Chapter 3

LATTICE ON PRE A∗-ALGEBRA

This Chapter analyzes the concept of *Lattice on Pre A*-algebra.* We define the lattice on Pre A*-algebra and we call such a defined lattice on Pre A*-algebra as Pre A*-lattice and we derive the properties of the Lattice on Pre A*-algebra. We define greatest lower bound and least upper bound on Pre A*-algebra .We observe that 2 acts as greatest lower bound(g.l.b) with respect to meet whereas 2 acts as least upper bound(l.u.b) with respect to join. We define atoms, dual atoms, irreducible elements with respect to meet as well as join on Pre A*-algebra. We obtain various theorems on these atoms, dual atoms, irreducible elements on Pre A*-algebra We establish the atomic lattices, dual atomic lattices *on* Pre A*-algebra.

This chapter consists of four sections. In the first section, we recall partial ordering $\leq$ on Pre A∗-algebra and recall that Pre A*-algebra as a poset. We recollect if A is a Pre A*- algebra then (A,$\leq$) is a lattice. We furnish the definition of semilattice, and its properties.

In the second section we identify for any subset L of a Pre A*-algebra, a Pre A*- lattice. We present various examples of Pre A*-lattices. We offer several properties of Pre A*-lattices. We confer axioms for a Pre A*-algebra to become a Pre A*-lattice. We define sub Pre A*-lattice and bound elements, bounded Pre A*-lattice,

In the third section we classify a number of kinds of Pre A*-lattices like distributive Pre A*-lattice, modular Pre A*-lattice and we establish some theorems related with this.

In the fourth section we characterize atoms, dual atoms, irreducible elements in Pre A* - algebra and we confirm a number of theorems in these. We delineate Pre A*-homomorphism and demonstrate a theorem in Pre A∗-Algebra and also we prove f: A→P (B) is an isomorphism and a finite Pre A∗ – algebra has 3^n elements for some positive integer n.

3.1Pre A* - algebra as a Poset:

In this section we recall the definition of a partial ordering ≤ on Pre A* - algebra and recall the theorem Pre A* - algebra as a Poset. Also we recall the theorem that if A is a Pre A* - algebra then (A, ≤) is a Lattice.

3.1.1 Definition[7]: Let A be a Pre A* - algebra. Define ≤ on A by $x \le y$ if and only if $x \wedge y = y \wedge x = x$, $\forall\, x, y \in A$. The defined ≤ is said to be partial ordering on Pre A* - alge**bra A.**

3.1.2 Lemma [7]: If A is a Pre A* - algebra then (A, ≤) is a Poset.

3.1.3 Theorem [7]: In a Poset (A, ≤) with 1, for any $x,y \in A$, $\inf\{x,y\} = x \wedge y$

3.1.4 Theorem [7]: In a poset (A, ≤) with 1, for any $x,y \in B(A)$. $\text{Sup}\{x,y\} = x \vee y$ where $B(A) = \{x \in A / x \vee x^{\sim} = 1\}$.

3.1.5 Theorem [7]: If A is Pre A* - algebra and $x \wedge (x \vee y) = x$ for all $x,y \in A$ then (A, ≤) is a lattice.

3.2 Lattice on Pre A* - algebra:

In this section we define (for any subset L of a Pre A*-algebra) a Pre A*-lattice $(L, \wedge, \vee)$. We give some examples of Pre A* -lattices. We give some properties of Pre A* -lattices. We give axioms for a Pre A* - algebra to become a Pre A*-lattice. We define sub Pre A*-lattice and bound elements, bounded Pre A*-lattice.

3.2.1 Definition of a lattice on Pre A^* - algebra or Pre A^* - lattice:

Let A be a Pre A^* - algebra. A non-empty subset L of a Pre A^*-algebra A in which for each pair of elements $a \in A$, $b \in B(A)$ in L has greatest lower bound $a \wedge b$ and least upper bound $a \vee b$ exists in L .Such a defined set L in Pre A^* - algebra is said to be a Pre A^* - lattice.

3.2.2 Pre A^* -Lattice as algebraic system:

Definition: Let A be a Pre A^* - algebra. A non-empty subset L of a Pre A^*-algebra A, equipped with two binary operations meet ($\wedge$) and join ($\vee$) which assign to every pair $a \in A$, $b \in B(A)$ of the elements of L, uniquely an element $a \wedge b$ as well as an element $a \vee b$ in L in such a way that the following axioms holds.

(i) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$, $\forall$ a, b, c $\in$ L (associative)

(ii) $a \wedge b = b \wedge a$, $\forall$ a, b $\in$ L (commutative)

(iii) $a \wedge (a \vee b) = a$, $\forall$ a, b $\in$ L (absorption law)

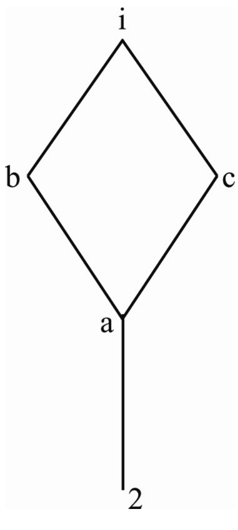
3.2.3 **Note**: The above axioms (i), (ii), (iii) holds with respect to $\vee$ also.

3.2.4 **Examples of Pre A^* - lattices:**

1. Let A be a Pre A^*-algebra and **2** = {0, 1}is a subset of A then **2** ={0,1}is a Pre A^* **-** lattice

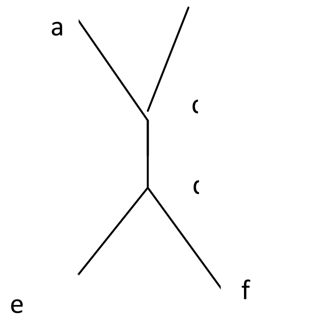
2. **3**= {0, 1, 2} is a subset of a Pre A^*-algebra then **3**= {0, 1, 2}is a Pre A^* **-**lattice

3. Fig(i),Fig(v)of chapter 2,Fig(vi) are examples of Pre A^* -lattices



Fig(vi)

4.Example of a poset which is shown in Fig(vii) is not a Pre A*-lattice:



fig(vii)

3.2.5 Properties of Pre A* -lattices:

(i)$a \wedge (b \wedge c) = (a \wedge b) \wedge c, \forall a, b, c \in L$ (associative)

(ii) $a \wedge b = b \wedge a, \forall a, b \in L$ (commutative)

(iii) $a \wedge (a \vee b) = a, \forall a, b \in L$ (absorption law)

(iv)$a \wedge a=a, \forall a\in L$ (idempotent)

3.2.6 Theorem: Let A be a Pre A∗–algebra. L is a subset of A Then (L, $\wedge$, $\vee$) is a Pre A* -lattice

Proof: Since A is a Pre A∗–algebra and L is a subset of A,

We have $a \wedge (b \wedge c) = (a \wedge b) \wedge c, \forall a, b, c \in A$ by1.2.1 (e)

$a \wedge b = b \wedge a, \forall a, b \in A$ by1.2.1 (c).

$a \wedge a = a, \forall a \in A$ by 1.2.1(b)

Therefore $a \wedge (b \wedge c) = (a \wedge b) \wedge c, \forall a, b, c \in L$,

$a \wedge b = b \wedge a, \forall a, b \in L$

$a \wedge a = a, \forall a \in L$

Hence (L, $\wedge$) is a semilattice.

Similarly we can prove (L, $\vee$) is a semilattice.

And since in a Pre A∗–algebra, $a \wedge (a \vee b) = a, \forall a \in A , b \in B(A)$

So, $a \wedge (a \vee b) = a$, $\forall$ a, b $\in$ L (absorption law).

Hence (L, $\wedge$, $\vee$) is a Pre A* -lattice.

3.2.7 Definition (Sub Pre A* -lattice): Let A be a Pre A* - algebra suppose L_1 be a subset a Pre A* **-** lattice L. We say L_1 is a sub Pre A* **-**lattice of L if L_1 itself is a Pre A* **-**lattice (with respect to the operations of L)

3.2.8 Example: 3= {0, 1, 2} is a Pre A* **-**lattice then 2={0,1}which is a subset of **3**= {0, 1, 2} is a sub Pre A***** **-**lattice.

3.2.9 Definition: (Bound elements in a Pre A* -lattice (L, ≤) and Bounded Pre A* -lattice L):

A Pre A* **-**lattice (L, ≤) is said to have a lower bound α if for any element a in L, we have $\alpha \leq a$. Analogously, L is said to have an upper bound β if for any a in L, we have $a \leq \beta$.

3.2.10 Example 1: For any subset L={0,1,2}**o**f a Pre A*-algebra with 1,0,2 which i**s** shown in fig(v) of chapter2,here 2 is the lower bound which is unique(least element)& 1 is the upper bound which is unique(greatest element) with respect to $\wedge$

Hence L= {0, 1, 2} is a bounded Pre A***** **-**lattice.

Example 2: The lattice shown in Fig(i) of chapter2 is the bounded Pre A* **-**lattice.

Here 2 is the lower bound for a, b, c which is the least element in this Pre A* -lattice and i is the upper bound for a, b, c which is the greatest element of this Pre A* -lattice.

3.2.11 Definition (Bounded Pre A* -Lattice (L, ≤)) : Let A be a Pre A* - algebra and L is a subset of A then we say that L is bounded Pre A* -lattice , $\forall$ a$\in$L if L has both unique greatest lower bound (least element)α and a unique least upper bound(greatest element) β

3.2.12 Example: The Hausse diagram shown in Fig(i) of chapter2, is the bounded Pre A*-lattice

3.2.13 Theorem: Let A be a Pre A* - algebra and L is a subset of A.Then every finite Pre A* **-**lattice L is bounded.

Proof: Let L = {a_1, a_2, ……. a_r} be a subset of a Pre A* - algebra with the binary operations $\wedge$, $\vee$ in L which is finite.

Since $\wedge(a_1,a_2)=a_1\wedge a_2=\inf\{a_1,a_2\}$ and since $\vee(a_1,a_2)=a_1\vee a_2=\sup\{a_1,a_2\}$,

for any $a_1,a_2\in A$

Then ($a_1 \vee a_2 \vee \ldots\ldots \vee a_n$) and ($a_1 \wedge a_2 \wedge \ldots\ldots\ a_n$) are upper bound and lower bound for L, let those be α, β respectively.

Hence L is Pre A* **-**lattice which is bounded in A.

3.3 Some kinds of Pre A*-lattices :

In this section we define some kinds of Pre A*-lattices like distributive Pre A*-lattice, modular Pre A*-lattice and we prove some theorems related with this.

3.3.1 Definition (Distributive Pre A* -lattice): Let A be a Pre A* – algebra and L is subset of A. Then a Pre A* -lattice (L, $\wedge$,$\vee$) is said to be distributive Pre A* -lattice if any elements a, b, c in L we have the distributive law

$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c), \forall\ a, b, c \in L$

3.3.2 Example of a distributive Pre A* -Lattice :

The chain shown in Fig(v) of chapter 2 is a distributive Pre A* -lattice

3.3.3 Theorem: Let A be a Pre A* - algebra and L is a subset of A which is Pre A* -lattice.Then L becomes a distributive Pre A* -Lattice

Proof : Since A is a Pre A*–algebra, and L is a subset of A which is a Pre A* -lattice

(by 3.2.6)and since distributive law holds in a Pre A*–algebra,

$a \wedge (b \vee c) = (a \wedge b)\vee (a\wedge c), \forall\ a, b, c \in L.$

Hence L becomes a distributive Pre A* -lattice.

3.3.4 Definition: (*-complement or complement of an element on a Pre A∗–algebra):

Let A be a Pre A∗ – algebra with the least element α and the greatest element β.

Then $a^{\sim} \in$ A is said to be complement of $a \in$ A if $a \wedge a^{\sim} = \alpha$, $a \vee a^{\sim} = \beta$ .Such a defined complement $a^{\sim}$ of the element a on a Pre A∗ – algebra is said to be the *-complement

3.3.5 Note: If α is least element with respect to $\vee$ is 0, β is greatest element with respect to $\wedge$ is 1 then $a \wedge a^{\sim} = \alpha$, $a \vee a^{\sim} = \beta$

therefore $0 \wedge 1 = 0$, $0 \vee 1 = 1$,so 0 is the complement of 1

Similarly $1 \wedge 0 = 0$, $1 \vee 0 = 1$, so 1 is the complement of 0

3.3.6 Note: Let A be a Pre A∗ – algebra. If α is the least element with respect to $\wedge$ i.e.,2, β is the greatest element with respect to $\vee$ i.e.,2 since $2 \wedge 2 = 2$, $2 \vee 2 = 2$

So 2 is the complement of 2

3.3.7 Definition(*-Complemented Pre A* -lattice):

Let A be a Pre A∗ – algebra then A is said to be ***-**complemented Pre A* -lattice if each element has a ***-**complement in it.

3.3.8 Example: The Pre A* -lattice shown in the fig(i) of chapter 2 is a ***-** complemented Pre A* -lattice

Here every element has a ***-**complement but these are not unique.

Here b,c are ***-**complements of a

Here a,c are ***-**complements of b

Here a,b are ***-**complements of c

3.3.9 Example: The fig**(ii)** of chapter2 is an example of a Pre A* -lattice which is not ***-** complemented

In fig(ii), the elements a,e c,d have ***-**complements but the element b has no ***-** complement.

3.3.10 Theorem: Let A be a Pre A*–algebra then for any subset L of A, a Pre A*- lattice L becomes a complemented distributive Pre A* -Lattice.

Proof: Since A is a Pre A*–algebra, then for any subset L of A, L is a distributive Pre - A* -lattice (by theorem 3.3.3). Since each element in L has complement in it.

Hence L is a complemented distributive Pre A* -lattice.

3.3.11 Lemma: In the Poset $(A, \leq)$, if $a \leq b \Rightarrow a \vee (b \wedge c) = b \wedge (a \vee c), \forall a, b, c \in A$

Proof : Define $\leq$ in A as $a \leq b \Leftrightarrow a \wedge b = a$ (i.e., $a \vee b = b$)

Suppose $a \leq b$ then $b \wedge a = a$.

Now $b \wedge (a \vee c) = (b \wedge a) \vee (b \wedge c) = a \vee (b \wedge c)$ (by 1.2.1 (f))

3.3.12 Definition (Modular Pre A* -Lattice): Let A be a Pre A* – algebra and L be subset of A. Then a Pre A* -lattice L is said to be a modular Pre A* -lattice if $a \leq b \Rightarrow$

$a \vee (b \wedge c) = b \wedge (a \vee c), \forall a,b,c \in L$

3.3.13 Example: The Pre A* -lattice shown in Fig (ii) of chapter 2, is the modular Pre A***** **-**lattice

Since $2 \leq a$**,** $2 \vee (a \wedge i) = a \wedge (2 \vee i)$.

3.3.14 Theorem: Let A be a Pre A^*–algebra. Then a sub set L of A is a modular Pre A^* - lattice

Proof: Since $(L, \leq)$is a Pre A^* -lattice. By lemma 3.3.12, if $a \leq b \Rightarrow a \vee (b \wedge c)$

$= b \wedge (a \vee c), \forall a, b, c \in L$. Hence L is a modular Pre A^* -lattice.

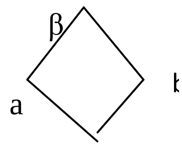
3.3.15 Definition (Unique *-complement of an element on a Pre A*–algebra):

Let A be a Pre A∗ – algebra. Then a∈ A is said to be a unique ***-**complement if a has exactly one ***-** complement in A.

3.3.16 Definition (Uniquely *-complemented Pre A* -lattice):

Let A be a Pre A∗ – algebra and L be a subset of A Then L is said to be uniquely ***-** complemented Pre A* -lattice if each element in L has unique ***-**complement in L.

3.3.17 Example: The Pre A*-lattice shown in fig (viii) is uniquely complemented



Fig(viii) α

3.3.18 Definition(Relative *-complement on a Pre A∗–algebra):

Let A be a Pre A∗ – algebra and L be a subset of A. Let [a, b]∈ L and u is an element of [a, b].An element $x^{\sim}$ of L is said to be relative ***-**complement of u in[a, b]

if $x \wedge x^{\sim} = a$, $x \vee x^{\sim} = b$

3.3.19 Note: If x is a relative ***-**complement of u in [a, b] then we have x ∈[a, b] and x is ***-**complement of u in[a, b]

3.3.20 Example: $0 \wedge 1 = 0$, $0 \vee 1 = 1$, so 0 is the relative ***-**complement in [0, 1]

$1 \wedge 0 = 0$, $1 \vee 0 = 1$,so 1 is the relative ***-**complement in [0,1]

Example: If a=2, b=2 since $2 \wedge 2 = 2$, $2 \vee 2 = 2$, so 2 is the relative ***-**complement of 2

3.3.21 Definition (Relatively *-complemented Pre A* -lattice):

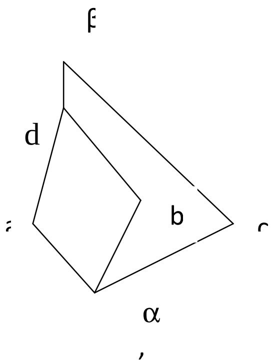
Let A be a Pre A∗ – algebra. Then a subset L of A is said to be relatively ***-** complemented Pre A* -lattice if for any triplet of elements a, b, u∈ L such that $a \le u \le b$ there exists at least one ***-**complement of u in [a,b]. That is every interval of L is a ***-** complemented Pre A* -lattice of A.

3.3.22 Example: The Pre A* -lattice shown in figure (i) of chapter 2 is an example of a Pre A* -lattice which is ***-**complemented as well as relatively ***-**complemented

3.3.23 Note: Let A be a Pre A* – algebra then every bounded relatively ***-**complemented Pre A* -lattice in A is ***-**complemented but converse is not true. That is a ***-**complemented Pre A* -lattice may or may not be relatively ***-**complemented Pre A* -lattice**.**

3.3.24 Example: The Pre A* -lattice shown in fig (iii) is ***-**complemented but not relatively ***-**complemented. Since [α, b], [a, β] are not ***-**complemented Pre A* -lattice since a has no ***-**complements in [α, b].

3.3.25 Example: The Pre A* -lattice shown in fig (iii) is not relatively ***-**complemented



Example of Pre A* -lattice which is not relatively *-complemented

Since [a, β] ={a, d, β} is not a ***-**complemented Pre A* -lattice since a has no ***-**complement hence it is not relatively ***-**complemented.

3.3.26 Definition (Section *-complemented Pre A* -lattice):

Let A be a Pre A* – algebra with least element α and L be a subset of A Then L is said to be section ***-**complemented Pre A* -lattice if every interval of the form [α, a] (a∈ L)is a ***-**complemented Pre A* -lattice of A.

That is for each pair of elements a, x with x ≤a there exists an element $x^{\sim} \in A$ such that $x \wedge x^{\sim} = \alpha$, $x \vee x^{\sim} = a$

3.3.27 Example: The Pre A* -lattice shown in this fig (iv) of chapter2 is not section ***-c**omplemented because [0, b] is a not ***-**complemented Pre A* -lattice

3.3.28 Example: The Fig. (i) of chapter2 is an example of a Pre A* -lattice which is section ***-**complemented

3.3.29 Theorem: Let A be a Pre A*–algebra. Then every relatively ***-**complemented Pre A* -lattice in A is Section ***-**complemented.

Proof: Since A is a Pre A*–algebra, if L is a relatively ***-**complemented Pre A* -lattice then by the definition L every interval of A is a ***-**complemented Pre A* -lattice of A.

Hence L is a section ***-**complemented Pre A* -lattice .

3.3.30 Note: Let A be a Pre A* – algebra every relatively ***-**complemented Pre A* -lattice in A is section ***-**complemented but converse is not true.

For example, in Fig(iii) of chapter2 is section ***-**complemented but not relatively ***-**complemented

Since [a, β] ={a, d, β} is not a ***-**complemented Pre A* -lattice since a has no ***-**complement hence it is not relatively ***-**complemented.

3.3.31 Definition(Semi-*-complemented Pre A* -lattice): Let A be a Pre A* – algebra with least element α and L be a subset of A.Then L is said to be semi-***-**complemented Pre A* -lattice if every inner element (other than least and greatest elements in A) has at least one proper semi-***-**complement.

3.3.32 Definition (Weakly *-complemented Pre A* -lattice on a Pre A*–algebra): Let A be a Pre A* – algebra with least element α and L be a subset of A Then L is said to be weakly ***-**complemented Pre A* -lattice if any pair of elements a, b (a < b) of L a has semi ***-**complement, that is however not a semi ***-**complement of b. That is $x^{\sim}$ is semi-***-**complement of a but not semi- ***-**complement of b.

3.3.33 Example: 2.2.22 of chapter2 is an example of a weakly ***-**complemented Pre A* -lattice.

Example: In the fig(B)of chapter 2, b is semi-***-**complemement of a however c is not semi-***-**complemement of a

Similarly a is semi-***-**complemement of b however c is not semi-***-**complemement of b

Hence it is weakly ***-**complemented

3.3.34 Example: The Pre A* -lattice shown in fig (iv) of chapter2 is example of a Pre A* -lattice which is not weakly ***-**complemented.

This is not weakly ***-**complemented since a<b and c is semi ***-**complement of

both a and b.

3.3.35 Theorem: Let A be a Pre A*–algebra.Then every weakly ***-**complemented Pre A* -lattice in A is semi-***-**complemented.

Proof: Let A be a Pre A*-algebra with least element α ,greatest element β

Let L be any weakly ***-**complemented Pre A* -lattice in A

Claim: L is semi-***-**complemented

Let $a \in L$ be an inner element

i.e.,$a \neq \alpha$,$a \neq \beta$

$\Rightarrow$ a is not a maximal element

$\Rightarrow \exists b \in L$ such that a<b

Since L is weakly ***-**complemented Pre A* -lattice in A,we have that there exists semi-complement x of a which is not a semi-***-**complement of b

i.e., $a \wedge x = \alpha \Rightarrow b \wedge x \neq \alpha$

then x is proper semi-***-**complement of a and hence L is semi-***-**complemented

Hence every weakly ***-**complemented semilattice in A is semi-***-**complemented

3.3.36 Theorem: Let A be a Pre A*–algebra with least element α .Then every section ***-**complemented Pre A* -lattice in A is weakly ***-**complemented.

Proof: Let A be a Pre A*-algebra with least element α

and let L be section ***-**complemented Pre A* -lattice in A

Claim: L is weakly ***-**complemented

Let $a,b \in L$ such that $a < b$

Now $[\alpha, b]$ is a ***-**complemented Pre A* -lattice of L

Since L is section ***-**complemented and $a \in [\alpha, b] \Rightarrow \exists\, x \in L$ such that

$a \wedge x = \alpha, a \vee x = b$

consider $b \wedge x = (a \vee x) \wedge x$

$= x$ (by absorption law in A)

When $x = \alpha$,

$b = a \vee \alpha$

$\Rightarrow b = a \vee (a \wedge x)$

$= a$

Therefore b=a which is a contradiction to $a < b$

Thus $x \neq \alpha$

Hence L is weakly ***-**complemented Pre A* -lattice

3.3.37 Theorem: Let A be a Pre A∗-algebra with least element α , greatest element β. Then every uniquely ***-**complemented Pre A* -lattice in A is weakly ***-**complemented

Proof: Let A be a Pre A∗-algebra with least element α , greatest element β and L be any uniquely ***-**complemented Pre A* -lattice in A

Claim: L is weakly ***-**complemented

Let a, b∈L such that there exists unique ***-**complement **a**~ of a such that

$a \wedge a^{\sim} = \alpha$, $a \vee a^{\sim} = \beta$

i.e., $a^{\sim}$ is semi-***-**complement of a since a < b we have $b \vee a^{\sim} > a \vee a^{\sim} = \beta$

$\Rightarrow b \vee a^{\sim} > \beta$

Since β is greatest element in A, $\beta > b \vee a^{\sim}$

$\Rightarrow b \vee a^{\sim} = \beta$

$b \wedge a^{\sim} \neq \alpha$ (suppose if $b \wedge a^{\sim} = \alpha$ then $a^{\sim}$ is ***-**complement of both a and b

which is a contradiction to our assumption that L is uniquely ***-**complemented Pre A* - lattice).

Therefore $a \wedge a^{\sim} = \alpha$, $b \wedge a^{\sim} \neq \alpha$

Hence L is weakly ***-**complemented Pre A* -lattice in A

3.4 Atoms and dual atoms for Pre A*-lattice on a Pre A* - algebra:

In this section first we define least and greatest elements in a Pre A* -lattice**,** we define atoms, dual atoms, irreducible elements in Pre A* - algebra and we prove some theorems in these. We define atomic, dual atomic Pre A*- lattice. We give **s**ome examples of these. We define Pre A* - homomorphism and we prove a theorem on atoms in Pre A∗-Algebra also we prove f: A→P (B) is an isomorphism.

3.4.1 Definition of least and greatest elements on a Pre A* -lattice: Let A be a Pre A* – algebra and L be any Pre A* -lattice in A. An element $\alpha \in$ L is called least element if $\alpha \leq x, \ \forall x \in L$. Similarly $\beta \in$ L is called greatest element if $x \leq \beta, \ \forall x \in L$.

3.4.2 Definition of atom for Pre A*-lattice: Let L be a subset of a Pre A* - algebra A. Then an element p of a bounded below Pre A*-lattice L with least element α is called an atom, if $\alpha \prec p$ (α is covered by p).

If there exists an atom p, for each element $a \neq \alpha$ of L such that $p \leq a$. Then we say that L is atomic Pre A* -lattice

That is, in a Pre A* - lattice,if (S, $\wedge$) is a semilattice with least element 2,then 2 is atom with respect to $\wedge$ if $2 \prec p$

if (S, $\vee$) is a semilattice with least element 0,then 0 is atom with respect to $\vee$

if $0 \prec p$.

3.4.3 Example: In the Pre A* -lattice shown in fig (i) of chapter2, a, b, c are atoms and this is atomic.

In the Pre A* -lattice shown in fig (vi) of chapter2 a is the only one atom and this Pre A*- lattice is also atomic**.**

In the Pre A* -lattice shown in fig (ii) of chapter2 a, c are atoms and this Pre A* - lattice is also atomic.

3.4.4 Theorem: Let A be a Pre A*-algebra and L be a subset of A then every finite Pre A* -lattice, which is bounded below is atomic.

Proof: Let L be a subset of a Pre A* - algebra A.

Then an element p of a bounded below Pre A* -lattice L with least element α is called an atom, if $\alpha \prec p$ (α is covered by p).

Since in a Pre A* - algebra A,if (S, $\wedge$) is a semilattice with least element 2,then 2 is atom with respect to $\wedge$ if $2 \prec p$

If there exists an atom p, for each element a ≠ α of S such that p≤a. Then we say that L is atomic

It is true for every such a finite Pre A* -lattice L

3.4.5 Definition (Dual atom for Pre A* -lattice): Let L be a subset of a Pre A* - algebra A. Then an element p of a bounded above Pre A* -lattice with greatest element β is called dual atom if if q $\prec$ β (q is covered by β)

If there exists a dual atom q for any element a ≠ β of L such that a ≤ q. Then we say that L is dual atomic Pre A* -lattice

3.4.6 Note: That is, in a Pre A* - lattice, if (S, ∨) is a semilattice with greatest element 2,then 2 is dual atom with respect to ∨ if 2 $\prec$ p

That is, in a Pre A* - lattice, if (S, ∧) is a semilattice with greatest element 1,then 1 is dual atom with respect to ∧ if 1 $\prec$ p

3.4.7 Example: Consider the diagrams: fig (i), fig (vi), fig (ii) of chapter 2

In fig (i), a, b, c are dual atoms; In fig (vi), b, c are dual atoms.

In fig (ii), d, e, b are the dual atoms. All these are dual atomic Pre A* -lattices.

3.4.8 Definition (Join irreducible elements for Pre A* - lattice): Let A be a Pre A*-algebra and Lbe a subset of A which is Pre A* -lattice in A with a lower bound α. An element a in L is said to be join irreducible if a = x ∨ y ⇒ a = x or a = y.

3.4.9 Example 1: 2 is join irreducible in a Pre A* - lattice.

Example 2: In Fig(v) of chapter2, every element in this chain is join irreducible.

3.4.10 Theorem: Let A be a Pre A*-algebra and L be a subset of A. Then in a finite Pre A* -lattice L if a∈ L then we can write a as the join of irredundant join irreducible elements.

Proof: Let L be a subset of a Pre A*-algebra A.

Assume that L be a finite Pre A* -lattice. Let H be the set of all elements of L which cannot be represented as the join of finite number of irredundant join irreducible elements.

Now we will show that H is empty.

Suppose if possible H is non-empty.

Then H does not contain any irredundant join irreducible elements, since if a is join irreducible element and $a \in H$ then $a = a \vee a$ and $a = a \vee \alpha$ (if α, $a \in H$) are two representations of the element a, which is contradicting the definition of H. Hence every element $a \in H$ is the join of finite number of join irreducible elements.

Since H is finite, then the set H contains at least one minimal element, say m.

Clearly m cannot be join irreducible.

So $m = m_1 \vee m_2$ where $m_1, m_2 \in L$ and $m_1, m_2 < m$

Since $m_1, m_2 < m$ we have m_1, m_2 does not in H.

So m_1, m_2 can be represented as $m_1 = q_1 \vee q_2 \vee \ldots \vee q_k$;

$m_2 = p_1 \vee p_2 \vee \ldots \vee p_l$ where each p_j, q_j are join irreducible elements and $p_j < m_2, q_j < m_1$.

Now $m = m_1 \vee m_2$

$$= \left(\sum_{j=1}^{k} q_j\right) \vee \left(\sum_{i=1}^{k} p_i\right)$$ which is a contradiction to $m \in H$

Hence H is empty.

Therefore in a finite Pre A* -lattice L if $a \in L$ then we can write a as the join of irredundant join irreducible elements.

3.4.11 Theorem: Let A be a Pre A* - algebra and L be a finite Pre A* -lattice with least element α .Then each $x \neq \alpha$ in L can be written uniquely as the join of atoms.

Proof: Let A be a finite Pre A* – algebra and L be a finite Pre A* -lattice.

Recall that an element a in a bounded below Pre A* -lattice L with least element α is called an atom, if $\alpha \prec a$ (α is covered by a).

Let B be the set of atoms of A and let P (B) be the Pre A* – algebra of all subsets of the set B of atoms.

Then by theorem 3.4.10, each $x \neq \alpha$ in L can be expressed uniquely as the join irreducible elements and since the join irreducible elements are atoms, i.e., elements of B.

Say $x = a_1 \vee a_2 \vee \ldots\ldots\ldots \vee a_r$

3.4.12 Theorem: Let L be a subset of a Pre A* - algebra A and L is finite distributive Pre A* -lattice. Then every element a in L can be written uniquely as the join of irredundant join irreducible elements.

Proof: Let a subset L of A be a finite distributive Pre A* -lattice.

Since L is finite we can write a as the join of irredundant join irreducible elements (By Theorem 3.4.10). Thus we need to prove uniqueness.

Suppose $a = b_1 \vee b_2 \vee \ldots\ldots \vee b_r = c_1 \vee c_2 \vee \ldots\ldots \vee c_s$. where the b's are irredundant and join irreducible and the c's are irredundant and join irreducible.

For any given i we have

$$b_i \leq (b_1 \vee b_2 \vee ,.. \vee b_r) = (c_1 \vee c_2 \vee \ldots\ldots \vee c_s).$$

Hence $b_i = b_i \wedge (c_1 \vee c_2 \vee \ldots\ldots \vee c_s) = (b_i \wedge c_1) \vee (b_i \wedge c_2) \vee \ldots\ldots \vee (b_i \wedge c_s)$

(Since L is distributive)

Since bi is join irreducible, there exists j such that $b_i = b_i \wedge c_j$, and so $b_i \leq c_j$.

By a similar argument, for c_j there exists b_k such that $c_j \leq b_k$.

Therefore $b_i \leq c_j \leq b_k$ which gives $b_i = c_j = b_k$ since the b's are irredundant

Thus the representation for a is unique.

3.4.13 Definition: Meet irreducible elements for Pre A* - lattice:

Let A be the Pre A* - algebra and a subset L of A be a Pre A* -lattice in A with an upper bound β. An element a in L is said to be meet irreducible

if $a = x \wedge y \Rightarrow a = x$ or $a = y$.

3.4.14 Example: 2 is meet irreducible in a Pre A* - lattice.

Example: In Fig(v) of chapter2**,**

Every element in this chain is meet irreducible

3.4.15 Theorem: Let A be a Pre A*-algebra and L be a subset of A. Then in a finite Pre A* -lattice L if $a \in L$ then we can write a as the meet of irredundant meet irreducible elements.

Proof: Let L be a subset of a Pre A*-algebra A.

Assume that L be a finite Pre A* -lattice.

Let H be the set of all elements of L which cannot be represented as the meet of finite number of irredundant meet irreducible elements.

Now we will show that H is empty.

Suppose if possible H is non-empty.

Then H does not contain any irredundant meet irreducible elements, since if a is meet irreducible element and $a \in H$ then $a = a \wedge a$ and $a = a \wedge \beta$ (if $\beta, a \in H$) are two representations of the element a, which is contradicting the definition of H.

Hence every element $a \in H$ is the meet of finite number of meet irreducible elements.

Since H is finite, then the set H contains at least one maximal element, say m

Clearly m cannot be meet irreducible

So $m = m_1 \wedge m_2$ where $m_1, m_2 \in L$ and $m_1, m_2 > m$

Since $m_1, m_2 > m$ we have m_1, m_2 does not in H

So m_1, m_2 can be represented as

$m_1 = q_1 \wedge q_2 \wedge \ldots \wedge q_k$

$m_2 = p_1 \wedge p_2 \wedge \ldots \wedge p_l$ where each p_j, q_j are meet irreducible elements and $p_j > m_2$, $q_j > m_1$

Now $m = m_1 \wedge m_2$

$$= \left(\bigcap_{j=1}^{k} q_j\right) \wedge \left(\bigcap_{i=1}^{k} p_i\right)$$ which is a contradiction to $m \in H$

Hence H is empty.

Therefore in a finite lattice L if $a \in L$ then we can write a as the meet of irredundant meet irreducible elements.

3.4.16 Theorem: Let A be a Pre A* - algebra and L be a finite Pre A* -lattice with greatest element β in L**.** Then each $x \neq \beta$ in L can be written uniquely as the meet of dual atoms.

Proof : Let A be a finite Pre A* – algebra and L be a finite Pre A* -lattice.

Then an element p of a bounded above Pre A* -lattice with greatest element β is called dual atom if if $q \prec \beta$ (q is covered by β)

If there exists a dual atom q for any element $a \neq \beta$ of L such that $a \leq q$. Then we say that L is dual atomic Pre A* -lattice

Let B be the set of dual atoms of L and let P (B) be the Pre A∗ – algebra of all subsets of the set B of dual atoms.

Then by theorem 3.4.15, each $x \neq \beta$ in L can be expressed uniquely as the meet of irreducible elements and since the meet of irreducible elements are dual atoms, i.e., elements of B.

Say $x = a_1 \wedge a_2 \wedge \ldots\ldots\ldots \wedge a_r$

3.4.17 Theorem: Let L be a subset of a Pre A* - algebra A and L is finite distributive Pre A* -lattice. Then every element a in L can be written uniquely as the meet of irredundant meet irreducible elements.

Proof: Let a subset L of A be a finite distributive Pre A* -lattice.

Since L is finite we can write a as the meet of irredundant meet irreducible elements (By Theorem 3.4.15). Thus we need to prove uniqueness.

Suppose $a = b_1 \wedge b_2 \wedge \ldots\ldots \wedge b_r = c_1 \wedge c_2 \wedge \ldots\ldots \wedge c_s$. where the b's are irredundant and meet irreducible and the c's are irredundant and meet irreducible

For any given i we have $b_i \leq b_1 \wedge b_2 \wedge \ldots\ldots \wedge b_r = c_1 \wedge c_2 \wedge \ldots\ldots \wedge c_s$.

Hence $bi = b_i \vee (c_1 \wedge c_2 \wedge \ldots\ldots \wedge c_{s)}$

$= (b_i \vee c1) \wedge (b_i \vee c_2) \wedge \ldots. \wedge (b_i \vee c_s)$ (Since L is distributive)

Since b_i is join irreducible, there exists j such that $b_i = b_i \wedge c_j$, and so $b_i \leq c_j$.

By a similar argument, for c_j there exists b_k such that $c_j \leq b_k$.

Therefore $b_i \leq c_j \leq b_k$ which gives $b_i = c_j = b_k$ since the b's are irredundant

Thus the representation for a is unique.

3.4.18 Definition (Pre A∗ – Homomorphism): Let (A_1, ∧, ∨, (-)˜) and (A_2, ∧, ∨, (-)˜) be two Pre A∗ – algebras. A mapping f: $A_1 \to A_2$ is called an Pre A∗ – homomorphism, if

(i) f (a ∧ b) = f (a) ∧ f (b)

(ii) f (a ∨ b) = f (a) ∨ f (b)

(iii) f(a˜) = (f(a)) ˜

The homomorphism f : $A_1 \to A_2$ is onto, then f is called epimorphism.

The homorphism f : $A_1 \to A_2$ is one–one, then f is called monomorphism.

The homomorphism f : $A_1 \to A_2$ is one-one and onto then f is called an isomorphism,

and A_1, A_2 are isomorphic, denoted by $A_1 \cong A_2$.

3.4.19 Theorem: Let A be a Pre A* - algebra and let B be the set of atoms (dual atoms)of A and let P (B) be the Pre A∗ – algebra of all subsets of the set B of atoms(dual atoms).

The mapping f : A → P(B) is an isomorphism.

Proof: Consider the mapping f : A → P(B) defined by f(x) = $\{a_1, a_2, \ldots\ldots a_r\}$.

The mapping is well defined since the representation is unique (by 3.4.11 or 3.4.16)

To verify that f is a homomorphism:

Since the mapping f: A → P (B) defined by

f (x) = $\{a_1, a_2, \ldots\ldots.. a_r\}$,

f (y) = $\{b_1, b_2, \ldots\ldots.. b_r\}$,

(i) f (x ∧ y) = $\{a_1 \wedge b_1, a_2 \wedge b_2, \ldots\ldots\ldots, a_r \wedge b_r\}$

=$\{a_1, a_2, \ldots\ldots\ldots\ldots\ldots.. a_r\} \wedge \{b_1, b_2, \ldots\ldots\ldots\ldots\ldots.. b_r\}$

=f (x) ∧ f (y)

Therefore f (x $\wedge$ y) =f (x) $\wedge$ f (y)

(ii) f (x $\vee$ y) = {$a_1 \vee b_1$, $a_2 \vee b_2$, ………, $a_r \vee b_r$}

={a_1, a_2, ……………..a_r} $\vee$ {b_1, b_2, ……………..b_r}

= f (x) $\vee$ f (y)

Therefore f (x $\vee$ y) = f (x) $\vee$ f (y)

(iii) Consider f($x^{\sim}$) = {$a_1^{\sim}$, $a_2^{\sim}$, ……………..$a_r^{\sim}$}= {a_1, a_2, ……………..a_r}$^{\sim}$

=$[f(x)]^{\sim}$

Therefore f($x^{\sim}$)=$[f(x)]^{\sim}$

Therefore f is a Pre A* - homomorphism.

Since f is one-one and onto.

Hence f is a Pre A* - isomorphism.

3.4.20 Corollary: A finite Pre A∗ – algebra has 3^n elements for some positive integer n.

Proof: If a set B has n elements, then its power set P (B) be the Pre A∗ – algebra of all subsets of the set B has 3^n elements.

Then by theorem 3.4.19 a finite Pre A∗ – algebra has 3^n elements for some positive integer n.

Conclusion: In this chapter, we defined the lattice on Pre A*-algebra and we call such a defined lattice on Pre A*-algebra as Pre A*-lattice and we established the properties of the Lattice on Pre A*-algebra. We defined greatest lower bound and least upper bound on Pre A*-algebra. We observed that 2 acts as greatest lower bound(g.l.b) with respect to the binary operation meet whereas 2 acts as least upper bound(l.u.b) with respect to the binary operation join. We defined atoms, dual atoms, irreducible elements with respect to meet as well as join on Pre A*-algebra. We obtained various theorems on these atoms, dual atoms, irreducible elements on Pre A*-algebra. We established the atomic lattices, dual atomic lattices on Pre A*-algebra. We proved that the mapping $f : A \to P(B)$(where a set B has n elements and its power set P (B)) is an isomorphism and a finite Pre A* – algebra has 3^n elements for some positive integer n.