Chapter 4

EUCLIDEAN PARTIAL SEMIRINGS

$4.1. **Euclidean partial semirings**

In this section we generalize the results of Golan[16] for semirings to partial semirings.

We denote the *set of all right divisors of* a in the partial monoid $(R, \cdot)$ by $RD(a)$. i.e., $RD(a) = \{b \in R \mid a \in Rb\} = \{b \in R \mid Ra \subseteq Rb\}$. We denote the set $\{b \in R \mid a \cdot b = 1 = b \cdot a\}$ by $U(R)$ and the set $\{a \in R \mid a \cdot a = a\}$ by $I^{\times}(R)$.

Remark 4.1.1. If R is a partial semiring then

(i). $b \in RD(a)$ if and only if $RD(b) \subseteq RD(a)$

(ii). $U(R) \subseteq RD(1_R) \subseteq RD(a)$.

Proof. (i). Suppose $b \in RD(a)$. Then $a \in Rb$.

Now for any $x \in RD(b)$, $Rb \subseteq Rx$ and hence $a \in Rx$.

$\Rightarrow x \in RD(a)$. Hence $RD(b) \subseteq RD(a)$.

Conversely suppose $RD(b) \subseteq RD(a)$. Since $b \in RD(b)$, $b \in RD(a)$.

(ii). Let $x \in U(R)$. Then $\exists\ y \in R \ni xy = 1 = yx \in Rx$ and hence $x \in RD(1_R)$.

Now let $x \in RD(1_R)$ then $1 \in Rx$. $\Rightarrow Rx = R$.

$\Rightarrow a \in Rx$ and hence $x \in RD(a)$.

Hence the remark. □

Definition 4.1.2. Let R be a partial semiring and $a \in R$. Then 'a' is said to be *irreducible from right* if and only if it satisfy

(i). $a \notin U(R)$

(ii). $RD(a) = U(R) \bigcup \{a\}$.

In the partial semirings $\mathbb{N}$ and $pfn(D, D)$, every nonzero element is irreducible from right.

Example 4.1.3. Consider the partial semiring $Mat_D(R)$, the set of $D \times D$ matrices over R. Take $D = \{a, b\}$ and $R = \mathbb{N}$. Then the only elements of $Mat_D(R)$ having determinant 1 which are irreducible from right are $[a_{ij}]$ and $[b_{ij}]$ where

$$a_{ij} = \begin{cases} 0, & if\ i = b\ and\ j = a, \\ 1, & otherwise. \end{cases}$$

and

$$b_{ij} = \begin{cases} 0, & if\ i = a\ and\ j = b, \\ 1, & otherwise. \end{cases}$$

Definition 4.1.4. Let A be a nonempty subset of a partial semiring R. Then the set of *common right divisors of* A is $CRD(A) = \bigcap\{RD(a) \mid a \in A\} = \{b \in R \mid RA \subseteq Rb\}$.

Definition 4.1.5. Let R be a partial semiring. Then an element $b \in CRD(A)$ is said to be a *greatest common right divisor of* A if and only if $CRD(A) = RD(b)$.

Theorem 4.1.6. If A is a nonempty subset of a partial semiring R then an element b of R is a greatest common right divisor of A if and only if the following conditions are satisfied:

(i). $RA \subseteq Rb$

(ii). if $c \in R$ satisfies $RA \subseteq Rc$ then $Rb \subseteq Rc$.

Proof. Suppose b is a greatest common right divisor of A.

(i). Since $b \in CRD(A)$, $b \in RD(a)\ \forall\ a \in A$. $\Rightarrow Ra \subseteq Rb\ \forall\ a \in A$.

Hence $RA \subseteq Rb$.

(ii). Suppose $c \in R \ni RA \subseteq Rc$. Then $c \in CRD(A) = RD(b)$ and hence $Rb \subseteq Rc$.

Conversely suppose that the conditions (i) and (ii) are satisfied.

By (i), $b \in CRD(A)$.

Now for any $x \in RD(b)$, $b \in Rx$. $\Rightarrow b = rx$ and $b \in CRD(A)$.

$\Rightarrow b = rx \in RD(a)\ \forall\ a \in A$. $\Rightarrow a \in Rrx \subseteq Rx\ \forall\ a \in A$.

$\Rightarrow x \in RD(a)\ \forall a \in A$. $\Rightarrow x \in CRD(A)$ and hence $RD(b) \subseteq CRD(A)$.

Now for any $c \in CRD(A)$, $RA \subseteq Rc$. $\Rightarrow Rb \subseteq Rc$ (by (ii)).

$\Rightarrow c \in RD(b)$ and hence $CRD(A) \subseteq RD(b)$.

Hence b is a greatest common right divisor of A. □

Corollary 4.1.7. If every left partial ideal of a partial semiring R is principal then every nonempty subset of R has a greatest common right divisor.

Proof. Let A be a nonempty subset of R.

Since RA is a left partial ideal of R, we have $RA = Rb$ for some $b \in R$.

Now let $c \in R \ni RA \subseteq Rc$. Then $Rb \subseteq Rc$.

Then by theorem 4.1.6, b is the greatest common right divisor of A. □

Theorem 4.1.8. Let a, b and c be elements of a partial semiring R. If d is a greatest common right divisor of $\{a, b\}$ and e is a greatest common right divisor of $\{c, d\}$ then e is a greatest common right divisor of $\{a, b, c\}$.

Proof. By definition, $RD(e) = RD(d) \bigcap RD(c) = RD(a) \bigcap RD(b) \bigcap Rd(c)$

$= CRD(\{a, b, c\})$.

Hence the theorem. □

Remark 4.1.9. If a and b are elements of a partial semiring R and (a, b) is a summable family in R then $CRD(\{a, b\}) \subseteq CRD(\{a + b, b\})$.

Proof. For any $x \in CRD(\{a, b\}) = RD(a) \bigcap RD(b)$, $a \in Rx$ and $b \in Rx$.

$\Rightarrow a + b \in Rx$ and hence $x \in RD(a + b) \bigcap RD(b) = CRD(\{a + b, b\})$.

Hence the remark. □

Theorem 4.1.10. Let R be a partial semiring. Then the following are equivalent

(i). $CRD(\{a,b\}) = CRD(\{a+b,b\})$ for all $a, b \in R \ni a+b$ exists in R

(ii). every principal left partial ideal of R is subtractive.

Proof. (i)⇒(ii): Suppose $CRD(\{a,b\}) = CRD(\{a+b,b\})$ for all $a, b \in R$ such that $a+b$ exists in R.

Let Rd be a principal left partial ideal of R and let $x, x+y \in Rd$.

Then $d \in RD(x)$ and $d \in RD(x+y)$. $\Rightarrow d \in CRD(\{x+y,x\}) = CRD\{x,y\}$.

$\Rightarrow d \in RD(y)$ and hence $y \in Rd$. Hence Rd is subtractive.

(ii)⇒(i): Suppose every principal left partial ideal of R is subtractive.

Let $a, b \in R \ni a+b$ exists in R and let $x \in CRD(\{a+b,b\})$.

Then $x \in RD(a+b)$ and $x \in RD(b)$. $\Rightarrow a+b \in Rx$ and $b \in Rx$.

$\Rightarrow a \in Rx$ and hence $x \in RD(a) \bigcap RD(b) = CRD(\{a,b\})$.

Hence $CRD(\{a,b\}) = CRD(\{a+b,b\})$ for all $a, b \in R \ni a+b$ exists in R. □

Definition 4.1.11. A partial semiring R is said to be *PLIS-semiring* if it satisfies any one of the following equivalent conditions:

(i). $CRD(\{a,b\}) = CRD(\{a+b,b\})$ for all $a, b \in R \ni a+b$ exists in R

(ii). every principal left partial ideal of R is subtractive.

Note that the partial semirings $\mathbb{N}, \mathbb{R}$ are PLIS-semirings. The following is an example of a partial semiring which is not PLIS-semiring.

Example 4.1.12. Let $S = (\mathbb{R}^+ \times \{0\}) \bigcup (\{0\} \times \mathbb{R}^+)$. Define Σ on S by

$$\Sigma x_i = \begin{cases} x_j,\ if\ x_i = 0\ \forall i \neq j,\ for\ some\ j, \\ (a + a', 0),\ if\ x_i = (a, 0),\ x_j = (a', 0)\ \&\ x_k = 0\ \forall\ k \neq i, j \\ (0, b + b'),\ if\ x_i = (0, b)\ or\ (b, 0),\ x_j = (0, b')\ \&\ x_k = 0\ \forall\ k \neq i, j \\ undefined,\ otherwise. \end{cases}$$

and '$\cdot$' defined on R by $(a, 0) \cdot (a', 0) = (aa', 0) = (0, a) \cdot (a', 0)$ and $(0, b) \cdot (o, b') = (0, bb') = (b, 0) \cdot (0, b')\ \forall a, a', b, b' \in \mathbb{R}^+$. Then $R = S \times \mathbb{N}$ is a partial semiring. Now $H = \{0\} \times \mathbb{R}^+ \times \{0\}$ is a principal left partial ideal of R. Since $(0, b, 0) \in H, (b, 0, 0) + (0, b, 0) = (0, 2b, 0) \in H$ and $(b, 0, 0) \notin H$, H is not subtractive. Hence R is not PLIS-semiring.

Definition 4.1.13. Let R be a partial semiring. Then a mapping $\delta : R \setminus \{0\} \to \mathbb{N}$ is said to be a *left Euclidean norm* on R if it satisfies the following condition:

If a and b are elements of R with $b \neq 0$ and $\delta(a) \geq \delta(b)$ then $\exists\ q, r \in R \ni a = qb + r$ with $r = 0$ or $\delta(r) < \delta(b)$.

Definition 4.1.14. A partial semiring R is said to be *left Euclidean* if and only if there exists a left Euclidean norm defined on R.

The partial semiring $\mathbb{N}$ is left Euclidean if we define the left Euclidean norm δ by $\delta : n \mapsto n$ or $\delta : n \mapsto n^2$.

Remark 4.1.15. If δ is a left Euclidean norm on a partial semiring R, then we can extend δ to δ' from R to $\mathbb{N} \bigcup \{\infty\}$ by defining $\delta'(0) = \infty$ and $\delta'(a) = \delta(a)\ \forall\ a \in R \setminus \{0\}$. Conversely if $\delta' : R \to \mathbb{N} \bigcup \{\infty\}$ is a function satisfy the condition:

for any $a, b \in R \ni \delta'(a) \geq \delta'(b)\ \exists\ q, r \in R \ni a = qb + r$ with $r = 0$ or $\delta'(r) < \delta'(b)$, then its restriction is a left Euclidean norm on R.

Theorem 4.1.16. If δ is a left Euclidean norm defined on a partial semiring R then there exists a left Euclidean norm δ^* satisfying

(i). $\delta^*(a) \leq \delta(a) \; \forall \; a \in R \setminus \{0\}$

(ii). $\delta^*(b) \leq \delta(rb) \; \forall \; b, r \in R \ni rb \neq 0$.

Proof. Define $\delta^* : R \setminus \{0\} \to \mathbb{N}$ by $\delta^*(a) = \min\{\delta(ra) \mid ra \neq 0\} \; \forall \; 0 \neq a \in R$.

Then $\delta^*(a) \leq \delta(a) \; \forall a \in R \setminus \{0\}$, and $\delta^*(b) \leq \delta(rb) \; \forall \; b, r \in R \ni rb \neq 0$.

Now we prove that δ^* is a left Euclidean norm.

Let $a, b \in R \setminus \{0\} \ni \delta^*(a) \geq \delta^*(b) = \min\{\delta(rb) \mid rb \neq 0\}$.

Then $\exists \; s \in R \ni \delta^*(b) = \delta(sb)$ and by (i), $\delta(a) \geq \delta^*(a) \geq \delta^*(b) = \delta(sb)$.

$\Rightarrow \exists \; q, r \in R \ni a = q(sb) + r$ where $r = 0$ or $\delta(r) < \delta(sb)$.

Suppose $\delta(r) < \delta(sb)$. Then $\delta^*(r) \leq \delta(r) < \delta(sb) = \delta^*(b)$.

$\Rightarrow \exists \; qs, r \in R \ni a = (qs)b + r$ where $r = 0$ or $\delta^*(r) < \delta^*(b)$.

Hence the theorem. □

Definition 4.1.17. Let (R, δ) be a left Euclidean partial semiring. Then δ is said to be *submultiplicative norm* if it satisfies the following condition

$\delta(b) \leq \delta(rb) \; \forall \; 0 \neq b \in R, \; r \in R \ni rb \neq 0$.

Definition 4.1.18. A left Euclidean norm δ defined on a partial semiring R is said to be *multiplicative norm* if and only if $\delta(ab) = \delta(a)\delta(b) \; \forall \; a, b \in R \ni ab \neq 0$.

In the left Euclidean partial semiring $\mathbb{N}$, δ defined by $\delta : n \mapsto n$ or $\delta : n \mapsto n^2$ is a submultiplicative and multiplicative norm.

Theorem 4.1.19. Let R be a partial semiring and $\delta : R \setminus \{0\} \to \mathbb{N}$ be a submultiplicative Euclidean norm. If $M_\delta = \{r \in R \mid \delta(r) \leq \delta(a) \; \forall \; 0 \neq a \in R\}$, a minimal element of $im(\delta)$ then

(i). $1_R \in M_\delta$

(ii). if $a \in M_\delta$ then $\exists \; q \in R \ni 1 = qa$

(iii). $M_\delta \bigcap I^\times(R) = \{1_R\}$

(iv). $U(R) \subseteq M_\delta$, with equality holding if R is commutative.

Proof. (i). Since δ is submultiplicative norm, $\delta(1_R) \leq \delta(a)\ \forall\ 0 \neq a \in R$.

Hence $1_R \in M_\delta$.

(ii). Let $a \in M_\delta$. Then $\delta(a) \leq \delta(b)\ \forall\ 0 \neq b \in R$.

$\Rightarrow \delta(a) \leq \delta(1_R)$. $\Rightarrow \exists\ q, r \in R \ni 1_R = qa + r$ with $r = 0$ or $\delta(r) < \delta(a)$.

Since $a \in M_\delta$, $\delta(a) \leq \delta(r)$ for $0 \neq r \in R$ and hence $r = 0$. Hence $1_R = qa$.

(iii). Let $c \in M_\delta \bigcap I^\times(R)$. Then $c \in M_\delta$ and $c^2 = c$.

By (ii), $\exists\ q \in R \ni 1_R = qc = qc^2 = 1_R c = c$. Hence $M_\delta \bigcap I^\times(R) = \{1_R\}$.

(iv). Let $a \in U(R)$. Then $\exists\ b \in R \ni 1_R = ba$.

Since δ is submultiplicative norm, $\delta(a) \leq \delta(ba) = \delta(1_R)$ and by (i), $\delta(1_R) \leq \delta(a)$.

$\Rightarrow \delta(a) = \delta(1_R) \leq \delta(b)\ \forall\ 0 \neq b \in R$ and hence $a \in M_\delta$.

Suppose R is commutative and let $a \in M_\delta$.

By (ii), $\exists\ q \in R \ni 1 = qa = aq$ and hence $a \in U(R)$.

Hence $M_\delta = U(R)$. □

Theorem 4.1.20. If $\gamma : R \to S$ is an epimorphism of partial semirings and δ is a left Euclidean norm on R then $\exists$ a left Euclidean norm δ' on S defined by $\delta'(c) = \min\{\delta(a) \mid a \in \gamma^{-1}(c)\}\ \forall\ c \in S \setminus \{0\}$.

Proof. Define $\delta' : S \setminus \{0\} \to \mathbb{N}$ by $\delta'(c) = \min\{\delta(a) \mid a \in \gamma^{-1}(c)\}\ \forall 0 \neq c \in S$.

Let $c, d \in S \ni d \neq 0$ with $\delta'(c) \geq \delta'(d)$.

$\Rightarrow \exists\ a, 0 \neq b \in R \ni \gamma(a) = c$, $\gamma(b) = d$ where b is such that

$\delta(b) = \min\{\delta(y) \mid y \in \gamma^{-1}(d)\}$.

Since $\delta'(c) \geq \delta'(d)$, $\min\{\delta(x) \mid x \in \gamma^{-1}(c)\} \geq \min\{\delta(y) \mid y \in \gamma^{-1}(d)\}$.

$\Rightarrow \delta(a) \geq \delta(b)$. $\Rightarrow \exists\ q, r \in R \ni a = qb + r$ where $r = 0$ or $\delta(r) < \delta(b)$.

$\Rightarrow c = \gamma(a) = \gamma(qb + r) = \gamma(q)d + \gamma(r)$ where $\gamma(r) = 0$ or

$\delta'(\gamma(r))$ =min$\{\delta(a) \mid a \in \gamma^{-1}(\gamma(r))\} \leq \delta(r) < \delta(b) = \delta'(d)$.

Hence δ' is a Euclidean norm on S. □

Theorem 4.1.21. If R is a left Euclidean partial semiring then every subtractive left partial ideal of R is principal.

Proof. Let δ be the Euclidean norm defined on R and I be a subtractive

left partial ideal of R.

If $I = \{0\}$ then I is principal.

So assume $I \neq \{0\}$. Then $\exists\ 0 \neq x \in I$.

Take $\mathcal{C} = \{\delta(a) \mid a \in I\}$.

Since $\delta(x) \in \mathcal{C}$, $\mathcal{C}$ is nonempty.

Then by Zorn's lemma, $\mathcal{C}$ has a minimal element. Let it be $\delta(b)$.

Suppose $I \neq Rb$. Then $\exists\ a \in I \ni a \notin Rb$.

$\Rightarrow \delta(b) \leq \delta(a)$ (by the minimality of $\delta(b)$).

$\Rightarrow \exists\ q, r \in R \ni a = qb + r$ with $r = 0$ or $\delta(r) < \delta(b)$.

Suppose $r = 0$ then $a = qb \in Rb$, a contradiction. $\Rightarrow \delta(r) < \delta(b)$.

Since $qb + r = a \in I$ and $b \in I$, we have $r \in I$.

$\Rightarrow r \in I$ and $\delta(r) < \delta(b)$, a contradiction.

Hence $I = Rb$, a principal left partial ideal of R. □

Theorem 4.1.22. The following conditions on a left Euclidean partial semiring are equivalent

(i). R is a PLIS-semiring

(ii). there exists a left Euclidean norm δ defined on R satisfying the condition that if $a = qb + r$ for $r \in R \setminus \{0\}$ and $\delta(r) < \delta(b)$ then $a \notin Rb$.

Proof. (i)$\Rightarrow$(ii): Suppose R is a PLIS-semiring.

Since R is left Euclidean partial semiring, $\exists$ a left Euclidean norm δ on R.

By theorem 4.1.20, $\exists$ a left Euclidean norm δ^* defined on $R \ni \delta^*(b) \leq \delta(rb)$

$\forall r, b \in R \ni rb \neq 0$.

Now suppose $a = qb + r \in Rb$ for $r \in R \setminus \{0\}$ and $\delta^*(r) < \delta^*(b)$.

Since R is PLIS-semiring, Rb is subtractive. $\Rightarrow r \in Rb$.

$\Rightarrow r = cb$ for some $c \in R$. $\Rightarrow \delta^*(r) = \delta^*(cb) = \delta(cb) \geq \delta^*(b)$, a contradiction.

Hence $a \notin Rb$.

(ii)$\Rightarrow$(i): Suppose the condition (ii) is valid and let $t \in CRD(\{a + b, b\})$.

$\Rightarrow a + b = dt$ and $b = et$ for some $d, e \in R$. $\Rightarrow a + et = dt \in Rt$.

Then by (ii), $\delta(r) \geq \delta(t)$ $\forall r \in R \setminus \{0\}$.

$\Rightarrow \delta(a) \geq \delta(t)$. $\Rightarrow \exists\ q, r \in R \ni a = qt + r$ where $r = 0$ or $\delta(r) < \delta(t)$.

$\Rightarrow dt = a + b = qt + r + et$.

Suppose $\delta(r) < \delta(t)$. Then by (ii), $dt \notin Rt$, a contradiction.

Hence $r = 0$ and hence $a = qt$. $\Rightarrow t \in RD(a) \bigcap RD(b) = CRD(\{a, b\})$.

Hence R is a PLIS-semiring. □

Theorem 4.1.23. If R is a left Euclidean PLIS-semiring then any nonempty finite subset A of R has a greatest common right divisor.

Proof. By theorem 4.1.8, it is enough to prove that $\exists$ a greatest common right divisor for $\{a, b\} \subseteq A$.

For $a = b = 0$, the greatest common right divisor is 0.

Suppose $b \neq 0$.

By theorem 4.1.22, $\exists$ a left Euclidean norm δ on $R \ni$ if $a = qb + r$ for $r \in R \setminus \{0\}$ and $\delta(r) < \delta(b)$ then $a \notin Rb$.

Since δ is a left Euclidean norm on R, $\exists\ q_1, r_1 \in R \ni a = q_1 b + r_1$ where $r_1 = 0$ or $\delta(r_1) < \delta(b)$.

If $r_1 = 0$ then $a = q_1 b \in Rb$, a contradiction.

Hence $\exists\ q_1, 0 \neq r_1 \in R \ni a = q_1 b + r_1$ where $\delta(r_1) < \delta(b)$.

Continuing this process, we get $q_1, q_2, ..., q_n, q_{n+1}$, $0 \neq r_1, 0 \neq r_2, ..., 0 \neq r_n \in R$ such that $a = q_1 b + r_1$, $b = q_2 r_1 + r_2$,..., $r_{n-2} = q_n r_{n-1} + r_n$, $r_{n-1} = q_{n+1} r_n$ and $\delta(b) > \delta(r_1) > ... > \delta(r_n)$.

This process of selecting q_i, r_i is terminated after a finitely many steps.

Then $r_{n-1} = q_{n+1} r_n$, $r_{n-2} = (q_n q_{n+1} + 1) r_n$, ..., $b = (q_2 q_3 ... q_n q_{n+1} + ... + q_2 + q_{n+1}) r_n$.

$\Rightarrow r_n \in RD(b)$.

Now $a = q' r_n$ for some $q' \in R$ and hence $r_n \in RD(a)$.

$\Rightarrow r_n \in RD(a) \bigcap RD(b) = CRD(\{a, b\})$. Let $d \in CRD(\{a, b\})$.

Then $d \in CRD(\{q_1 b + r_1, b\})$. $\Rightarrow d \in CRD(\{r_1, b_1\})$ and hence $d \in RD(r_1)$.

Similarly $d \in RD(r_2)$,..., $d \in RD(r_n)$. Hence $CRD(\{a, b\}) = RD(r_n)$.

Therefore r_n is the greatest common right divisor of $\{a, b\}$.

Hence the theorem. □

Remark 4.1.24. If R is a partial semiring then $P(R) = \{0_R\} \bigcup \{r + 1_R \mid r \in R\}$ is a partial subsemiring of R.

Proof. Clearly $0_R, 1_R \in P(R)$.

Let $(r_i : i \in I)$ be a summable family in $R \ni r_i \in P(R), i \in I$.

Then $\Sigma_i r_i$ exists and $r_i = s_i + 1_R$ for some $s_i \in R, i \in I$.

$\Rightarrow \Sigma_i r_i = \Sigma_i (s_i + 1_R) = (\Sigma_i s_i + \Sigma_{i \neq k} 1_R) + 1_R \in P(R)$.

Hence $\Sigma_i r_i \in P(R)$.

Let $r_1, r_2 \in P(R)$. Then $r_1 = s_1 + 1_R, r_2 = s_2 + 1_R$ for some $s_1, s_2 \in R$.

$\Rightarrow r_1 r_2 = (s_1 + 1_R)(s_2 + 1_R) = (s_1 s_2 + s_1 + s_2) + 1_R \in P(R)$.

Hence $P(R)$ is a partial subsemiring of R. □

Definition 4.1.25. A partial semiring R is said to be *antisimple* if and only if $P(R) = R$.

The partial semiring $\mathbb{N}$ is antisimple whereas $pfn(D, D)$ is not antisimple partial semiring.

Definition 4.1.26. Let R be a commutative antisimple partial semiring. Then a function $\delta : R \to \mathbb{N}$ is said to be *Dale norm* if and only if the following conditions are satisfied

(i). $\delta(a) = 0$ if and only if $a = 0_R$

(ii). if $\Sigma_i a_i$ exists then $\delta(\Sigma_i a_i) \geq \delta(a_i)$ for any $i \in I$

(iii). $\delta(ab) = \delta(a)\delta(b)$ for all $a, b \in R$

(iv). If $a \in R$ and $0 \neq b \in R$ then there exists $q, r \in R \ni a = qb + r$,

where $r = 0$ or $\delta(r) < \delta(b)$.

The function defined by $n \mapsto n$ or $n \mapsto n^2$ is a Dale norm on the partial semiring $\mathbb{N}$.

Remark 4.1.27. If R is a commutative antisimple partial semiring and δ is a Dale norm on R then R is entire.

Proof. Let $a, b \in R \ni ab = 0_R$. Then $\delta(ab) = \delta(0_R) = 0$.

$\Rightarrow \delta(a)\delta(b) = 0$. $\Rightarrow \delta(a) = 0$ or $\delta(b) = 0$.

$\Rightarrow a = 0_R$ or $b = 0_R$ and hence R is entire. □

Clearly every Dale norm defined on a partial semiring R is a left Euclidean norm. The following is an example of a partial semiring R in which δ is a left Euclidean norm but not Dale norm.

Example 4.1.28. Consider the partial semiring $R = \{0, a, b, 1\}$ in which Σ defined on R by

$$\Sigma x_i = \begin{cases} x_j,\ if\ x_i = 0\ \forall i \neq j,\ for\ some\ j, \\ 0,\ if\ x_i = x_j = a\ for\ some\ i,\ j\ \&\ x_k = 0\ \forall\ k \neq\ i, j \\ 1,\ if\ x_i = a,\ x_j = b\ for\ some\ i,\ j\ \&\ x_k = 0\ \forall\ k \neq\ i, j \\ a,\ if\ x_i = x_j = 1\ or\ b\ for\ some\ i,\ j\ \&\ x_k = 0\ \forall\ k \neq\ i, j \\ b,\ if\ x_i = 1,\ x_j = a\ for\ some\ i,\ j\ \&\ x_k = 0\ \forall\ k \neq\ i, j \\ undefined,\ otherwise. \end{cases}$$

and '$\cdot$' defined on R by the following table:

.	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	1	b
1	0	a	b	1

Then R is a commutative antisimple partial semiring. Now $\delta : R \setminus \{0\} \to \mathbb{N}$ defined by $\delta(1) = \delta(b) = 2$ and $\delta(a) = 3$ is a left Euclidean norm which cannot be converted to a Dale norm.

Theorem 4.1.29. If R is a commutative antisimple partial semiring and δ is a Dale norm defined on R then

(i). $U(R) = \{a \in R \mid \delta(a) = 1\}$

(ii). R is a division partial semiring if and only if $\delta(R)$ is finite.

Proof. (i). Note that $\delta(1_R) = \delta(1_R \cdot 1_R) = \delta(1_R) \cdot \delta(1_R)$ and hence $\delta(1_R) = 1$.

Let $a \in U(R)$. Then $\exists\ b \in R \ni ab = 1_R$.

$\Rightarrow \delta(ab) = \delta(a)\delta(b) = \delta(1_R) = 1$.

$\Rightarrow \delta(a) = 1$ and $\delta(b) = 1$ and hence $a \in \{c \in R \mid \delta(c) = 1\}$.

Now let $a \in \{c \in R \mid \delta(c) = 1\}$. Then $\delta(a) = 1 = \delta(1_R)$.

$\Rightarrow \exists\ q, r \in R \ni 1_R = qa + r$, where $r = 0_R$ or $\delta(r) < \delta(a)$.

Suppose $\delta(r) < \delta(a) = 1$. Then $\delta(r) = 0$ and hence $r = 0_R$.

$\Rightarrow 1_R = qa$ and hence $a \in U(R)$. Hence $U(R) = \{a \in R \mid \delta(a) = 1\}$.

(ii). Suppose R is a division partial semiring and let $0 \neq \delta(a) \in \delta(R)$.

Then $0_R \neq a \in R$. $\exists\ b \in R \ni ab = 1_R$.

$\Rightarrow \delta(ab) = \delta(1_R) = 1$. $\Rightarrow \delta(a) = 1$ and $\delta(b) = 1$.

Hence $\delta(R) = \{0, 1\}$, a finite set.

Conversely suppose that $\delta(R)$ is a finite subset of $\mathbb{N}$.

Suppose $\exists$ a nonunit $r \in R \setminus \{0\}$.

Then $\delta(r) > 1$ and r^k is nonunit for all $k \geq 1$.

$\Rightarrow \delta(r^k) = \delta(r)\delta(r^{k-1}) > \delta(r^{k-1})\ \forall k > 1$.

Hence $\delta(R)$ is not finite, a contradiction.

Hence R is division partial semiring. □