Chapter 2

PRE A∗-ALGEBRA AS A SEMILATTICE

This Chapter describes the concept of *Pre A*-algebra as a semilattice.* In this Chapter, we define semilattice on a Pre A*-algebra with respect to the binary operation $\wedge$ (meet) and as well as $\vee$ (join) and obtain the properties of semilattice on a Pre A*-algebra. We establish Pre A*-algebra as a semilattice. We prove necessary conditions for a semilattice to become a lattice with respect to meet and as well, as join. We define greatest lower bound of an element on Pre A* - algebra and least upper bound of an element on Pre A* - algebra and we provide examples of these. We define semi-*-complement for semilattice on Pre A* - algebra and we prove some theorems on these. We define atoms, dual atoms, irreducible elements with respect to meet as well as join for semilattice on Pre A*-algebra. We obtain various theorems on these atoms, dual atoms, irreducible elements for semilattice on Pre A*-algebra. We establish the atomic, dual atomic semilattices *on* Pre A*-algebra.

This chapter consists of two sections. In the first section we recall the definition of partial ordering $\leq$ on Pre A∗-algebra. We recall if A is a Pre A*-algebra then $(A, \leq)$ is a poset. We define a semilattice on Pre A*-algebra. We prove Pre A*-algebra as a semilattice. Next we prove some theorems on semilattice over a Pre A*-algebra.

In the second section we describe greatest lower bound of an element in Pre A* - algebra and least upper bound of an element in Pre A* - algebra and we provide examples of these. We define semi-*-complement for semilattice on Pre A* - algebra and we prove some theorems on these. We define atoms, dual atoms, irreducible elements with respect to meet as well as join for semilattice on Pre A*-algebra. We obtain various theorems on these atoms, dual atoms, irreducible elements for semilattice on Pre A*-algebra. We establish the atomic, dual atomic semilattices on Pre A*-algebra.

2.1 Pre A* - algebra as a semilattice

2.1.1 Definition [9]: Let A be a Pre A* - algebra. Define the binary operation $\leq$ on A by $x \leq y$ if and only if $x \wedge y = y \wedge x = x$, for all $x, y \in A$. The defined operation $\leq$ is said to be partial ordering on Pre A* - algebra A.

2.1.2 Lemma [9]: If A is a Pre A* - algebra then $(A, \leq)$ is a Poset.

2.1.3 (a)Definition (Semi lattice($S, \wedge(\vee)$ **on a Pre A* - algebra) :** A non-empty subset S of a Pre A*- algebra A is said to be a semilattice if **(**$S, \leq$) is an ordered set in which every pair of elements has a greatest lower bound,(g.l.b) (least upper bound,(l.u.b)) (This will be proved in Theorem 2.1.8)

(b) Alternate Definition(Semi lattice($S, \wedge(\vee)$ **on a Pre A* - algebra) :** A non-empty subset S of a Pre A*- algebra A equipped with a binary operation $\wedge(\vee)$ is said to be a semi lattice, if the following semi lattice axioms are satisfied.

(i) $\wedge(\vee)$ is associative i.e. $a \wedge (b \wedge c) = (a \wedge b) \wedge c, \forall a, b, c \in S$

(ii) $\wedge(\vee)$ is commutative i.e., $a \wedge b = b \wedge a, \forall a, b \in S$

(iii) $\wedge(\vee)$ is idempotent i.e., $a \wedge a = a, \forall a \in S$

2.1.4 Theorem: In Pre A$*$ – algebra A, $(S, \wedge)$ & $(S, \vee)$ are semilattices.

Proof: In Pre A*-algebra, $a \wedge (b \wedge c) = (a \wedge b) \wedge c, \forall a, b, c \in A$ (1.2.1(e))

$a \wedge b = b \wedge a, \forall a, b \in A$ (1.1(c)) and $a \wedge a = a, \forall a \in A$ (1.2.1 (b)).

Hence $(S, \wedge)$ is a semi lattice. By the duality in A $(S, \vee)$ is a semilattice.

2.1.5 Theorem: In a Pre A*-algebra A, the class of semilattices can be equationally defined as the class of all semi group satisfying the commutative and idempotent laws.

Proof: Let (S, $\wedge$,($\vee$)) be a semi lattice on a Pre A* - algebra A. By the definition of semi lattice we have $\wedge$($\vee$) is associative

i.e. $a \wedge (b \wedge c) = (a \wedge b) \wedge c$, $\forall$ a, b, c $\in$ S

(ii) $\wedge$($\vee$) is commutative

i.e., $a \wedge b = b \wedge a$, $\forall$ a, b $\in$ S

(iii) $\wedge$($\vee$) is idempotent

i.e., $a \wedge a = a$, $\forall$ a $\in$ S.

Hence (S, $\wedge$) as well as (S, $\vee$) is a semigroup satisfying commutative and idempotent laws.

Therefore (S, $\wedge$, ($\vee$)) is a semigroup satisfying the commutative and idempotent laws

Converse:

Suppose (S,$\wedge$) as well as (S,$\vee$) is a semi-group satisfying commutative and idempotent laws.

By the definition of Pre A$*$ – algebra A, we have

$a \wedge (b \wedge c) = (a \wedge b) \wedge c$, $\forall$ a, b, c $\in$ A (1.2.1(e)) ; $a \wedge b = b \wedge a$, $\forall$ a, b $\in$ A (1.2.1(c))

and $a \wedge a = a$, $\forall$ a $\in$ A (1.2.1 (b)).

Hence $\wedge$($\vee$) is associative, commutative and idempotent.

Hence (S, $\wedge$($\vee$)) is a Semilattice on a Pre A*-algebra A.

2.1.6 Theorem (Pre A* - algebra as a semilattice): Let A be a Pre A$*$–algebra.Then A is a semilattice.

Proof: Since A is a Pre A$*$–algebra, $a \wedge (b \wedge c) = (a \wedge b) \wedge c$, $\forall$ a, b, c $\in$ A by1.2.1 (e)

$a \wedge b = b \wedge a$, $\forall$ a, b $\in$ A by1.2.1 (c). Hence $a \wedge a = a$, $\forall$ a $\in$ A by 1.2.1(b).

Hence A is a semilattice.

2.1.7 Note 1: We can define Pre A* - algebra in terms of semilattice as follows:

An algebra $(A, \wedge, \vee, \tilde{})$ is said to be Pre A* - algebra where A is non-empty set with 1 and $\wedge, \vee$ are binary operations ~ is a unary operation satisfying:

(i) $A, \wedge)$ is a semilattice

(ii) $x^{\sim\sim} = x, \forall, x \in A$,

(iii) $(x \wedge y)^{\sim} = x^{\sim} \vee y^{\sim}, \forall\ x, y \in A$

(iv) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \forall\ x,y,z \in A$

(v) $x \wedge y = x \wedge (x^{\sim} \vee y), \forall x, y, z \in A$

Note 2: We can define Pre A* - algebra in terms of semilattice with respect to $\vee$ also.

2.1.8 Theorem: Let A be a Pre A∗ – algebra. In a semilattice S of a Pre A∗ – algebra A, define

$x \leq y$ if and only if $x \wedge y = x$. Then $(S, \leq)$ is an ordered set in which every pair of elements has a greatest lower bound.

Conversely, given an ordered set S with that property, define $x \wedge y$ = g.l.b. (x, y).

Then $(S, \wedge)$ is a semilattice.

Proof: Let $(S, \wedge)$ be a semilattice, and define $\leq$ as above.

First we check that $\leq$ is

a partial order.

(1) $x \wedge x = x$ implies $x \leq x$.

(2) If $x \leq y$ and $y \leq x$, then $x = x \wedge y = y \wedge x = y$.

(3) If $x \leq y \leq z$, then $x \wedge z = (x \wedge y) \wedge z = x \wedge (y \wedge z) = x \wedge y = x$, so $x \leq z$.

Since $(x \wedge y) \wedge x = x \wedge (x \wedge y) = (x \wedge x) \wedge y = x \wedge y)$, we have $x \wedge y \leq x$

similarly $x \wedge y \leq y$. Thus $x \wedge y$ is a lower bound for $\{x, y\}$.

To see that it is the greatest lower bound, suppose $z \leq x$ and $z \leq y$.

Then $z \wedge (x \wedge y) = (z \wedge x) \wedge y = z \wedge y = z$, so $z \leq x \wedge y$

Converse: Suppose $(S, \leq)$ is an ordered set.

Define $x \wedge y$ = g.l.b. (x, y).

Since $(S, \leq)$ is an ordered set,

(i) $x \leq x$ implies $x \wedge x = x$

(ii) $x \leq y$ and $y \leq x$, then $x \wedge y = y \wedge x$.

(iii) $z \leq x$ and $z \leq y$ implies $z \wedge (x \wedge y) = (z \wedge x) \wedge y$.

Hence $(S, \wedge)$ is a semilattice.

Note: We can prove the above with respect to $\vee$ also. Then the theorem is as follows:

2.1.9 Theorem: Let A be a Pre A$*$ – algebra. In a semilattice S of a Pre A$*$ – algebra A, define

$x \leq y$ if and only if $x \vee y = y$. Then $(S, \leq)$ is an ordered set in which every pair of elements has a least upper bound.

Conversely, given an ordered set S with that property, define $x \vee y$ = l.u.b (x, y).

Then $(S, \vee)$ is a semilattice.

Proof: By duality of the above theorem, we can prove this theorem.

2.1.10Theorem: Let A be a Pre A$*$ – algebra. In a semilattice S of a Pre A$*$ – algebra A, define $x \leq y$ if and only if $x \wedge y = x$ ($x \vee y = y$) Then $(S, \leq)$ is an ordered set in which every pair of elements has a greatest lower bound (least upper bound).

Conversely, given an ordered set S with that property, define $x \wedge y$ = g.l.b. (x, y)($x \vee y$ = l.u.b (x, y))

Then (S, $\wedge$) ((S, $\vee$)) is a semilattice and hence (S, $\wedge$,$\vee$) is a lattice.

Proof: By theorems 2.1.8 & 2.1.9

2.2 Atoms, dual atoms for semilattice on Pre A*-algebra:

In this section we describe greatest lower bound of an element in Pre A* - algebra and least upper bound of an element in Pre A* - algebra and we provide examples of these. We define semi-*-complement for semilattice on Pre A* - algebra and we prove some theorems on these. We define atoms, dual atoms, irreducible elements with respect to meet as well as join for semilattice on Pre A*-algebra. We obtain various theorems on these atoms, dual atoms, irreducible elements for semilattice on Pre A*-algebra. We establish the atomic, dual atomic semilattices *on* Pre A*-algebra.

2.2.1 Definition of a greatest lower bound of an element on Pre A^* - algebra:

Let A be a Pre A^* - algebra. An element a$\in$ A is said to be a lower bound if $a \leq x$, $\forall$ x $\in$ A. And a is said to be a greatest lower bound if there exists a lower bound b such that b$\leq$ a. The greatest lower bound (g.l.b) of the elements a, b is denoted by $a \wedge b$.

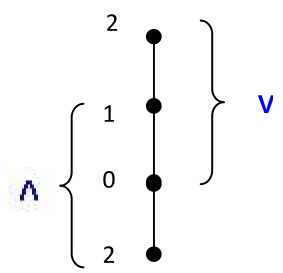
2.2.2 Example: Since in a Pre A^* - algebra, $0 \wedge 1 = 0$ so 0 is g.l.b of {0, 1}. Also since $2 \wedge 0 = 2$ so 2 is g.l.b of {0, 2}. Hence in a Pre A^* - algebra, 2 if exists then it is the g.l.b.

2.2.3 Definition of least upper bound of an element on Pre A^* - algebra:

Let A be a Pre A^* - algebra. An element a$\in$ A is said to be an upper bound if $x \leq a$, for all x $\in$ A. And a is said to be a least upper bound if there exists an upper bound b such that $a \leq b$. The least upper bound (l.u.b) of the elements a, b is denoted by $a \vee b$.

2.2.4 Example: Since in a Pre A^* - algebra, $0 \vee 1 = 1$. So 1 is l.u.b of {0, 1}. Also since $2 \vee 0 = 2$, 2 is l.u.b of {0, 2}. Hence in a Pre A^* - algebra, 2 if exists then it is the l.u.b.

2.2.5 Note: Since in a Pre A*–algebra. we have



Observe that element 2 acts as the least element with respect to “meet”, whereas the same element (2) acts as the greatest element with respect to “join”

2.2.6 Definition of least element on a (S, $\wedge$) semilattice: Let A be a Pre A$*$ – algebra and (S, $\wedge$) be any semilattice on A. An element $\alpha \in$ S is called least element if $\alpha \leq$ x, $\forall$ x $\in$ S

2.2.7 Note: The least element on a Pre A$*$–algebra with respect to $\wedge$ is 2

The least element on a Pre A$*$–algebra with respect to $\vee$ is 0

2.2.8 Definition of greatest element on a (S, $\vee$) semilattice: Let A be a Pre A$*$ – algebra and (S, $\vee$) be any semilattice on A. An element $\beta \in$ S is called greatest element if x $\leq \beta$, $\forall$ x $\in$ S

2.2.9 Note: The greatest element on a Pre A$*$–algebra with respect to $\wedge$ is 1

The greatest element on a Pre A$*$–algebra with respect to $\vee$ is 2

2.2.10 Note: In a Pre A$*$ – algebra,

the least element with respect to $\wedge$ is 2, least element with respect to $\vee$ is 0

and greatest element with respect to $\wedge$ is 1, greatest element with respect to $\vee$ is 2 .

α may be 0 or 2 i.e., $\alpha \in [2,1)$

β may be 1 or 2 $\beta \in (0,2]$

2.2.11 Note: We define semi-*****-complement for semilattice on Pre A$*$ – algebra

2.2.12 Definition (Semi-*-complement of an element on a Pre A$*$–algebra):

Let A be a Pre A$*$ – algebra with the least element α and x$\in$ A. An element $x^{\sim} \in$ A is said to be semi-*****-complement of x if $x \wedge x^{\sim} = \alpha$

2.2.13 Example: Let A be a Pre $A*$ – algebra with least element with respect to $\vee$ is 0 and

since $0 \wedge 1 = 0$,so 1 is semi-*****-complement of 0

Since$1 \wedge 0 = 0$, 0 is the semi-*****-complement of 1

2.2.14 Example: Let A be a Pre $A*$ – algebra with least element with respect to $\wedge$ is 2 and

$2 \in A$, since $2 \wedge 2 = 2$, hence 2 is semi-*****-complement of 2

2.2.15 Definition: Let A be a Pre $A*$ – algebra with least element is α. The semi-*****-complements of an element other than the least element α is called a proper semi-*****-complement in A. If in addition the proper semi-*****-complement is maximal then it is called maximal proper semi- *****-complement in A.

2.2.16 Example 1: In a Pre A^{*} – algebra with least element with respect to $\vee$,

since $0 \wedge 1 = 0$, $1 \wedge 0 = 0$, so 0, 1 are semi-*****-complements to one another

and 1 is the proper semi-*****-complement

Example 2: Let A be a Pre $A*$ – algebra with least element with respect to $\wedge$(i.e.,2) ,

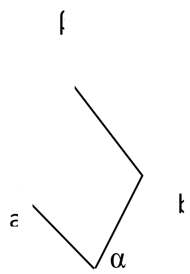
since $2 \wedge 2 = 2$, 2 is semi-*****-complement but proper semi-*****-complement not exists for 2

2.2.17 Definition(Semi-*-complemented semilattice on a Pre A*–algebra): Let A be a Pre $A*$ – algebra with least element α.Then A is said to be semi-*****-complemented semilattice if every inner element (other than least and greatest elements in A) has at least one proper semi-*****-complement.

2.2.18 Example: Let **3**={0,1,2} be a Pre $A*$ – algebra with least element with respect to $\vee$ is 0 and greatest element with respect to $\vee$ is 2 which is a semi-*****-complemented semilattice since 1 has proper semi-*****-complement.

2.2.19 Note: Let A be a Pre A∗ – algebra with least element. Then every ***-**complement of an element is a semi-*****-complement but semi-*****-complement need not be ***-**complement

2.2.20 For example,



Fig(A)

In this fig(A), b is semi-***-**complemement of a as well as, a is semi-***-**complement of b

But b is not ***-**complemement of a as well as, a is not ***-**complement of b.

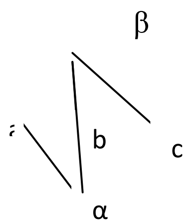
2.2.21 Definition (Weakly *-complemented semilattice on a Pre A∗–algebra): Let A be a Pre A∗ – algebra with least element α. Then A is said to be weakly ***-**complemented semilattice if any pair of elements a, b (a < b) of A a has semi ***-**complement, that is however not a semi ***-**complement of b. That is $x^{\sim}$ is semi-***-** complement of a but not semi- ***-**complement of b.

2.2.22 Example: Let **3** = {0, 1, 2} be a Pre A∗ – algebra with least element with respect to $\vee$ is 0 and for the pair of elements 0, 1 since $0 \wedge 1 = 0$, so 1 is semi-***-**complement of 0

Since1 $\wedge$ 0 =0, 0 is the semi-***-**complement of 1.

Henc9 **3** = {0,1,2} is a weakly ***-**complemented semilattice.

2.2.23 Example:



Fig(B) weakly *-complemented semilattice.

In this fig(B), b is semi-***-**complemement of a however c is not semi-***-**complemement of a

Similarly a is semi-***-**complemement of b however c is not semi-***-**complemement of b

Hence it i**s** weakly ***-**complemented

2.2.24 Example: The semilattice shown in fig (iv) is example of a semilattice which is not weakly ***-**complemented.

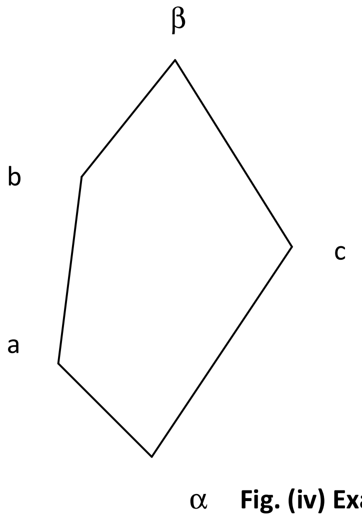


Fig. (iv) Example of a semilattice which is not weakly *-complemented

This is not weakly ***-**complemented since a<b and c is semi ***-**complement of both a and b.

2.2.25 Theorem: Let A be a Pre A∗–algebra. Then every weakly ***-**complemented semilattice in A is semi-***-**complemented.

Proof: Let A be a Pre A∗-algebra with least element α , greatest element β

Let S be any weakly ***-**complemented semilattice in A

Claim: S is semi-***-**complemented

Let a∈ S be an inner element

i.e., $a \neq \alpha, a \neq \beta$

$\Rightarrow$ a is not a maximal element

$\Rightarrow \exists b \in S$ such that $a < b$

Since S is weakly ***-**complemented semilattice in A, we have that there exists semi-complement x of a which is not a semi-***-**complement of b.

i.e., $a \wedge x = \alpha \Rightarrow b \wedge x \neq \alpha$

Then x is proper semi-***-**complement of a and hence S is semi-***-**complemented

Hence every weakly ***-**complemented semilattice in A is semi-***-**complemented.

2.2.26 Absorption law on Pre A*–algebra:

If $a \in A$, $b \in B(A)$ then $a \vee (a \wedge b) = a$

2.2.27 Definition of atom for a semilattice on Pre A* - algebra:: Let S be a subset of a Pre A* - algebra A. Then an element p of a bounded below semilattice $(S, \wedge)$ with least element α is called an atom, if $\alpha \prec p$ (α is covered by p). If there exists an atom p, for each element $a \neq \alpha$ of S such that $p \leq a$. Then we say that S is atomic

2.2.28 Note: That is, in a Pre A* - algebra A,

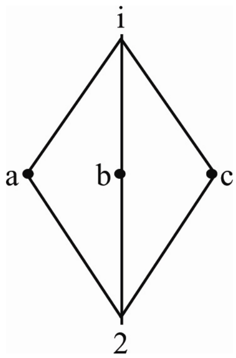
if $(S, \wedge)$ is a semilattice with least element 2,then 2 is atom with respect to $\wedge$ if $2 \prec p$

if $(S, \vee)$ is a semilattice with least element 0,then 0 is atom with respect to $\vee$ if $0 \prec p$.

2.2.29 Example: In fig (i), a, b, c are atoms and this is atomic.

In fig (vi) a is the only one atom and this is also atomic.

In fig (ii) a, c are atoms and this is also atomic.



Fig(i)

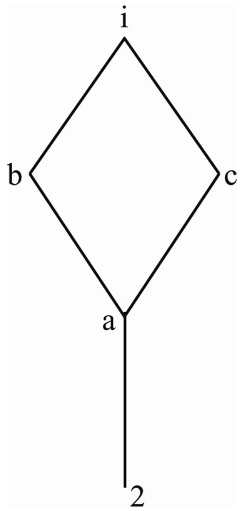


Fig (vi)

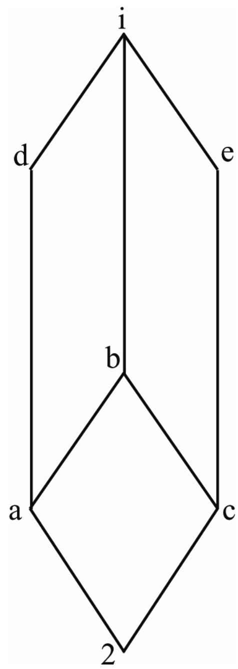


Fig (ii)

Examples of atomic & dual atomic posets

2.2.30 Theorem: Let A be a Pre A*-algebra and S be a subset of A. Then every finite semilattice, which is bounded below is atomic.

Proof: Let S be a subset of a Pre A* - algebra A.

Then an element p of a bounded below semilattice (S, $\wedge$) with least element α is called an atom, if $\alpha \prec p$ (α is covered by p).

Since in a Pre A* - algebra A, if (S, $\wedge$) is a semilattice with least element 2,then 2 is atom with respect to $\wedge$ if $2 \prec p$

If there exists an atom p, for each element $a \neq \alpha$ of S such that $p \leq a$. Then we say that S is atomic.

It is true for every such a finite semilattice(S, $\wedge$).

2.2.31 Definition (Dual atom for semilattice on Pre A* - algebra): Let S be a subset of a Pre A* - algebra A. Then an element p of a bounded above semilattice $(S,\vee)$ with greatest element β is called dual atom if $q \prec \beta$ (q is covered by β)

If there exists a dual atom q for any element $a \neq \beta$ of S such that $a \leq q$. Then we say that S is dual atomic semilattice

Note: That is, in a Pre A* - algebra A, if $(S, \vee)$ is a semilattice with greatest element 2,then 2 is dual atom with respect to $\vee$ if $2 \prec p$

2.2.32 Example: Consider the diagrams fig (i),(ii),(iv):

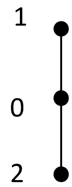
In fig (i), a, b, c are dual atoms; In fig (vi), b, c are dual atoms.

In fig (ii), d, e, b are the dual atoms. All these are dual atomic semilattices.

2.2.33 Definition (**Join irreducible elements for semilattice on Pre A* - algebra)**: Let A be a Pre A*- algebra and $(S, \vee)$ be a semilattice in A with a lower bound α. An element a in S is said to be join irreducible if $a = x \vee y \Rightarrow a = x$ or $a = y$.

2.2.34 Example 1: 2 is join irreducible in a Pre A* - algebra.

Example 2: In Fig(v), every element in this chain is join irreducible.



Fig(v)

2.2.35 Theorem: Let A be a Pre A*-algebra Then in a finite semilattice S of A, if $a \in S$ then we can write a as the join of irredundant join irreducible elements.

Proof: Assume that $(S,\vee)$ be a finite semilattice of a Pre A*-algebra A.

. Let H be the set of all elements of S which cannot be represented as the join of finite number of irredundant join irreducible elements.

Now we will show that H is empty.

Suppose if possible H is non-empty.

Then H does not contain any irredundant join irreducible elements, since if a is join irreducible element and $a \in H$ then $a=a \vee a$ and $a=a \vee \alpha$ (if $\alpha, a \in H$) are two representations of the element a, which is contradicting the definition of H. Hence every element $a \in H$ is the join of finite number of join irreducible elements.

Since H is finite, then the set H contains at least one minimal element, say m.

Clearly m cannot be join irreducible.

So $m=m_1 \vee m_2$ where $m_1, m_2 \in S$ and $m_1, m_2 < m$

Since $m_1, m_2 < m$ we have m_1, m_2 does not in H.

So m_1, m_2 can be represented as $m_1=q_1 \vee q_2 \vee \ldots \vee q_k$;

$m_2 = p_1 \vee p_2 \vee \ldots \vee p_l$ where each p_j, q_j are join irreducible elements and $p_j < m_2, q_j < m_1$.

Now $m=m_1 \vee m_2$

$=(\sum_{j=1}^{k} q_j) \vee (\sum_{i=1}^{k} p_i)$ which is a contradiction to $m \in H$

Hence H is empty.

Therefore in a finite semilattice S if $a \in S$ then we can write a as the join of irredundant join irreducible elements.

2.2.36 Theorem: Let A be a Pre A* - algebra and S be a finite semilattice with least element α in S**.** Then each $x \neq \alpha$ in S can be written uniquely as the join of atoms.

Proof: Let A be a finite Pre A* – algebra and S be a finite semilattice.

Recall that an element a in a bounded below semilattice $(S, \wedge)$ with least element α is called an atom, if $\alpha \prec a$ (α is covered by a).

Let B be the set of atoms of S and let P (B) be the Pre A$*$ – algebra of all subsets of the set B of atoms.

Then by theorem 2.2.35, each $x \neq \alpha$ in S can be expressed uniquely as the join of irreducible elements and since the join irreducible elements are atoms, i.e., elements of B.

Say $x = a_1 \vee a_2 \vee \ldots\ldots\ldots \vee a_r$

2.2.37 Meet irreducible elements for semilattice on Pre A* - algebra:

Definition : Let A be the Pre A* - algebra and a subset S of A be a semilattice in A with an upper bound β . An element a in S is said to be meet irreducible

if $a = x \wedge y \Rightarrow a = x$ or $a = y$.

2.2.38 Example: 2 is meet irreducible

Example: In the above drawn Fig v**,** every element in this chain is meet irreducible

2.2.39 Theorem: Let A be a Pre A*-algebra and S be a subset of A.Then in a finite semilattice S if $a \in S$ then we can write a as the meet of irredundant meet irreducible elements.

Proof: Let S be a subset of a Pre A*-algebra A.

Assume that S be a finite semilattice.

Let H be the set of all elements of S which cannot be represented as the meet of finite number of irredundant meet irreducible elements.

Now we will show that H is empty.

Suppose if possible H is non-empty.

Then H does not contain any irredundant meet irreducible elements, since if a is meet irreducible element and $a \in H$ then $a = a \wedge a$ and $a = a \wedge \beta$ (if $\beta, a \in H$) are two representations of the element a, which is contradicting the definition of H.

Hence every element $a \in H$ is the meet of finite number of meet irreducible elements.

Since H is finite, then the set H contains at least one maximal element, say m

Clearly m cannot be meet irreducible

So $m = m_1 \wedge m_2$ where $m_1, m_2 \in S$ and $m_1, m_2 > m$

Since $m_1, m_2 > m$ we have m_1, m_2 does not in H

So m_1, m_2 can be represented as

$m_1 = q_1 \wedge q_2 \wedge \ldots \wedge q_k$

$m_2 = p_1 \wedge p_2 \wedge \ldots \wedge p_l$ where each p_j, q_j are meet irreducible elements and $p_j > m_2$, $q_j > m_1$

Now $m = m_1 \wedge m_2$

$$= \left(\bigcap_{j=1}^{k} q_j\right) \wedge \left(\bigcap_{i=1}^{k} p_i\right)$$ which is a contradiction to $m \in H$

Hence H is empty.

Therefore in a finite semilattice S if $a \in S$ then we can write a as the meet of irredundant meet irreducible elements.

2.2.40 Theorem: Let A be a Pre A* - algebra and S be a finite semilattice with greatest element β in S**.** Then each $x \neq \beta$ in S can be written uniquely as the meet of dual atoms.

Proof: Let A be a finite Pre A* – algebra and S be a finite semilattice.

Then an element p of a bounded above semilattice $(S, \vee)$ with greatest element β is called dual atom if if $q \prec \beta$ (q is covered by β)

If there exists a dual atom q for any element $a \neq \beta$ of S such that $a \leq q$. Then we say that S is dual atomic semilattice

Let B be the set of dual atoms of S and let P (B) be the Pre A∗ – algebra of all subsets of the set B of dual atoms.

Then by theorem 2.2.39, each $x \neq \beta$ in S can be expressed uniquely as the meet of irreducible elements and since the meet of irreducible elements are dual atoms, i.e., elements of B.

Say $x = a_1 \wedge a_2 \wedge \ldots\ldots\ldots \wedge a_r$

Conclusion: In this Chapter, we studied the concept of semilattice on a Pre A*-algebra with respect to the binary operation (meet) and as well as the binary operation (join) and obtained the properties of semilattice on a Pre A*-algebra. We established Pre A*-algebra as a semilattice. We proved necessary conditions for a semilattice to become a lattice with respect to meet and as well, as join.

In the second section we defined greatest lower bound of an element in Pre A* - algebra and least upper bound of an element in Pre A* - algebra and we provided examples of these. We defined semi-*-complement for semilattice on Pre A* - algebra and we proved some theorems on these. We defined atoms, dual atoms, irreducible elements with respect to meet as well as join for semilattice on Pre A*-algebra. We obtained various theorems on these atoms, dual atoms, irreducible elements for semilattice on Pre A*-algebra. We established the atomic, dual atomic semilattices on Pre A*-algebra.