Then by theorem 2.2.27, $R \setminus I$ is p-system.

Hence by corollary 2.2.26, I is semiprime. □

Theorem 2.2.29. If I, H are ideals of a complete so-ring R, then

(i). $I \subseteq H \Rightarrow \sqrt{I} \subseteq \sqrt{H}$

(ii). $\sqrt{\sqrt{I}} = \sqrt{I}$

(iii). $\sqrt{I \bigvee H} = \sqrt{\sqrt{I} \bigvee \sqrt{H}}$.

Proof. (i). Suppose $I \subseteq H$.

Then $\{P \in \text{Spec}(R) \mid I \subseteq P\} \supseteq \{P \in \text{Spec}(R) \mid H \subseteq P\}$.

$\Rightarrow \bigcap \{P \in \text{Spec}(R) \mid I \subseteq P\} \subseteq \bigcap \{P \in \text{Spec}(R) \mid H \subseteq P\}.$

Hence $\sqrt{I} \subseteq \sqrt{H}$.

(ii). Since every prime ideal is semiprime and intersection of prime ideals is semiprime, we have $\sqrt{I}$ is semiprime.

By theorem 2.2.28, $\sqrt{\sqrt{I}} = \sqrt{I}$.

(iii). Since $I \subseteq \sqrt{I}$ and $H \subseteq \sqrt{H}$, we have $I \bigvee H \subseteq \sqrt{I} \bigvee \sqrt{H}$.

Hence $\sqrt{I \bigvee H} \subseteq \sqrt{\sqrt{I} \bigvee \sqrt{H}}$.

Also $I \subseteq I \bigvee H$, $H \subseteq I \bigvee H$. $\Rightarrow \sqrt{I} \bigvee \sqrt{H} \subseteq \sqrt{I \bigvee H}$.

$\Rightarrow \sqrt{\sqrt{I} \bigvee \sqrt{H}} \subseteq \sqrt{\sqrt{I \bigvee H}} = \sqrt{I \bigvee H}$ (by (ii)).

Hence $\sqrt{I \bigvee H} = \sqrt{\sqrt{I} \bigvee \sqrt{H}}$. □

Theorem 2.2.30. If I is an ideal of a commutative complete so-ring R then $\sqrt{I} = \{a \in R \mid a^n \in I \text{ for some positive integer } n\}$.

Proof. Take $K = \{a \in R \mid a^n \in I \text{ for some positive integer } n\}$.

First we prove K is ideal of R:

Let $(a_i : i \in \Delta)$ be a family in K. Then $\exists\, n_i \in \mathbb{Z}^+ \ni a_i^{n_i} \in I$, $\forall i \in \Delta$.

Take $n = \Sigma n_i$. Then $(\Sigma a_i)^{n-1} \in I$ and hence $\Sigma a_i \in K$.

Let $x \in R, y \in K \ni x \leq y$. Then $x \leq y$ & $y^n \in I$ for some $n \in \mathbb{Z}^+$.

$\Rightarrow x^n \in I$ and hence $x \in K$.

Let $r \in R, x \in K$. Then $r \in R$ & $x^n \in I$ for some $n \in \mathbb{Z}^+$.

$\Rightarrow (rx)^n = r^n x^n \in I$ and hence $rx \in K$.

Hence K is an ideal of R.

Now we prove that $\sqrt{I} = K$:

Let $c \in R \setminus K$ and suppose $c^2 \in K$.

$\Rightarrow c \in R \setminus K$ and $\exists\, n \in \mathbb{Z}^+ \ni (c^2)^n \in I$.

$\Rightarrow c \in R \setminus K$ and $\exists\, 2n \in \mathbb{Z}^+ \ni c^{2n} \in I$.

$\Rightarrow c \in R \setminus K$ and $c \in K$, a contradiction.

Hence $c^2 \in R \setminus K$. $\Rightarrow R \setminus K$ is a p-system and hence K is semiprime.

$\Rightarrow K = \sqrt{K}$ (by theorem 2.2.28).

Since $I \subseteq K$, $\sqrt{I} \subseteq \sqrt{K} = K$ (by theorem 2.2.29(i)).

Now let P be a prime ideal containing I. For any $a \in K$, $a^n \in I$ for some $n \in \mathbb{Z}^+$.

$\Rightarrow a^n \in P$ and hence $a \in P$.

$\Rightarrow K \subseteq P\ \forall$ prime ideal $P \ni I \subseteq P$. $\Rightarrow K \subseteq \bigcap V(I) = \sqrt{I}$.

Hence $\sqrt{I} = K = \{a \in R \mid a^n \in I \text{ for some positive integer } n\}$. □

Definition 2.2.31. An ideal I of a so-ring R is said to be *primary* if and only if $\forall a \in R \setminus I$ and $b \in R$, $ab \in I$ implies $b^k \in I$ for some positive integer k.

In the so-ring $R = [0,1]$ as defined in example 0.6.2, every ideal $[0,x]$ is primary ideal of R. However the following is an example of a so-ring in which an ideal is not primary.

Example 2.2.32. Consider the so-ring $R = \{0,u,v,x,y\}$ as in example 2.1.13. For $x \in R \setminus \{0,u\}$ and $v \in R$, $xv = 0 \in \{0,u\}$ but $v^k = v$ for any $k \in \mathbb{Z}^+$ and $v \notin \{0,u\}$. Hence the ideal $\{0,u\}$ is not primary.

Corollary 2.2.33. If I is a primary ideal of a commutative complete so-ring R then $\sqrt{I}$ is a prime ideal of R.

Proof. Let $a,b \in R \ni ab \in \sqrt{I}$.

Then $(ab)^n = a^n b^n \in I$ for some $n \in \mathbb{Z}^+$.

Suppose $a \notin \sqrt{I}$. Then $a^n \notin \sqrt{I}$.

$\Rightarrow a^n \in R \setminus \sqrt{I}$ and hence $a^n \in R \setminus I$.

Now $a^n \in R \setminus I$, $b^n \in R \ni a^n b^n \in I$ and I is primary.

$\Rightarrow (b^n)^k \in I$ for some $n,k \in \mathbb{Z}^+$ and hence $b \in \sqrt{I}$.

Hence $\sqrt{I}$ is a prime ideal of R. □

Theorem 2.2.34. If I, H are ideals of a commutative complete so-ring R then $\sqrt{IH} = \sqrt{I \bigcap H} = \sqrt{I} \bigcap \sqrt{H}$.

Proof. Since $IH \subseteq I \bigcap H \subseteq I,\ H$, we have $\sqrt{IH} \subseteq \sqrt{I \bigcap H} \subseteq \sqrt{I} \bigcap \sqrt{H}$.

Now for any $a \in \sqrt{I} \bigcap \sqrt{H}$, $a^n \in I$ and $a^m \in H$ for some $n,m \in \mathbb{Z}^+$.

$\Rightarrow a^{n+m} \in IH$ for some $n+m \in \mathbb{Z}^+$ and hence $a \in \sqrt{IH}$.

Hence $\sqrt{I} \bigcap \sqrt{H} = \sqrt{IH}$.

Hence the theorem. □

Theorem 2.2.35. If I is an ideal of a commutative complete so-ring R such that $\sqrt{I}$ is finitely generated then $\exists\ n \in \mathbb{Z}^+ \ni (\sqrt{I})^n \subseteq \sqrt{I}$.

Proof. Suppose $\sqrt{I} =< \{a_1, ..., a_k\} >$ for $a_1, ..., a_k \in R$ and $k \in \mathbb{Z}^+$.

$\Rightarrow \exists\ n_i \in \mathbb{Z}^+$, $1 \leq i \leq k \ni a_i^{n_i} \in I$.

Take $n = \Sigma_{i=1}^k n_i$. Then for any $b \in \sqrt{I}$, $b \leq \Sigma_{i=1}^k r_i a_i$.

$\Rightarrow b^n \leq \Sigma \frac{n!}{h_1!...h_k!}(r_1 a_1)^{h_1}...(r_k a_k)^{h_k}$ where the sum is taken over all k-tuples $(h_1, ..., h_k) \ni \Sigma_{i=1}^k h_i = n$.

In each summand, we must have $h_j \geq n_j$ for atleast one j

Hence each sum is in I.

$\Rightarrow b^n \in I$ for some $n \in \mathbb{Z}^+$ and hence $b \in \sqrt{I}$.

Hence the theorem. □

Definition 2.2.36. A so-ring R is said to be *semiprime* if (0) is a semiprime ideal.

$pfn(D, D)$, $Mfn(D, D)$ and $Mset(D, D)$ are semiprime so-rings for any nonempty set D.

Theorem 2.2.37. If K is an ideal of a semiprime complete so-ring R then

(i). $K \bigcap K^* = 0$

(ii). $K \bigvee K^*$ is dense ideal.

Proof. (i). Since $(K \bigcap K^*)^2 = (K \bigcap K^*)(K \bigcap K^*) \subseteq KK^* = 0$, we have $K \bigcap K^* = 0$.

(ii). Since $(K \bigvee K^*)^* = K^* \bigcap K^{**} = 0$, $K \bigvee K^*$ is dense ideal of R.

(By theorem 2.1.53). □

Definition 2.2.38. An ideal I of a so-ring R is said to be *annihilator ideal* if and only if $\exists\ J \subseteq R \ni I = J^*$.

Theorem 2.2.39. The annihilator ideals in a commutative semiprime complete so-ring R form a complete Boolean algebra $B^*(R)$, with 'intersection' as infimum and '$*$' as complementation.

Proof. Take $B^*(R) = \{I^* \mid I \subseteq R\}$.

Since $0^* = R \in B^*(R)$, $B^*(R) \neq \emptyset$.

Moreover $(B^*(R), \subseteq)$ is an ordered set which has greatest element R and

$\inf K_i^* = \bigcap K_i^* = (\bigvee K_i)^* \in B^*(R)$ exists for any family $\{K_i^* \mid i \in I\}$ of $B^*(R)$.

Hence $(B^*(R), \subseteq)$ is a complete lattice.

To prove $B^*(R)$ is a Boolean algebra, it remains to prove that

$J \bigcap K^* = 0 \Leftrightarrow J \subset K$ for any $J, K \in B^*(R)$.

Suppose $J \subset K$. Then $J \bigcap K^* \subset K \bigcap K^* = 0$ and hence $J \bigcap K^* = 0$.

Suppose $J \bigcap K^* = 0$. Then $JK^* = 0$. $\Rightarrow J \subset K^{**} = K$.

Hence $B^*(R)$ is a complete Boolean algebra. □

Definition 2.2.40. An ideal I of a so-ring R is said to be *irreducible* if and only if for any ideals H and K of R, $I = H \cap K$ implies $I = H$ or $I = K$.

Definition 2.2.41. An ideal I of a so-ring R is said to be *strongly irreducible* if and only if for any ideals H and K of R, $H \cap K \subseteq I$ implies $H \subseteq I$ or $K \subseteq I$.

In the so-ring $R = [0, 1]$ as in example 0.6.2, every ideal $[0, x]$ is strongly irreducible.

Clearly every strongly irreducible ideal is irreducible. The following is an example of a so-ring R in which an irreducible ideal is not a strongly irreducible ideal.

Example 2.2.42. Let $(R, \Sigma, \cdot)$ be the so-ring as in example 2.1.9. For the ideals $\{0, a\}$, $\{0, b\}$ and $\{0, c\}$ of R, $\{0, b\} \bigcap \{0, c\} = \{0\} \subseteq \{0, a\}$ and $\{0, b\} \nsubseteq \{0, a\}$, $\{0, c\} \nsubseteq \{0, a\}$. Hence $\{0, a\}$ is not strongly irreducible. However the ideal $\{0, a\}$ is irreducible.

Definition 2.2.43. A non empty subset A of so-ring R is said to be an i-*system* if and only if for any $a, b \in A$, $(a) \bigcap (b) \bigcap A \neq \emptyset$.

Example 2.2.44. Let $(R, \Sigma, \cdot)$ be the so-ring as in example 2.1.13. Then the subset $\{0, u\}$ of R is an i-system whereas the subset $\{x, y\}$ is not an i-system since $(x) = \{0, x\}$, $(y) = \{0, y\}$ and $(x) \bigcap (y) \bigcap A = \emptyset$.

Theorem 2.2.45. If I is an ideal of a complete so-ring R then the following are equivalent

(i). I is strongly irreducible

(ii). if $a, b \in R$ satisfy $(a) \cap (b) \subseteq I$ then $a \in I$ or $b \in I$

(iii). $R \setminus I$ is an i-system.

Proof. (i)$\Rightarrow$(ii): Suppose I is strongly irreducible.

Then for any $a,\ b \in R \ni (a) \bigcap (b) \subseteq I$, $(a) \subseteq I$ or $(b) \subseteq I$.

Hence $a \in I$ or $b \in I$.

(ii)$\Rightarrow$(iii): Suppose $a,\ b \in R \ni (a) \bigcap (b) \subseteq I$ imply $a \in I$ or $b \in I$.

Let $a,\ b \in R \setminus I$. Then $(a) \bigcap (b) \nsubseteq I$.

$\Rightarrow (a) \bigcap (b) \bigcap (R \setminus I) \neq \emptyset$.

Hence $R \setminus I$ is an i-system.

(iii)$\Rightarrow$(i): Suppose $R \setminus I$ is an i-system.

Let $H,\ K$ be ideals of $R \ni H \bigcap K \subseteq I$ and suppose $H \nsubseteq I$ & $K \nsubseteq I$.

$\Rightarrow \exists\ x,\ y \in R \setminus I \ni x \in H$ and $y \in K$.

$\Rightarrow \exists\ z \in (x) \bigcap (y)$ and $z \notin I$.

$\Rightarrow \exists\ z \in H \bigcap K$ and $z \notin I$ and hence $H \bigcap K \nsubseteq I$, a contradiction.

Hence I is strongly irreducible. □

Theorem 2.2.46. Let a be a nonzero element of a so-ring R and let I be an ideal of R not containing a. Then there exists an irreducible ideal H of R containing I and not containing a.

Proof. Take $\mathcal{C} = \{J \in \text{ideal}(R) \mid I \subseteq J \;\&\; a \notin J\}$. Clearly $I \in \mathcal{C}$.

Then by Zorn's lemma, $\mathcal{C}$ has a maximal element. Let it be H.

Now we prove that H is irreducible.

Let $A,\ B$ be ideals of $H \ni H = A \bigcap B$ and suppose $H \subset A \;\&\; H \subset B$.

$\Rightarrow a \in A$ and $a \in B$ and hence $a \in A \bigcap B = H$, a contradiction.

Hence H is irreducible and hence the theorem. □

Theorem 2.2.47. Any proper ideal I of a so-ring R is the intersection of all irreducible ideals containing it.

Proof. Since I is proper, $1 \notin I$.

Then by theorem 2.2.46, $\exists$ an irreducible ideal of R containing I and not containing 1.

Take $I' = \bigcap\{J \in \text{ideal}(R) \mid J \text{ is irreducible } \& I \subseteq J\}$.

Then $I \subseteq I'$. Suppose $I \subset I'$.

$\Rightarrow \exists\; x \in I' \ni x \notin I$.

Again by theorem 2.2.46, $\exists$ an irreducible ideal $H \ni I \subseteq H$ and $x \notin H$.

Then $I' \subseteq H$. Since $x \in I'$, $x \in H$, a contradiction.

Hence $I = I' = \bigcap\{J \in \text{ideal}(R) \mid J \text{ is irreducible } \& \; I \subseteq J\}$. □

Theorem 2.2.48. An ideal I of a complete so-ring R is prime if and only if it is semiprime and strongly irreducible.

Proof. Suppose I is prime. Then I is semiprime.

Let $H,\ K$ be ideals of $R \ni H \bigcap K \subseteq I$. Then $HK \subseteq I$.

$\Rightarrow H \subseteq I$ or $K \subseteq I$ and hence I is strongly irreducible.

Conversely suppose that I is semiprime and strongly irreducible.

Let $H,\ K$ be ideals of $R \ni HK \subseteq I$.

Then $(H \bigcap K)^2 = (H \bigcap K)(H \bigcap K) \subseteq HK \subseteq I$.

Since I is semiprime, $H \bigcap K \subseteq I$.

$\Rightarrow H \subseteq I$ or $K \subseteq I$ (since I is strongly irreducible).

Hence I is a prime ideal of R. □

Definition 2.2.49. An element a of a partial semiring R is said to be *multiplicatively regular* if and only if $\exists\ b \in R \ni aba = a$. A partial semiring R is said to be *multiplicatively regular* if and only if each element of R is multiplicatively regular.

The so-ring $R = [0, 1]$ as defined in example 0.6.2 is multiplicatively regular. However the so-ring $\mathbb{N}$ is not multiplicatively regular since for $2 \in \mathbb{N}\ \exists$ no $x \in \mathbb{N} \ni 2x2 = 2$.

Remark 2.2.50. If R is multiplicatively regular then $HI = H \cap I$ for any ideals H and I of R.

Proof. For any ideals H and I of R, $HI \subset H \bigcap I$ by theorem 2.1.12.

For any $x \in H \bigcap I$, $\exists\ y \in R \ni x = xyx \in HI$ and hence $H \bigcap K \subseteq HK$.

Hence the remark. □

Theorem 2.2.51. Let R be a complete so-ring which is multiplicatively regular. Then an ideal I of R is prime if and only if it is irreducible.

Proof. Suppose I is prime, then by theorem 2.2.48, I is strongly irreducible and hence I is irreducible.

Conversely suppose that I is irreducible. Let H, K be ideals of R such that $HK \subseteq I$.

Since R is multiplicatively regular, $HK = H \cap K$ and hence $H \cap K \subseteq I$.

Let $x \in (H \bigvee I) \bigcap (K \bigvee I)$. Then $x \in (H \bigvee I)(K \bigvee I)$.

$\Rightarrow x \leq \Sigma_i y_i z_i$ for $y_i \in H \bigvee I, z_i \in K \bigvee I$.

$\Rightarrow x \leq \Sigma_i[\Sigma_j(h_{i_j} + a_{i_j})][\Sigma_k(p_{i_k} + b_{i_k})]$ for $h_{i_j} \in H, p_{i_k} \in K, a_{i_j}, b_{i_k} \in I$.

$\Rightarrow x \leq \Sigma_i \Sigma_j \Sigma_k [h_{i_j} p_{i_k} + h_{i_j} b_{i_k} + a_{i_j} p_{i_k} + a_{i_j} b_{i_k}]$ and hence $x \in HK \bigvee I = I$.

$\Rightarrow (H \bigvee I) \bigcap (K \bigvee I) = I$.

$\Rightarrow H \bigvee I = I$ or $K \bigvee I = I$.

$\Rightarrow H \subseteq I$ or $K \subseteq I$.

Hence I is prime ideal of R. □

It can be easily observed that the theorems 2.2.3, 2.2.19, 2.2.23, 2.2.30, 2.2.39 and 2.2.51 are true for partial semirings.

$2.3. **Right strongly prime partial semiring**

In this section we introduce the notion of a right strongly prime partial semiring and characterize strongly prime radicals by using the notion of super sp-system.

Throughout this section R stands for a partial semiring.

Definition 2.3.1. Let A be a nonempty subset of a partial semiring R. Then the *right annihilator of* A in R, denoted by A^*, is defined by $A^* = \{r \in R \mid Ar = (0)\}$.

Remark 2.3.2. A^* is a right subtractive partial ideal of R.

Definition 2.3.3. Let $r \in R$ such that $r \neq 0$. Then a finite subset S_r of R is said to be a *right insulator for* r if and only if $(rS_r)^* = (0)$.

Definition 2.3.4. A partial semiring R is said to be a *right strongly prime* if and only if every nonzero element of R has a right insulator. i.e., for each nonzero $r \in R$ $\exists$ a finite subset S_r of $R \ni$ for $t \in R$, $(rS_r)t = (0) \Rightarrow t = 0$.

Theorem 2.3.5. Every right strongly prime partial semiring is prime.

Proof. Let R be a right strongly prime partial semiring and H, K be partial ideals of $R \ni HK = (0)$.

Suppose $H \neq (0)$. Then $\exists\ a \in H \ni a \neq 0$.

$\Rightarrow a$ has a right insulator S_a.

Let $b \in K$. Then $(aS_a)b \subseteq Hb \subseteq HK = (0)$.

Since R is right strongly prime, $b = 0$. $\Rightarrow K = (0)$.

Hence R is prime partial semiring. □

Theorem 2.3.6. A partial semiring R is right strongly prime if and only if every nonzero partial ideal of R contains a finitely generated left partial ideal whose right annihilator is (0).

Proof. Suppose R is a right strongly prime partial semiring and I is a nonzero partial ideal of R.

Then $\exists$ a nonzero element $a \in I \ni a$ has a right insulator S_a. $\Rightarrow aS_a \subseteq I$.

Since S_a is finite subset of R, aS_a is also finite.

Let $L = R(aS_a)$, the finitely generated left partial ideal.

Since $aS_a \subseteq I$, $L \subseteq I$.

Let $t \in R \ni Lt = (0)$. Now $(aS_a)t \subseteq R(aS_a)t = Lt = (0)$.

Since S_a is right insulator, $t = 0$.

Thus I contains a finitely generated left partial ideal L such that $L^* = (0)$.

Conversely suppose the condition holds and let $r \in R \ni r \neq 0$.

Then $< r >$ is a nonzero partial ideal of R.

$\Rightarrow \exists$ a finite subset F of $< r > \ni L = RF$ and $L^* = (0)$.

Since $F \subseteq < r >$, every element of F is of the form $\Sigma_i r_i' r r_i$, $r_i', r_i \in R$.

Take $S_r = \{r_1, r_2, ..., r_k \mid \Sigma_{i=1}^{m} r_i' r r_i \in F \text{ for some } m \leq k\}$.

Now we prove that S_r is a right insulator of r.

Let $t \in R \ni rS_rt = (0)$. Then $rr_it = 0$ $\forall i = 1$ to k. $\Rightarrow Lt = RFt = 0$.

Since $L^* = (0)$, $t = 0$. $\Rightarrow (rS_r)^* = (0)$.

Hence S_r is a right insulator for r.

Hence the theorem. □

Theorem 2.3.7. A partial semiring R is right strongly prime if and only if every nonzero partial ideal of S contains a finite subset $G \ni G^* = (0)$.

Proof. Suppose R is a right strongly prime partial semiring and I is a nonzero partial ideal of R.

Then $\exists$ a nonzero element $a \in I \ni a$ has a right insulator S_a.

Let $G = aS_a$. Then G is a finite subset of I.

Let $t \in R \ni Gt = (0)$. Then $(aS_a)t = (0)$. $\Rightarrow t = 0$.

Hence $G^* = (0)$.

Conversely suppose the condition holds and let $a \in R \ni a \neq 0$.

Then $< a >$ contains a finite subset $G \ni G^* = (0)$.

Since $G \subseteq < a >$, every element of G is of the form $\Sigma_i r'_i a r_i$, $r'_i, r_i \in R$.

Suppose $ay = 0$ $\forall 0 \neq y \in R$. Then $Gy \subseteq < a > y = 0$.

$\Rightarrow y \in G^* = (0)$, a contradiction.

Hence $ay \neq 0$ for some $0 \neq y \in R$.

$\Rightarrow \exists$ a finite subset H of $< ay > \ni H^* = (0)$.

Now the elements of H are of the form $\Sigma_i x'_i ayx_i$, $x'_i, x_i \in R$.

Take $H' = \{ay, ayx_1, ..., ayx_k \mid \Sigma_{i=1}^{m} x'_i ayx_i \in H$ for some $m \leq k\}$.

Suppose $(H')^* \neq (0)$. Then $\exists$ $0 \neq x \in (H')^*$.

$\Rightarrow ayx = 0 = ayx_ix$ $\forall i = 1$ to k. $\Rightarrow Hx = (0)$.

$\Rightarrow 0 \neq x \in H^*$, a contradiction.

Hence $(H')^* = (0)$.

Let $F = \{y, yx_1, ..., yx_k\}$.

Since $(H')^* = (0)$, F is a right insulator for a.

Hence the theorem. □

Remark 2.3.8. Every simple partial semiring is right strongly prime.

Proof. Let R be a simple partial semiring.

Then R is the only nonzero partial ideal of R.

Now $G = \{1_R\}$ is a finite subset of $R \ni G^* = (0)$.

Then by theorem 2.3.7, R is right strongly prime partial semiring. □

Definition 2.3.9. A class ρ of partial semirings is said to be *hereditary* if and only if it satisfies the following condition

If I is a partial ideal of a partial semiring R and $R \in \rho$ then $I \in \rho$.

Theorem 2.3.10. The class of all right strongly prime partial semirings is hereditary.

Proof. Let R be a right strongly prime partial semiring, I be a partial ideal of R and a be a nonzero element of I.

Suppose $aI = (0)$.

Then for any finite subset G of R, we have $(0) \neq I \subseteq (aG)^*$ which is a contradiction by theorem 2.3.7.

Hence $\exists\ y \in I \ni ay \neq 0$.

Now by theorem 2.3.7, $\exists$ a finite subset H of $< ay > \ni H^* = (0)$ and

$H = \{ay, ayx_1, ..., ayx_k\}$ for some $x_1, ..., x_k \in R$ (as constructed in theorem 2.3.7).

Hence the subset $F = \{y, yx_1, ..., y_k\}$ of I is a right insulator for $a \in I$.

Hence I is a right strongly prime partial semiring.

Hence the theorem. □

Definition 2.3.11. A finite subset F of a partial semiring R is said to be a *uniform insulator for* R if and only if it is a right insulator for every nonzero element of R.

Definition 2.3.12. A partial semiring R is said to be a *uniformly strongly prime* if it contains a uniform insulator.

Definition 2.3.13. A partial semiring R is said to be a *bounded right strongly prime partial semiring of bound* n, denoted by $SP_r(n)$ if and only if it satisfies the following conditions:

(i). every nonzero element of R has an insulator $S \ni |S| \leq n$

(ii). $\exists$ atleast one nonzero element $r \in R \ni$ every right insulator for r have a cardinality n.

Here n is called the uniform bound of S.

Remark 2.3.14. A partial semi-integral domain is a bounded right strongly prime of bound 1.

Proof. Let R be a partial semi-integral domain and $r \in R \ni r \neq 0$.

Now we prove that $\{r\}$ is a right insulator for r.

Let $t \in R \ni r\{r\}t = 0$. Then $r = 0$ or $t = 0$.

Since $r \neq 0$, $t = 0$ and hence $(r\{r\})^* = (0)$.

Hence the remark. □

Theorem 2.3.15. If R is a prime partial semiring with descending chain condition (DCC) on right annihilators, then R is a right strongly prime partial semiring.

Proof. Let $s \in R \ni s \neq 0$ and $\mathcal{C} = \{(sJ)^* \mid J$ is a finite subset of $R\}$.

Then by DCC, the family $\mathcal{C}$ has a minimal element, say M and let I be the corresponding finite subset of R.

Suppose $M \neq 0$. Then $\exists\ m \in M \ni m \neq 0$.

Since R is prime, $\exists\ q \in R \ni sqm \neq 0$.

Let $I' = I \bigcup \{q\}$ and $M' = (sI')^*$.

Since $sI \subseteq sI'$, $M' = (sI')^* \subseteq (sI)^* = M$.

Since $sqm \neq 0$, $m \notin M'$. Hence $M' \subsetneq M$, a contradiction.

Hence $M = (sI)^* = (0)$. $\Rightarrow I$ is a right insulator for s.

Hence the theorem. □

Theorem 2.3.16. Let R be a partial semi-integral domain and D be a finite set of cardinality n. Then $Mat_D(R)$, the set of all $D \times D$-matrices over R, is bounded right strongly prime partial semiring of bound n. Also, $Mat_D(R)$ has uniform bound n^2.

Proof. Let $A = [a_{ij}]$ be a nonzero element in $Mat_D(R)$.

Then $\exists$ atleast one $a_{ij}, i, j \in D$ is nonzero, let it be a_{pq}.

Now define the matrices E_{ij} by

$$E_{ij}(m,n) = \begin{cases} 1, \ if \ (m,n) = (i,j), \\ 0, \ otherwise. \end{cases}$$

Now we prove that $\{E_{qi} \mid i \in D\}$ is a right insulator for a_{pq}.

Let $B = [b_{ij}] \in Mat_D(R) \ni AE_{qi}B = 0$. Then $a_{pq}b_{ij} = 0\ \forall j \in D$.

$\Rightarrow b_{ij} = 0\ \forall i, j \in D$. $\Rightarrow (a_{pq}\{Eqi \mid i \in D\})^* = (0)$.

$\Rightarrow \{E_{qi} \mid i \in D\}$ is a right insulator for $a_{pq} \ni |\{E_{qi} \mid i \in D\}| \leq n$.

Clearly $E_{11} \in Mat_D(R)$ has an insulator $\{E_{1j} \mid j \in D\} \ni |\{E_{1j} \mid j \in D\}| = n$ and no insulator of E_{11} contains less than n elements.

Hence $Mat_D(R)$ is bounded right strongly prime partial semiring of bound n.

Moreover $\{E_{ij} \mid i, j \in D\}$ is a uniform insulator for every nonzero element of $Mat_D(R)$.

Hence $Mat_D(R)$ has uniform bound n^2. □

Remark 2.3.17. If R is a partial division semiring then $Mat_D(R)$ is a bounded strongly prime with bound n and $Mat_D(R)$ is also uniformly strongly prime of bound n^2.

Remark 2.3.18. If R is right strongly prime partial semiring and e is a nonzero element of R then eRe is also right strongly prime partial semiring.

Proof. First we prove eRe is a partial subsemiring.

Let $(es_ie : i \in I)$ be a summable family in $R \ni es_ie \in eRe, i \in I$.

Then $\Sigma_i es_ie = e(\Sigma_i s_i)e \in eRe$. Now let $es_1e, es_2e \in eRe$.

Then $es_1e \cdot es_2e = e(s_1ees_2)e \in eRe$. Hence eRe is a partial semiring.

Now let $ese \in eRe \ni ese \neq 0$. Then $ese \in R \ni ese \neq 0$.

$\Rightarrow ese$ has a right insulator $\{f_i \mid i = 1 \text{ to } n\}$ in R.

Now we prove $\{ef_ie \mid i = 1 \text{ to } n\}$ is a right insulator for ese in eRe.

Let $ete \in eRe \ni (ese)(ef_ie)(ete) = 0\ \forall i$. Then $[(ese)f_i](ete) = 0\ \forall i$.

$\Rightarrow ete \in \{(ese)f_i \mid i = 1 \text{ to } n\}^* = (0)$. $\Rightarrow ete = 0$.

Hence eRe is right strongly prime partial semiring. □

Remark 2.3.19. If R is a right strongly prime partial semiring and D is a finite set then $Mat_D(R)$ is also a right strongly prime partial semiring.

Proof. Let $B = [b_{ij}]$ be a nonzero element in $Mat_D(R)$.

Then $\exists\ p, q \in D \ni 0 \neq b_{pq} \in R$.

$\Rightarrow \exists$ a right insulator $\{t_k \mid k = 1 \text{ to } n\}$ for b_{pq} in R.

Now we prove $\{t_kE_{ij} \mid i, j \in D, k = 1 \text{ to } n\}$ is a right insulator for B.

Let $A = [a_{ij}] \in Mat_D(R) \ni B(t_kE_{qj})A = 0\ \forall k$.

Then $b_{pq}t_ka_{ij} = 0\ \forall k$. $a_{ij} = 0\ \forall i, j \in D$. $\Rightarrow A = 0$.

Hence $Mat_D(R)$ is a right strongly prime partial semiring. □

Definition 2.3.20. A nonzero partial ideal of a partial semiring R is said to be an *essential partial ideal* of R if and only if for any nonzero partial ideal J of R, $I \bigcap J \neq (0)$.

Definition 2.3.21. A partial ideal I of a partial semiring R is said to be *singular partial ideal* if and only if the right annihilator of every element in I is an essential right partial ideal.

Theorem 2.3.22. The singular partial ideal of a right strongly prime partial semiring is (0).

Proof. Let R be a right strongly prime partial semiring and I be a nonzero partial ideal of R.

Suppose $\exists$ a nonzero element s of I.

Then s has a right insulator $\{s_i \mid i = 1 \text{ to } k\}$.

Since I is a partial ideal, $ss_i \in I \ \forall i$.

$\Rightarrow (ss_i)^*$ is an essential right partial ideal of $R \ \forall i$.

$\Rightarrow (ss_i)[\bigcap_{i=1}^{k}(ss_i)^*] \subseteq (ss_i)(ss_i)^* = (0) \ \forall i$.

$\Rightarrow \bigcap_{i=1}^{k}(ss_i)^* = (0)$, a contradiction to $(ss_i)^*$ an essential partial ideal of R.

Hence $I = 0$. □

Definition 2.3.23. A partial ideal I of a partial semiring R is said to be *right strongly prime* if and only if it satisfies the following condition:

if $a \notin I$, then there is a finite set $F \subseteq < a > \ni Fb \subseteq I$ implies $b \in I$.

Definition 2.3.24. Let I be a proper partial ideal of a partial semiring R. Then the relation on R, denoted by ρ_I and defined by $s\rho_I s'$ if and only if $s + a = s' + a'$ for some $a, a' \in I$, is called the *Bourne relation* on R defined by I.

Remark 2.3.25. The Bourne relation ρ_I defined on a complete partial semiring R is a congruence relation on R.

We denote the Bourne congruence ρ_I class of an element r of R by r/I and denote the set of all such congruence classes of R by R/I.

It may be noted that for any $s \in R$ and for any proper partial ideal of a complete partial semiring R, s/I is not necessarily equal to $s + I = \{s + a \mid a \in R\}$ but surely contains it.

Definition 2.3.26. Let I be a proper partial ideal of a complete partial semiring R. Then the set R/I is a complete partial semiring with the summation and multiplication defined on R/I by $\Sigma_i(a_i/I) = (\Sigma_i a_i)/I$ and $(a/I)(b/I) = (ab)/I$ for all $a_i, a, b \in R$, called the *Bourne factor partial semiring* or simply the *factor partial semiring.*

Theorem 2.3.27. A proper partial ideal I of a complete partial semiring R is a right strongly prime if and only if R/I is a right strongly prime partial semiring.

Proof. Suppose I is a right strongly prime partial ideal of R and let P/I be any nonzero partial ideal of R/I.

Then $\exists\ a/I \in P/I \ni a/I \neq 0/I.$ $\Rightarrow a \notin I.$

$\Rightarrow \exists$ a finite subset F of $< a > \ni Fb \subseteq I$ implies $b \in I$.

$\Rightarrow F/I$ is a finite subset of P/I.

Now let $b/I \in R/I \ni (F/I)(b/I) = (0/I)$.

$\Rightarrow Fb \subseteq I.$ $\Rightarrow b \in I.$

$\Rightarrow b/I = 0/I$ and hence $(F/I)^* = (0/I)$.

Then by theorem 2.3.7, R/I is a right strongly prime partial semiring.

Conversely suppose R/I is a right strongly prime partial semiring and let $a \notin I$.

Then $a/I \in R/I \ni a/I \neq 0/I.$ $\Rightarrow < a > /I$ is a nonzero partial ideal of R/I.

$\Rightarrow$ a finite subset F/I of $< a > /I \ni (F/I)^* = (0/I)$.

Let $F/I = \{f_1/I, ..., f_k/I\}$ and $F' = \{f_1, ..., f_k\}$.

Then F' is a finite subset of $< a >$. Let $b \in R \ni F'b \subseteq I$.

$\Rightarrow (F'/I)(b/I) = (0/I)$. $\Rightarrow (F/I)(b/I) = (0/I)$.

$\Rightarrow b/I = 0/I$ and hence $b \in I$.

Hence I is a right strongly prime partial ideal of R. $\square$

Definition 2.3.28. A subset G of a partial semiring R is said to be an *sp-system* if and only if for any $g \in G$, $\exists$ a finite subset F of $< g > \ni Fz \bigcap G \neq \emptyset$ for all $z \in G$.

Theorem 2.3.29. A partial ideal I of a partial semiring R is right strongly prime if and only if $R \setminus I$ is an *sp*-system.

Proof. Suppose I is a right strongly prime partial ideal of R and let $g \in R \setminus I$.

Then $g \notin I$. $\Rightarrow \exists$ a finite subset F of $< g > \ni Fb \subseteq I$ implies $b \in I$.

i.e., $Fz \bigcap (R \setminus I) \neq \emptyset \ \forall z \in R \setminus I$.

Hence $R \setminus I$ is an *sp*-system.

Conversely suppose $R \setminus I$ is an *sp*-system and let $a \notin I$.

Then $a \in R \setminus I$. $\exists$ a finite subset F of $< a > \ni Fz \bigcap (R \setminus I) \neq \emptyset \ \forall z \in R \setminus I$.

Suppose $Fb \subseteq I$ for any $b \in R$. Then $Fb \bigcap (R \setminus I) = \emptyset$.

Suppose $b \notin I$. Then $b \in R \setminus I$.

$\Rightarrow Fb \bigcap (R \setminus I) \neq \emptyset$, a contradiction. Hence $b \in I$.

Hence I is a right strongly prime partial semiring. $\square$

Definition 2.3.30. Let R be a partial semiring. Then the *right strongly prime radical* of R is defined by $SP(R) = \bigcap \{I \mid I$ is a right strongly prime partial ideal of $R\}$.

Definition 2.3.31. Let R be a partial semiring, P be a partial ideal of R and G be any subset of R. Then the pair (G, P) is said to be a *super sp-system* of R if and only if the following conditions are hold (i). $G \bigcap P \subseteq (0)$

(ii). for any $g \in G$, there is a finite subset F of $< g > \ni Fz \bigcap G \neq \emptyset \ \forall z \notin P$.

Remark 2.3.32. A partial ideal I of a partial semiring R is right strongly prime if and only if the pair $(R \setminus I, I)$ is a super *sp*-system of R.

Theorem 2.3.33. For any partial semiring R, $SP(R) = \{x \in R \mid (G, P)$ is a super *sp*-system $\ni x \in G \Rightarrow 0 \in G\}$.

Proof. Let $T = \{x \in R \mid (G, P)$ is a super *sp*-system $\ni x \in G \Rightarrow 0 \in G\}$

and let $x \in SP(R)$.

Suppose $x \in G$, (G, P) is a super *sp*-system and $0 \notin G$. Then $G \bigcap P = \emptyset$.

Take $\mathcal{C} = \{I \mid I$ is a partial ideal of $R, P \subseteq I$ & $G \bigcap I = \emptyset\}$. Clearly $P \in \mathcal{C}$

Then by Zorn's lemma, $\mathcal{C}$ has a maximal element. Let it be Q.

Now we prove that Q is a right strongly prime partial ideal of R.

Let $a \notin Q$. Then $\exists\ g \in G \ni < g > \subseteq Q + < a >$.

Since (G, P) is a super *sp*-system, $\exists$ a finite subset $F = \{f_1, ..., f_k\} \subseteq < g >$

$\ni Fz \bigcap G \neq \emptyset \ \forall z \notin P$.

$\Rightarrow F \subseteq Q + < a >$.

$\Rightarrow$ every element f_i is of the form $f_i = q_i + a_i$ for some $q_i \in Q$ and $a_i \in < a >$.

Let $F' = \{a_1, ..., a_k\}$. Then $F' \subseteq < a >$.

Let $z \in R \ni F'z \subseteq Q$. Then $Fz \subseteq Q$.

Suppose $z \notin Q$. Then $Fz \bigcap G \neq \emptyset$, a contradiction to $G \bigcap Q = \emptyset$.

Hence $z \in Q$. $\Rightarrow Q$ is a right strongly prime partial ideal of R.

Since $x \in G$ and $G \bigcap Q = \emptyset$, we have $x \notin Q$, a contradiction to $x \in SP(R)$.

Hence $0 \in G$ and hence $x \in T$.

Conversely let $x \in T$.

Suppose $x \notin SP(R)$. Then $\exists$ a right strongly prime partial ideal I of $R \ni x \notin I$.

$\Rightarrow (R \setminus I, I)$ is a super *sp*-system and $x \in R \setminus I$.

$\Rightarrow 0 \in R \setminus I$, a contradiction. Hence $x \in SP(R)$.

Hence the theorem. $\square$