Chapter 1

Φ-REPRESENTATION OF SO-RINGS

$1.1. **Congruence relations**

In this section we obtain a necessary and sufficient condition for R/θ to be a so-ring. We begin with the following.

Definition 1.1.1. Let $((R^i, \Sigma^i, \cdot^i) : i \in I)$ be a family of partial semirings. Then the partial semiring $(\prod R^i, \Sigma, \cdot)$ is called the *product partial semiring* where $\prod R^i$ is the cartesian product of the R^is, Σ and '$\cdot$' are defined as follows:

Let $(x_j : j \in J)$ be a family in $\prod R^i$. Then each $x_j = (x_j^i : i \in I)$, where $x_j^i \in R^i$. We say the family $(x_j : j \in J)$ is summable in $\prod R^i$ and we write $\Sigma_j(x_j : j \in J) = (\Sigma_j(x_j^i : j \in J) : i \in I)$ if for each $i \in I$, $(x_j^i : j \in J)$ is summable in R^i.

For any $(x_i : i \in I)$, $(y_i : i \in I)$ in $\prod R^i$, $(x_i : i \in I)\cdot(y_i : i \in I) = (x_i \cdot^i y_i : i \in I)$.

Definition 1.1.2. Let R be a partial semiring and θ be an equivalence relation on R. Then θ is said to be a *partial semiring congruence* on R if and only if θ is closed under the additive and multiplicative operations of the product partial semiring $R \times R$.

Definition 1.1.3. Let $(R, \Sigma, \cdot)$ be a partial semiring and θ be a partial semiring congruence on R. Then their *quotient* is the structure $(R/\theta, \Sigma', \cdot)$ where
$R/\theta = \{\overline{x} \mid x \in R\}$ ($\overline{x}$ is the equivalence class containing x with respect to θ), Σ', '$\cdot$' are defined as follows:

The family $(\overline{x_i} : i \in I)$ is summable in R/θ if and only if the family $(x_i : i \in I)$ is summable in R and we write $\Sigma'\overline{x_i} = \overline{\Sigma x_i}$.

We define $\overline{x} \cdot \overline{y} = \overline{x \cdot y}$ for any $\overline{x_i}, \overline{x}, \overline{y} \in R/\theta$.

The following example shows that R/θ need not be a partial semiring.

Example 1.1.4. We know that $(P(D), \Sigma, \cdot)$ is a partial semiring, where

$$\Sigma A_i = \begin{cases} \bigcup A_i, \ if \ A_i \bigcap A_j = \phi \ \forall i \neq j \\ undefined, \ otherwise \end{cases}$$

and $A \cdot B = A \bigcap B$. Take $D = \{x, y\}$. Then $\theta = \{(\emptyset, \emptyset), (\{x\}, \{x\}), (\{y\}, \{y\}), (D, D), (\{x\}, D), (D, \{x\}), (\Phi, \{y\}), (\{y\}, \Phi)\}$ is partial semiring congruence on $P(D)$. Now $P(D)/\theta = \{\overline{\Phi}, \overline{\{x\}}\}$, where $\overline{\Phi} = \{\Phi, \{y\}\} = \overline{\{y\}}$, $\overline{\{x\}} = \{\{x\}, D\} = \overline{D}$. Here $\{x\} + \{y\}$ is defined but $\{x, y\} + \{y\}$ is not defined, hence $\overline{\{x\}} + \overline{\{y\}}$ is not well defined. Hence $P(D)/\theta$ is not a partial semiring.

Remark 1.1.5. Let θ be a partial semiring congruence on a partial semiring R. Then a necessary and sufficient condition for R/θ to be partial semiring is that:

$(y_i : i \in I)$ is summable whenever $(x_i : i \in I)$ is summable and $x_i \theta y_i$, $i \in I$.

Proof. Suppose R/θ is partial-semiring, $(x_i : i \in I)$ is summable and $x_i \theta y_i$, $i \in I$.

Then $(\overline{x_i} : i \in I)$ is summable and $\overline{x_i} = \overline{y_i}$, $i \in I$.

$\Rightarrow (\overline{y_i} : i \in I)$ is summable.

Hence $(y_i : i \in I)$ is summable.

Conversely suppose the condition is true and let $(\overline{x_i} : i \in I)$, $(\overline{y_i} : i \in I)$ in R/θ such that $(\overline{x_i} : i \in I)$ is summable and $\overline{x_i} = \overline{y_i}$, $i \in I$.

Then $(x_i : i \in I)$ is summable and $x_i \theta y_i$, $i \in I$.

By condition, we get $(y_i : i \in I)$ is summable.

$\Rightarrow (\overline{y_i} : i \in I)$ is summable and $\Sigma' \overline{x_i} = \Sigma' \overline{y_i}$.

Hence R/θ is a partial semiring. □

The following is an example of a so-ring R in which θ is a partial semiring congruence on R such that R/θ is a partial semiring but not a so-ring.

Example 1.1.6. Let R be the set of all real numbers with finite support addition and ordinary multiplication. Take $M = R \times R$ and define Σ on M as componentwise addition, $*$ as $(a, b) * (c, d) = (ac + bd, ad + bc)$. Then $(M, \Sigma, *)$ is a so-ring. Define θ on M as $(a, b)\theta(c, d)$ if and only if $a + d = b + c$.

Now we prove that θ is a partial semiring congruence on M: For any $a, b \in R$, $(a, b)\theta(a, b)$. Let $(a, b)\theta(c, d)$. Then $a + d = b + c$. $\Rightarrow c + b = d + a$ and hence $(c, d)\theta(a, b)$. Let $(a, b)\theta(c, d)$ and $(c, d)\theta(e, f)$. Then $a + d = b + c$ and $c + f = d + e$. $\Rightarrow a + d + c + f = b + c + d + e$. $\Rightarrow a + f = b + e$ and hence $(a, b)\theta(e, f)$. Hence θ is an equivalence relation on $M = R \times R$.

Let $((a_i, b_i) : i \in I)$, $((c_i, d_i) : i \in I)$ be summable families such that $(a_i, b_i)\theta(c_i, d_i)$, $i \in I$. Then $a_i + d_i = b_i + c_i$, $i \in I$. $\Sigma(a_i + d_i) = \Sigma(b_i + c_i)$. $\Rightarrow \Sigma a_i + \Sigma d_i = \Sigma b_i + \Sigma c_i$. $\Rightarrow (\Sigma a_i, \Sigma b_i)\theta(\Sigma c_i, \Sigma d_i)$ and hence $[\Sigma(a_i, b_i)]\theta[\Sigma(c_i, d_i)]$. Let $(a, b)\theta(c, d)$ and $(e, f)\theta(g, f)$. Then $a + d = b + c$ and $e + h = f + g$. Consider $ae+bf+ch+dg = be+ce-de+af+df-cf+cf+cg-ce+de+dh-df = be+af+cg+dh$. $\Rightarrow (ae + bf, af + be)\theta(cg + dh, ch + dg)$ and hence $[(a, b) * (e, f)]\theta[(c, d) * (g, h)]$. Hence θ is a partial semiring congruence on M.

Clearly M/θ is a partial semiring. Since $\overline{(1, 1)} = \overline{(2, 2)} = \overline{(1, 2)} + \overline{(1, 0)}$ and $\overline{(1, 2)} = \overline{(1, 1)} + \overline{(0, 1)}$, we have $\overline{(1, 2)} \leq \overline{(1, 1)}$ and $\overline{(1, 1)} \leq \overline{(1, 2)}$ but $\overline{(1, 1)} \neq \overline{(1, 2)}$. Hence M/θ is not a so-ring.

Definition 1.1.7. A partial semiring congruence θ on a so-ring R is said to have the *diagonal property* if it satisfies the following:

$a\theta(b + k)$ and $b\theta(a + h) \Leftrightarrow a\theta b \; \forall a, b, h, k \in R$.

Theorem 1.1.8. Let θ be a partial semiring congruence on a so-ring R such that R/θ is a partial semiring. Then R/θ is a so-ring if and only if θ has the diagonal property.

Proof. Suppose R/θ is a so-ring. Then

$x\theta(y + k)$ & $(x + h)\theta y \Leftrightarrow \bar{x} = \bar{y} + \bar{k}$ & $\bar{x} + \bar{h} = \bar{y} \Leftrightarrow \bar{y} \leq \bar{x}$ & $\bar{x} \leq \bar{y} \Leftrightarrow \bar{x} = \bar{y} \Leftrightarrow x\theta y$.

Hence θ has the diagonal property.

Conversely suppose θ has the diagonal property. Then

$\bar{x} \leq \bar{y}$ & $\bar{y} \leq \bar{x} \Leftrightarrow \bar{x} + \bar{h} = \bar{y}$ & $\bar{y} + \bar{k} = \bar{x}$ for some $h, k \in R \Leftrightarrow (x + h)\theta y$ & $x\theta(y + k)$

$\Leftrightarrow x\theta y \Leftrightarrow \bar{x} = \bar{y}$. Hence R/θ is a so-ring. □

Definition 1.1.9. A partial semiring congruence θ on a so-ring R is said to be a *congruence relation* on R if and only if θ has the diagonal property.

Remark 1.1.10. $ConR$, the set of all congruence relations on a so-ring R forms a complete lattice with 0_R and 1_R, the smallest and largest congruences, respectively.

Definition 1.1.11. Let $(R, \Sigma, \cdot)$, $(R_1, \Sigma_1, *)$ be partial semirings. Then a mapping $f : R \to R_1$ is said to be a *homomorphism* if it satisfies the following

(i). whenever $(x_i : i \in I)$ is summable in R, then $(f(x_i) : i \in I)$ is summable in R_1 and $f(\Sigma x_i) = \Sigma_1 f(x_i)$

(ii). $f(x \cdot y) = f(x) * f(y)$ for all x, y in R.

Remark 1.1.12. If $f : R \to R_1$ is a homomorphism then $f(0_R) = 0_{R_1}$, $f(1_R) = 1_{R_1}$.

Definition 1.1.13. The homomorphism $p_i : (\prod R^i, \Sigma, \cdot) \to (R^i, \Sigma^i, \cdot^i)$ defined by $x \mapsto x^i$ is called the *i-th projection map.*

$1.2. **Structure theorem**

In this section we show that any so-ring is a subdirect product of subdirectly irreducible so-rings.

Definition 1.2.1. Let $\{R_i \mid i \in I\}$ be a family of so-rings. Then a subset $R \subset \prod_{i\in I} R_i$ is said to be a *subdirect product of so-rings* R_i, $i \in I$ if it satisfies the following

(i). R is subso-ring of $\prod_{i\in I} R_i$

(ii). all the projections of $\prod_{i\in I} R_i$ are onto.

Definition 1.2.2. A so-ring R is said to be *subdirectly irreducible* if and only if

$$\bigcap_{i\in I} \theta_i = 0_R \Rightarrow \theta_i = 0_R \text{ for some } i \in I.$$

Lemma 1.2.3. Let R be a so-ring and θ be a congruence on R. Then there is a one to one correspondence between congruence relations of R containing θ and congruence relations of R/θ.

Proof. Let ϕ be a congruence of R containing θ.

Define a relation ϕ/θ on R/θ by $[a]^\theta(\phi/\theta)[b]^\theta$ if and only if $a\phi b$, where $[a]^\theta$ denotes the equivalence class containing a relative to θ.

Since ϕ is congruence on R, ϕ/θ is an equivalence relation on R/θ.

Now we prove ϕ/θ is congruence relation on R/θ:

Let $([a_i]^\theta : i \in I)$ and $([b_i]^\theta : i \in I)$ be summable families in R/θ such that $[a_i]^\theta(\phi/\theta)[b_i]^\theta$, $i \in I$.

Then $(a_i : i \in I)$ and $(b_i : i \in I)$ are summable families in R such that $a_i\phi b_i$, $i \in I$.

$\Rightarrow (\Sigma a_i)\phi(\Sigma b_i)$. $\Rightarrow [\Sigma a_i]^\theta(\phi/\theta)[\Sigma b_i]^\theta$. Hence $\overline{\Sigma}[a_i]^\theta(\phi/\theta)\overline{\Sigma}[b_i]^\theta$.

Let $[a]^\theta(\phi/\theta)[b]^\theta$ and $[c]^\theta(\phi/\theta)[d]^\theta$. Then $a\phi b$ and $c\phi d$. $\Rightarrow (ac)\phi(bd)$.

$\Rightarrow ([a]^\theta[c]^\theta)(\phi/\theta)([b]^\theta[d]^\theta)$.

Hence ϕ/θ is a partial semiring congruence on R/θ.

Now $[a]^\theta(\phi/\theta)([b]^\theta + [h]^\theta)$ and $([a]^\theta + [k]^\theta)(\phi/\theta)[b]^\theta$ for some $h, k \in R$

$\Leftrightarrow a\phi(b + h)$ and $(a + k)\phi b \Leftrightarrow a\phi b \Leftrightarrow [a]^\theta(\phi/\theta)[b]^\theta$.

Hence ϕ/θ is a congruence relation on R/θ.

Let ϕ' be a congruence relation on R/θ.

Define a relation ϕ_θ on R by $a\phi_\theta b$ if and only if $[a]^\theta\phi'[b]^\theta$.

Since ϕ' is congruence on R/θ, ϕ_θ is an equivalence relation on R.

Now we prove ϕ_θ is a congruence relation on R containing θ:

Let $(a_i : i \in I)$ and $(b_i : i \in I)$ be summable families in R such that $a_i\phi_\theta b_i$, $i \in I$.

Then $([a_i]^\theta : i \in I)$ and $([b_i]^\theta : i \in I)$ are summable families in R/θ such that $[a_i]^\theta\phi'[b_i]^\theta$, $i \in I$. $\Rightarrow \overline{\Sigma}[a_i]^\theta\phi'\overline{\Sigma}[b_i]^\theta$ and hence $(\Sigma a_i)\phi_\theta(\Sigma b_i)$.

Let $a\phi_\theta b$ and $c\phi_\theta d$. Then $[a]^\theta\phi'[b]^\theta$ and $[c]^\theta\phi'[d]^\theta$.

$\Rightarrow [ac]^\theta\phi'[bd]^\theta$ and hence $(ac)\phi_\theta(bd)$.

Hence ϕ_θ is partial semiring congruence on R.

Now $a\phi_\theta(b+h)$ and $(a+k)\phi_\theta b$ for some $h, k \in R \Leftrightarrow [a]^\theta\phi'([b]^\theta + [h]^\theta)$ and $([a]^\theta + [k]^\theta)\phi'[b]^\theta \Leftrightarrow [a]^\theta\phi'[b]^\theta \Leftrightarrow a\phi_\theta b$.

Let $a\theta b$. Then $[a]^\theta = [b]^\theta$. $\Rightarrow [a]^\theta\phi'[b]^\theta$. $\Rightarrow a\phi_\theta b$. Hence $\theta \subseteq \phi_\theta$.

Hence ϕ_θ is a congruence relation on R containing θ.

Let ϕ be a congruence relation of R containing θ.

Then ϕ/θ is a congruence relation on R/θ.

$\Rightarrow (\phi/\theta)_\theta$ is a congruence relation on R containing θ.

Note that $a\phi b \Leftrightarrow [a]^\theta(\phi/\theta)[b]^\theta \Leftrightarrow a(\phi/\theta)_\theta b$.

Hence $(\phi/\theta)_\theta = \phi$.

Let ϕ' be a congruence relation on R/θ.

Then ϕ_θ is a congruence relation of R containing θ.

$\Rightarrow (\phi_\theta)/\theta$ is a congruence relation of R/θ.

Note that $[a]^\theta\phi'[b]^\theta \Leftrightarrow a\phi_\theta b \Leftrightarrow [a]^\theta(\phi_\theta)/\theta[b]^\theta$.

Hence $(\phi_\theta)/\theta = \phi'$.

Hence the lemma. $\square$

Lemma 1.2.4. Let $\{\theta_i \mid i \in I\}$ be a family of congruences of a so-ring R such that $\bigcap_{i\in I}\theta_i = 0_R$. Then R is isomorphic to a subdirect product of so-rings R/θ_i, $i \in I$.

Proof. Define a mapping $f : R \to \prod_{i\in I} R/\theta_i$ by $f(a) = f_a\ \forall a \in R$, where $f_a \in \prod_{i\in I} R/\theta_i$ and $if_a = [a]^{\theta_i}$, $i \in I$. Take $M = \{f_a \mid a \in R\}$.

Now we prove f is a one-one homomorphism.

Let $(x_j : j \in J)$ be a summable family in R.

Then for any $i \in I$, $if_{(\Sigma x_j)} = [\Sigma_j x_j]^{\theta_i} = \overline{\Sigma_j}[x_j]^{\theta_i} = i(\overline{\Sigma_j} f_{x_j})$.

Hence $f(\Sigma_j x_j) = \overline{\Sigma_j} f(x_j)$.

For any $x, y \in R$ and for any $i \in I$, $if_{xy} = [xy]^{\theta_i} = [x]^{\theta_i}[y]^{\theta_i} = if_x \cdot if_y = i(f_x f_y)$.

Hence $f(xy) = f(x)f(y)$.

Let $x, y \in R$ such that $f(x) = f(y)$.

Then $if_x = if_y \ \forall i \in I$. $\Rightarrow [x]^{\theta_i} = [y]^{\theta_i} \ \forall i \in I$. $\Rightarrow x(\bigcap_{i \in I} \theta_i)y$. $\Rightarrow x = y$.

Hence $f : R \to \prod_{i \in I} R/\theta_i$ is a one-one homomorphism.

Hence $f : R \to M$ is an isomorphism.

Now we prove M is subdirect product of R/θ_i, $i \in I$:

It can be noted that M is subso-ring of $\prod_{i \in I} R/\theta_i$.

Now $p_i(M) = \{p_i(f_x) \mid f_x \in M\} = \{if_x \mid x \in R\} = \{[x]^{\theta_i} \mid x \in R\} = R/\theta_i$, $i \in I$.

Hence M is a subdirect product of so-rings R/θ_i, $i \in I$.

Hence the lemma. □

Lemma 1.2.5. Let $\{\theta_i \mid i \in I\}$ be a simply ordered family of congruences of a so-ring R. Then $\bigcup(\theta_i : i \in I) = \bigvee(\theta_i : i \in I)$.

Proof. We know that $\bigvee(\theta_i : i \in I) = \bigcap\{\theta' \mid \theta' \in \mathrm{Con}R, \theta' \supseteq \bigcup \theta_i\}$.

$\Rightarrow \bigcup(\theta_i : i \in I) \subseteq \bigvee(\theta_i : i \in I)$.

Let $a(\bigvee \theta_i)b$. Then $\exists\ a = z_0, ..., z_{n-1}, z_n = b$ and $\theta_1, ..., \theta_{n-1}, \theta_n \in \{\theta_i \mid i \in I\}$ such that $z_i(\theta_{i+1})z_{i+1}$ for all $0 \le i < n$.

Since $\{\theta_i \mid i \in I\}$ is simply ordered, $\exists\ \theta \in \{\theta_i \mid i \in I\} \ni \theta_i \subseteq \theta \ \forall 1 \le i \le n$.

$\Rightarrow z_i \theta z_{i+1}$ for all $0 \le i < n$ and $\theta \in \mathrm{Con}R$.

$\Rightarrow z_0 \theta z_n$. $\Rightarrow a\theta b$. $\Rightarrow a(\bigcup \theta_i)b$.

Hence $\bigvee(\theta_i \mid i \in I) \subseteq \bigcup(\theta_i \mid i \in I)$.

Hence the lemma. □

Lemma 1.2.6. Let R be a so-ring and $a \neq b$. Then there is a congruence relation $\theta(a,b)$ on R such that $a \not\equiv b(\theta(a,b))$($a$ is not congruent to b under $\theta(a,b)$) and $\theta(a,b)$ is maximal with respect to this property.

Proof. For any $a, b \in R$ such that $a \neq b$, define $\mathcal{C} = \{\phi \in ConR \mid a \not\equiv b(\phi)\}$.

Since $a \not\equiv b(0_R)$, we have $0_R \in \mathcal{C}$.

Let $\{\phi_i \mid i \in I\}$ be a simply ordered family in $\mathcal{C}$ and take $\theta = \bigvee(\phi_i : i \in I)$.

Then by lemma 1.2.5, $\theta = \bigvee(\phi_i : i \in I) = \bigcup(\phi_i : i \in I)$.

Since $a \not\equiv b(\phi_i)$ $\forall i \in I$, we have $a \not\equiv b(\theta)$.

$\Rightarrow \theta = \bigcup(\phi_i : i \in I)$ is an upper bound of $\{\phi_i \mid i \in I\}$ in $\mathcal{C}$.

Hence by Zorn's lemma, $\mathcal{C}$ has a maximal element, say $\theta(a,b)$.

Hence the lemma. □

Lemma 1.2.7. Let R be a so-ring. Then R is subdirectly irreducible if and only if $ConR$ has one and only one atom which is contained in every congruence relation other than 0_R, the zero congruence.

Proof. Suppose R is subdirectly irreducible and Con$R$ has more than one atom.

Then Con$R \setminus \{0_R\}$ has no least element. $\Rightarrow \bigcap(\text{Con}R \setminus \{0_R\}) = 0_R$.

$\Rightarrow \theta_i = 0_R$ for some $i \in I$, where $\theta_i \in$Con$R \setminus \{0_R\}$, a contradiction.

Hence Con$R$ has one and only one atom which is contained in every congruence relation other than 0_R.

Conversely suppose that Con$R$ has one and only one atom which is contained in every congruence relation other than 0_R.

Let $\{\theta_i \mid i \in I\}$ be a family of congruences on $R \ni \bigcap \theta_i = 0_R$.

Suppose $\theta_i \neq 0_R$ for every $i \in I$. Then $\bigcap \theta_i = \theta'$ for some $\theta' \neq 0_R$, a contradiction.

Hence A is subdirectly irreducible. □

Theorem 1.2.8. Any so-ring is a subdirect product of subdirectly irreducible so-rings.

Proof. Let R be a so-ring. Consider the family of congruences $C = \{\theta(a,b) \mid a \neq b\}$ as constructed in the lemma 1.2.6.

Suppose $x \equiv y(\bigcap_{a \neq b} \theta(a,b))$. Then $x \equiv y(\theta(a,b))\ \forall a \neq b$.

Suppose $x \neq y$. Then $x \equiv y(\theta(x,y))$, a contradiction.

Hence $\bigcap_{a \neq b} \theta(a,b) = 0_R$.

By lemma 1.2.4, R is isomorphic to a subdirect product of so-rings $R/\theta(a,b), a \neq b$.

Now we prove $R/\theta(a,b), a \neq b$ is subdirectly irreducible.

Let $[\theta(a,b))$ denote the set of congruences of R containing $\theta(a,b)$.

Denote $\psi(a,b)$ be the smallest congruence such that $a \equiv b$.

Then $\theta(a,b) \subseteq \psi(a,b) \vee \theta(a,b)$.

Suppose $\theta(a,b) = \psi(a,b) \vee \theta(a,b)$. Then $\psi(a,b) \subseteq \theta(a,b)$.

$\Rightarrow a \equiv b(\theta(a,b))$ for $a \neq b$, a contradiction.

Hence $\theta(a,b) \subset \psi(a,b) \vee \theta(a,b)$.

$\Rightarrow \psi(a,b) \vee \theta(a,b) \in [\theta(a,b))$ and $\psi(a,b) \vee \theta(a,b) \neq \theta(a,b)$.

Let ϕ be another congruence in $[\theta(a,b))$ other than $\theta(a,b)$.

Then $a \equiv b(\phi)$. $\Rightarrow \psi(a,b) \subseteq \phi$. Hence $\psi(a,b) \vee \theta(a,b) \subseteq \phi$.

Therefore $\psi(a,b) \vee \theta(a,b)$ is the only atom which is contained in every congruence in $[\theta(a,b))$ other than $\theta(a,b)$.

By lemma 1.2.3, there is a one to one correspondence between congruences of $R/\theta(a,b)$, $a \neq b$ and $[\theta(a,b))$.

$\Rightarrow Con(R/\theta(a,b))$ has one and only one atom which is contained in every congruence other than $\overline{\theta(a,b)}$ (the zero congruence of $R/\theta(a,b)$).

By lemma 1.2.7, $R/\theta(a,b)$, $a \neq b$ is subdirectly irreducible.

Hence the theorem. □

$1.3. **Φ-representation of so-rings**

Walendziak [35] studied the Φ-representation of algebras as a common generalization of subdirect and direct product of algebras. In this section we show that $< (R_i : i \in I), f >$ is a ϕ-representation of R if and only if $0_R = \prod_\phi(\theta_i : i \in I)$ where $R_i = R/\theta_i, i \in I$.

Definition 1.3.1. Let $\{R_i \mid i \in I\}$ be a family of so-rings and R be a subset of $\prod_{i \in I} R_i$, and let $\phi \in ConR$. Then R is said to be a *ϕ-product* of the so-rings $R_i, i \in I$ if it satisfies the following

(i). R is a subdirect product of the so-rings $R_i, i \in I$

(ii). for every $\overline{x} = (x_i : i \in I) \in A^I$, if $(x_i, x_j) \in \phi\ \forall i, j \in I$ then

$< x_i(i) : i \in I >\in A$.

Example 1.3.2. Take $R_1 = \{0, 1\}$. Define Σ as

$$\Sigma x_i = \begin{cases} x_j,\ if\ x_i = 0\ \forall i \neq j\ for\ some\ j \\ 1,\ if\ x_h = 1,\ x_k = 1\ for\ some\ h, k,\ x_i = 0\ \forall i \neq h, k \\ undefined,\ otherwise. \end{cases}$$

and '$\cdot$' as $0 \cdot 0 = 1 \cdot 0 = 0 \cdot 1 = 0, 1 \cdot 1 = 1$. Then $(R_1, \Sigma, \cdot)$ is a so-ring.

Take $R_2 = \{0, a, 1\}$. Define Σ as

$$\Sigma x_i = \begin{cases} x_j,\ if\ x_i = 0\ \forall i \neq j\ for\ some\ j \\ a,\ if\ x_h = x_k = a\ for\ some\ h, k,\ x_i = 0\ \forall i \neq h, k \\ 1,\ if\ x_h = 1,\ x_k = a\ or\ 1,\ for\ some h, k,\ x_i = 0\ \forall i \neq h, k \\ undefined,\ otherwise. \end{cases}$$

and '$\cdot$' as $0 \cdot 0 = 1 \cdot 0 = 0 \cdot 1 = a \cdot 0 = 0 \cdot a = 0$, $a \cdot a = a \cdot 1 = 1 \cdot a = a, 1 \cdot 1 = 1$. Then $(R_2, \Sigma, \cdot)$ is a so-ring. Also $R_1 \times R_2$ is a so-ring. Take $R = \{< 0, 0 >, < 1, 0 >, < 1, a >,$

$< 1, 1 >\}$ and $\alpha = \{(< 0, 0 >, < 0, 0 >), (< 1, 0 >, < 1, 0 >), (< 1, a >, < 1, a >),$
$(< 1, 1 >, < 1, 1 >), (< 0, 0 >, < 1, 0 >), (< 1, 0 >, < 0, 0 >)\}$. Then α is a congruence on R. Now for every $\overline{x} = (x_1, x_2) \in R^2$, if $(x_1, x_1), (x_1, x_2), (x_2, x_1), (x_2, x_2) \in \alpha$ then $< x_1(1), x_2(2) > \in R$. Also $p_1(R) = R_1, p_2(R) = R_2$. Hence R is an α-product of R_1, R_2.

Now take $\beta = \{(< 0, 0 >, < 0, 0 >), (< 1, 0 >, < 1, 0 >), (< 1, a >, < 1, a >),$
$(< 1, 1 >, < 1, 1 >), (< 0, 0 >, < 1, 0 >), (< 1, 0 >, < 0, 0 >), (< 0, 0 >, < 1, a >),$
$(< 1, a >, < 0, 0 >), (< 1, 0 >, < 1, a >), (< 1, a >, < 1, 0 >)\}$ a congruence on R and take $\overline{x} = (< 0, 0 >, < 1, a >) \in R^2$. Then $(< 0, 0 >, < 0, 0 >), (< 0, 0 >, < 1, a >),$
$(< 1, a >, < 0, 0 >), (< 1, a >, < 1, a >) \in \beta$. But $<< 0, 0 > (1), < 1, a > (2) >$
$= < 0, a > \notin R$, thus R is not a β-product of R_1, R_2.

Remark 1.3.3. Let R and $R_i (i \in I)$ be so-rings. Then

(i). R is a subdirect product of R_i if and only if R is a 0_R-product of $R_i, i \in I$.

(ii). R is a 1_R-product of $R_i, i \in I$ if and only if $R = \prod_{i \in I} R_i$.

Proof.

(i). Suppose R is a 0_R-product of so-rings R_i, $i \in I$. Then R is a subdirect product of so-rings R_i, $i \in I$.

Conversely suppose R is a subdirect product of R_i, $i \in I$.

Let $\overline{x} =< x_i : i \in I >\in R^I \ni (x_i, x_j) \in 0_R \ \forall i, j \in I$.

$\Rightarrow < x_i(i) : i \in I >= x_i \in R$.

Hence R is a 0_R-product of so-rings R_i, $i \in I$.

(ii). Suppose R is a 1_R-product of so-rings R_i, $i \in I$. Then $R \subseteq \prod_{i \in I} R_i$.

Let $x \in \prod_{i \in I} R_i$. $\Rightarrow x =< x_i : i \in I >\in \prod_{i \in I} R_i$ where $x_i \in R_i$, $i \in I$.

Since $p_i \mid_R : R \to R_i$ is a surjective homomorphism, $\exists \ a_i \in R \ni$

$p_i \mid_R (a_i) = x_i$, $i \in I$.

$\Rightarrow \exists\ \overline{a} =< a_i : i \in I >\in R^I \ni a_i(i) = x_i\ \forall i \in I.$

Since $1_R = R^I$, $(a_i, a_j) \in 1_R\ \forall i, j \in I$.

$\Rightarrow < a_i(i) : i \in I >\in R. \Rightarrow x =< x_i : i \in I >\in R.$

Hence $R = \prod_{i \in I} R_i$.

Conversely suppose that $R = \prod_{i \in I} R_i$.

Then R is clearly a subdirect product of R_i, $i \in I$.

Let $\overline{x} =< x_i : i \in I >\in R^I \ni (x_i, x_j) \in 1_R\ \forall i, j \in I$.

$\Rightarrow < x_i(i) : i \in I >\in \prod_{i \in I} R_i = R.$

Hence R is a 1_R-product of so-rings R_i, $i \in I$. □

Theorem 1.3.4. Let $\{R_i \mid i \in I\}$ be a family of so-rings, let R be a subso-ring of $\prod_{i \in I} R_i$ and let $\phi \in ConR$. For $i \in I$, let θ_i be the kernel of the projection at i, restricted to R. If R is a ϕ-product of so-rings $R_i, i \in I$, then

(i). $0_R = \bigcap_{i \in I} \theta_i$

(ii). for every $\overline{x} = (x_i : i \in I) \in R^I$, if $(x_i, x_j) \in \phi\ \forall i, j \in I$, then $\exists\ x \in R \ni$

$(x, x_i) \in \theta_i, \forall i \in I$

(iii). $R/\theta_i \cong R_i\ \forall i \in I$.

Proof.

(i). Let $(x, y) \in \bigcap_{i \in I} \theta_i$ for some $x, y \in R$. Then $p_i \mid_R (x) = p_i \mid_R (y)\ \forall i \in I$.

$\Rightarrow p_i(x) = p_i(y)\ \forall i \in I. \Rightarrow x = y.$

Hence $0_R = \bigcap_{i \in I} \theta_i$.

(ii). Let $\overline{x} = (x_i : i \in I) \in R^I \ni (x_i, x_j) \in \phi\ \forall i, j \in I$.

Then $x =< x_i(i) : i \in I >\in R. \Rightarrow x(i) = x_i(i), x, x_i \in R,\ \forall i \in I.$

$\Rightarrow p_i \mid_R (x) = p_i \mid_R (x_i)\ \forall i \in I. \Rightarrow (x, x_i) \in \theta_i\ \forall i \in I.$

Hence $\exists\ x \in R \ni (x, x_i) \in \theta_i, \forall i \in I$.

(iii). Define $f : R/\theta_i \to R_i$ by $[a]_{\theta_i} \mapsto a(i)$.

For any $[a]_{\theta_i}, [b]_{\theta_i} \in R/\theta_i$, $[a]_{\theta_i} = [b]_{\theta_i} \Leftrightarrow (a,b) \in \theta_i$

$\Leftrightarrow p_i \mid_R (a) = p_i \mid_R (b) \Leftrightarrow a(i) = b(i) \Leftrightarrow f([a]_{\theta_i}) = f([b]_{\theta_i})$.

Hence f is well defined and one-one.

For any $a_i \in R_i$, $\exists\ a \in R \ni a(i) = p_i \mid_R (a) = a_i$.

$\Rightarrow \exists\ [a]_{\theta_i} \in R/\theta_i \ni f([a]_{\theta_i}) = a(i) = a_i$.

Hence f is onto.

Let $([x_j]_{\theta_i} : j \in I)$ be a summable family in R/θ_i.

Then $f(\Sigma_{j\in I}[x_j]_{\theta_i}) = f([\Sigma_j x_j]_{\theta_i}) = (\Sigma_j x_j)(i) = \Sigma_j f([x_j]_{\theta_i})$.

For any $[a]_{\theta_i}, [b]_{\theta_i} \in R/\theta_i$, $f([a]_{\theta_i} \cdot [b]_{\theta_i}) = f([a \cdot b]_{\theta_i}) = (a \cdot b)(i)$

$= a(i) \cdot b(i) = f([a]_{\theta_i}) \cdot f([b]_{\theta_i})$ and hence f is a homomorphism.

Hence $R/\theta_i \cong R_i$, $i \in I$. □

Definition 1.3.5. Let R be a so-ring and let $\phi \in ConR$. For any system $\{\theta_i \mid i \in I\}$ of congruences of R, we write $0_R = \prod_\phi(\theta_i : i \in I)$ if and only if the conditions (i) and (ii) of theorem 1.3.4 are satisfied.

Remark 1.3.6. Let R be a so-ring and let $\{\theta_i \mid i \in I\}$ be a system of congruences of R. Then

(i). $0_R = \prod_{0_R}(\theta_i : i \in I)$ if and only if $0_R = \bigcap_{i\in I} \theta_i$

(ii). $0_R = \prod_{1_R}(\theta_i : i \in I)$ if and only if $0_R = \bigcap_{i\in I} \theta_i$ and for every $(x_i : i \in I) \in R^I$, $\exists\ x \in R \ni (x, x_i) \in \theta_i\ \forall i \in I$.

Proof.

(i). Suppose $0_R = \prod_{0_R}(\theta_i : i \in I)$ then $0_R = \bigcap(\theta_i : i \in I)$.

Conversely suppose that $0_R = \bigcap(\theta_i : i \in I)$.

Let $\overline{x} =< x_i : i \in I >\in R^I \ni (x_i, x_j) \in 0_R\ \forall i, j \in I$.

$\Rightarrow x_i = x_j \ \forall i, j \in I. \Rightarrow \exists\ x = x_i \in R \ni (x, x_i) \in \theta_i \ \forall i \in I.$

Hence $0_R = \prod_{0_R}(\theta_i : i \in I)$.

(ii). $0_R = \prod_{1_R}(\theta_i : i \in I) \Leftrightarrow 0_R = \bigcap_{i \in I} \theta_i$ and for every

$\overline{x} =< x_i : i \in I >\in R^I, \exists\ x \in R \ni (x, x_i) \in \theta_i \ \forall i \in I.$ □

Definition 1.3.7. Let $f : R \to R'$ be an epimorphism of so-rings R, R' and α be any congruence relation on R. Then we define $f(\alpha) = \{(f(x), f(y)) \mid (x, y) \in \alpha\}$.

Remark 1.3.8. $f(\alpha)$ is a congruence relation on R'.

Proof. Note that $f(\alpha) = \{(f(x), f(y)) \mid (x, y) \in \alpha\}$.

It can be easily obtained that $f(\alpha)$ is an equivalence relation on R'.

Let $(f(x_i) : i \in I)$ and $(f(y_i) : i \in I)$ be summable families such that

$(f(x_i), f(y_i)) \in f(\alpha) \ \forall i \in I.$

$\Rightarrow (x_i, y_i) \in \alpha \ \forall i \in I. \Rightarrow (\Sigma x_i, \Sigma y_i) \in \alpha.$

$\Rightarrow (f(\Sigma x_i), f(\Sigma y_i)) \in f(\alpha). \Rightarrow (\Sigma f(x_i), \Sigma f(y_i)) \in f(\alpha).$

Let $(f(x), f(y))$ and $(f(x'), f(y')) \in f(\alpha)$. Then (x, y) and $(x', y') \in \alpha$.

$\Rightarrow (xx', yy') \in \alpha. \Rightarrow (f(x)f(x'), f(y)f(y')) \in f(\alpha).$

Now let k', $h' \in R'$.

Since f is an epimorphism, $\exists\ k, h \in R \ni f(k) = k'$ and $f(h) = h'$.

Now $(f(x), f(y) + k') \in f(\alpha)$ and $(f(x) + h', f(y)) \in f(\alpha)$

$\Leftrightarrow (f(x), f(y) + f(k)) \in f(\alpha)$ and $(f(x) + f(h), f(y)) \in f(\alpha)$

$\Leftrightarrow (f(x), f(y + k)) \in f(\alpha)$ and $(f(x + h), f(y)) \in f(\alpha)$

$\Leftrightarrow (x, y + k) \in \alpha$ and $(x + h, y) \in \alpha$

$\Leftrightarrow (x, y) \in \alpha \Leftrightarrow (f(x), f(y)) \in f(\alpha).$

Hence $f(\alpha)$ is a congruence relation on R'. □

Lemma 1.3.9. Let R and R' be so-rings and $\phi, \theta_i (i \in I)$ be congruences of R. If f is an isomorphism from R onto R', then

$$0_R = \prod_{\phi}(\theta_i : i \in I) \text{ if and only if } 0_{R'} = \prod_{f(\phi)}(f(\theta_i) : i \in I).$$

Proof. Suppose $0_R = \prod_{\phi}(\theta_i : i \in I)$.

By remark 1.3.8, $f(\phi), f(\theta_i)(i \in I)$ are congruence relations on R'.

Let $(f(x), f(y)) \in \bigcap\limits_{i \in I} f(\theta_i)$. Then $(x, y) \in \bigcap\limits_{i \in I} \theta_i = 0_R$.

$\Rightarrow x = y$. $\Rightarrow f(x) = f(y)$.

Hence $\bigcap\limits_{i \in I} f(\theta_i) = 0_{R'}$.

Let $\overline{y} = (y_i : i \in I) \in {R'}^I \ni (y_i, y_j) \in f(\phi)\ \forall i, j \in I$.

Since f is onto, $\exists\ \overline{x} = (x_i : i \in I) \in R^I \ni f(x_i) = y_i\ \forall i \in I\ \&\ (x_i, x_j) \in \phi\ \forall i, j \in I$.

$\Rightarrow \exists\ x \in R \ni (x, x_i) \in \theta_i\ \forall i \in I$.

$\Rightarrow \exists\ y = f(x) \in R' \ni (y, y_i) \in f(\theta_i)\ \forall i \in I$.

Hence $0_{R'} = \prod_{f(\phi)}(f(\theta_i) : i \in I)$.

Conversely suppose $0_{R'} = \prod_{f(\phi)}(f(\theta_i) : i \in I)$.

Let $(x, y) \in \bigcap\limits_{i \in I} \theta_i$.

Then $(f(x), f(y)) \in \bigcap\limits_{i \in I} f(\theta_i) = 0_{R'}$.

$\Rightarrow f(x) = f(y)$. $\Rightarrow x = y$.

Hence $\bigcap\limits_{i \in I} \theta_i = 0_R$.

Let $\overline{x} = (x_i : i \in I) \in R^I \ni (x_i, x_j) \in \phi\ \forall i, j \in I$.

Then $\overline{y} = (f(x_i) : i \in I) \in {R'}^I \ni (f(x_i), f(y_i)) \in f(\phi)\ \forall i, j \in I$.

$\Rightarrow \exists\ y \in R' \ni (y, f(x_i)) \in f(\phi)\ \forall i \in I$.

$\Rightarrow \exists\ x \in R \ni f(x) = y\ \&\ (x, x_i) \in \phi\ \forall i \in I$.

Hence $0_R = \prod_{\phi}(\theta_i : i \in I)$. □

Theorem 1.3.10. Let R be a so-ring, $\phi \in ConR$ and $\{\theta_i \mid i \in I\}$ be a system of congruences of $R \ni 0_R = \prod_\phi(\theta_i : i \in I)$. If the mapping $f : R \to \prod\limits_{i \in I} R_i$ is defined by $f(x) = ([x]_{\theta_i} : i \in I)\ \forall x \in R$ where $R_i = R/\theta_i$, then $f(R)$ is a $f(\phi)$-product of so-rings $R_i, i \in I$.

Proof. By lemma 1.2.4, $f(R)$ is a subdirect product of so-rings $R_i = R/\theta_i, i \in I$.

Let $(y_i : i \in I) \in f(R)^I \ni (y_i, y_j) \in f(\phi)\ \forall i, j \in I$.

Then $\exists\ x_i \in R\ \forall i \in I \ni f(x_i) = y_i\ \&\ (x_i, x_j) \in \phi\ \forall i, j \in I$.

$\Rightarrow \exists\ x \in R \ni (x, x_i) \in \theta_i\ \forall i \in I$. $\Rightarrow [x]_{\theta_i} = [x_i]_{\theta_i}\ \forall i \in I$.

$\Rightarrow f(x)(i) = f(y_i)(i) = y_i(i)\ \forall i \in I$ and hence $(y_i(i) : i \in I) = f(x) \in f(R)$.

Hence $f(R)$ is a $f(\phi)$-product of so-rings $R_i, i \in I$. □

Definition 1.3.11. Let R, $R_i(i \in I)$ be so-rings, $\phi \in ConR$ and f be an embedding of R into $\prod\limits_{I \in I} R_i$. If $f(R)$ is a $f(\phi)$-product of so-rings $R_i, i \in I$, then the ordered pair $< (R_i : i \in I), f >$ is called as a *ϕ-representation* of the so-ring R.

Remark 1.3.12. Let R and $R_i(i \in I)$ be so-rings. Then

(i). $< (R_i : i \in I), f >$ is a 0_R-representation of R if and only if $f(R)$ is a subdirect product of $R_i, i \in I$

(ii). $< (R_i : i \in I), f >$ is a 1_R-representation of R if and only if $f(R)$ is the direct product of $R_i, i \in I$.

Proof.

(i). By remark 1.3.3(i), we have R is a subdirect product of so-rings R_i, $i \in I$ if and only if R is a 0_R-product of so-rings R_i, $i \in I$.

Since $R \cong f(R)$, we have $f(R)$ is a subdirect product of so-rings R_i, $i \in I$ if and only if $f(R)$ is a $f(0_R)$-product of so-rings R_i, $i \in I$.

Hence $< (R_i : i \in I), f >$ is a 0_R- representation of R if and only if $f(R)$ is a subdirect product of R_i, $i \in I$.

(ii). By remark 1.3.3(ii), we have R is a 1_R-product of R_i, $i \in I$ if and only if $R = \prod_{i \in I} R_i$.

Then $f(R)$ is a $f(1_R)$-product of R_i, $i \in I$ if and only if $f(R) = \prod_{i \in I} R_i$.

Hence $< (R_i : i \in I), f >$ is the direct product of R_i, $i \in I$. □

Theorem 1.3.13. Let R be a so-ring, ϕ, $\theta_i (i \in I) \in ConR$. Define $f : R \to \prod_{i \in I} R_i$ by $x \mapsto ([x]_{\theta_i} : i \in I)$ where $R_i = R/\theta_i, i \in I$. Then $< (R_i : i \in I), f >$ is a ϕ-representation of R if and only if $0_R = \prod_\phi (\theta_i : i \in I)$.

Proof. Suppose $< (R_i : i \in I), f >$ is a ϕ-representation of R.

Then $f(R)$ is a $f(\phi)$-product of so-rings $R_i, i \in I$.

$(f(x), f(y)) \in f(\theta_i) \Leftrightarrow (x, y) \in \theta_i \Leftrightarrow [x]_{\theta_i} = [y]_{\theta_i}$

$\Leftrightarrow p_i \mid_{f(R)} (f(x)) = p_i \mid_{f(R)} (f(y)) \Leftrightarrow (f(x), f(y)) \in ker(p_i \mid_{f(R)})$.

Hence $f(\theta_i), i \in I$ is the kernel of the projection at i restricted to $f(R)$.

Then by theorem 1.3.4, $0_{f(R)} = \bigcap (f(\theta_i) : i \in I)$.

Hence $0_{f(R)} = \prod_{f(\phi)} (f(\theta_i) : i \in I)$.

By lemma 1.3.9, $0_R = \prod_\phi (\theta_i : i \in I)$.

The converse part is trivial in view of theorem 1.3.10.

Hence the theorem. □

Corollary 1.3.14. (i). A system $(\theta_i : i \in I)$ of congruences of so-ring R gives a subdirect representation if and only if $\bigcap_{i \in I} \theta_i = 0_R$.

(ii). A system $(\theta_i : i \in I)$ of congruences of so-ring R constitutes a direct representation if and only if $0_R = \prod_{1_R} (\theta_i : i \in I)$.