Chapter 6

Nonparametric estimators for survivor function of paired recurrent events

6.1 Introduction

In many applications, pair of study subjects may experiences recurrent events of same type or different types. The recurrences of infectious diseases on left and right eyes are common in practice. In such contexts, the analysis of recurrent event data would be difficult as it involves two kinds of dependence, namely, dependence among study subjects in the same pair and dependence between sequence of observations of individuals in the pairs. The estimators for bivariate survivor function available in literature are not appropriate in such contexts due to these two kinds of dependence.

The analysis of paired recurrent event data is, therefore, an area of research to be explored. Motivated by this, nonparametric estimators of bivariate survivor function for pair of recurrent times are developed. The proposed estimators generalize the well known estimators for bivariate survivor function developed by Dabrowska (1988) and the estimator of Burke (1988).

The rest of the chapter is organized as follows. In Section 6.2, a new stochastic model for the analysis of paired recurrent event data is proposed. Nonparametric estimator of the survivor

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1The results in this chapter have been accepted for publication in Statistical Methodology (see Sankaran, Manoharan and Anisha, 2012c).
function using cumulative hazard vector is discussed in Section 6.3. In Section 6.4, an estimator of the bivariate survivor function using inverse probability weighted technique is derived. Section 6.5 discusses various asymptotic properties of the estimators. The procedure is illustrated with a real life data in Section 6.6. In Section 6.7, a simulation study is carried out to assess the finite sample properties of the estimators. Finally, Section 6.8 gives a brief conclusion of the study.

6.2 The Model

Suppose that a sequence of events is to be observed over a pair of individuals in a study and the starting time of follow up is the time of initial event. Let \((T_{1j}, T_{2j})\) denotes the pair of recurrent times from \((j - 1)\)th to \(j\)th event. Let \(N = (N_1, N_2)\), where \(N_k = \{T_{kj}, j = 1, 2, \ldots\}\), \(k = 1, 2\) be the collection of the pair of recurrence times on study subjects. Let \(D = (D_1, D_2)\) be the pair of censoring times which is the time between the initial event and the end of follow up. Assume that pair of individuals are sampled independently, but correlation among recurrence times from the same individual and correlation between the times of two individuals in the same pair are allowed. The estimators of bivariate survivor function available in literature are valid when the gap times are independent and identically distributed sample from some underlying distribution \(F(t_1, t_2)\). One type of generalization that provides association between gap times is a frailty model.

The model involves the following assumptions;

1. There exists a latent variable \(\nu\) such that conditioning on \(\nu\), the bivariate vectors \((T_{1j}, T_{2j})\), \(j = 1, 2 \ldots\) are independent and identically distributed (i.i.d.).

2. The censoring vector \(D\) is independent of \((N, \nu)\).

Assumption 1 implies that the recurrence times \(T_{1j}\) and \(T_{2j}\) are allowed to be correlated when conditioning on \(\nu\). The variable \(\nu\) characterizes the association between bivariate vectors
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\((T_{1j}, T_{2j})\) within a pair of subjects, \(j = 1, 2, \ldots\). Assumption 2 is the usual independent censoring condition. The validity of the assumptions depends on the nature of recurrence times as well as on the specific censoring pattern in a study. The objective here is to develop estimators for the survivor function of the recurrent times for a pair of subjects, which is referred as the bivariate recurrent survivor function. It is assumed that the recurrence times for pair of subjects from different episodes have the same distribution.

Denoting the common survivor function of \((T_{1j}, T_{2j})\) as \(S(t_1, t_2)\) and the conditional density of \((T_{1j}, T_{2j})\) given \(\nu\) as \(f(t_1, t_2|\nu)\), we obtain

\[
S(t_1, t_2) = P(T_{1j} > t_1, T_{2j} > t_2) = \int_t^{\infty} \int_t^{\infty} f(u_1, u_2|\nu) d\mu(u_1, u_2) dP_\nu(\nu),
\]

where \(P_\nu(\nu)\) is the distribution function of \(\nu\) and \(\mu\) is the \(\sigma\)-finite dominating measure of the density \(f\). Let \(m_k\) be the index of censoring time for \(k\)th individual in the pair such that

\[
\sum_{j=1}^{m_k-1} T_{kj} \leq D_k \quad \text{and} \quad \sum_{j=1}^{m_k} T_{kj} > D_k, \quad k = 1, 2.
\]

(6.1)

Obviously \(m = (m_1, m_2)\) is a random vector and note that last recurrence time for \(k\)th individual is subject to right censoring with censoring time

\[
D_k^* = D_k - \sum_{j=1}^{m_k-1} T_{kj}, \quad k = 1, 2.
\]

(6.2)

Further, the integer \(m_1\) may not be equal to \(m_2\), which implies that the same pair may involves one censoring time and one observed recurrence time. The study will continue until both satisfies the condition (6.1). Thus, the observed vector is

\[
\left\{ (Y_{11}, Y_{21}), (Y_{22}, Y_{22}), \ldots, (Y_{1m_1}, Y_{2m_2}), (D_1, D_2) \right\},
\]

where \(Y_{kj} = T_{kj}\) if \(j = 1, 2, \ldots, m_k - 1\) and \(Y_{kj} = T_{km_k}^*\) if \(j = m_k\) with \(T_{km_k}^*\) as the time from
event $m_k - 1$ to the event of follow up, $k = 1, 2$.

Let $G(t_1, t_2) = P[D_1 > t_1, D_2 > t_2]$ be the survivor function of the censoring time vector. Our interest is to estimate $S(t_1, t_2)$ nonparametrically.

### 6.3 Generalized Dabrowska’s Estimator

Firstly, a representation of $S(t_1, t_2)$ in terms of cumulative hazard rate vector is given. Let $a = a(D_1, D_2)$ be a positive valued function of $(D_1, D_2)$ with $E(a^2) < \infty$.

Now define functions

\[
H_1(t_1, t_2) = E[aI(T_{1j} > t_1, T_{2j} > t_2)I(D_1 > t_1, D_2 > t_2)],
\]

\[
H_2(t_1, t_2) = E[aI(T_{1j} \geq t_1, T_{2j} > t_2)I(D_1 \geq T_{1j})],
\]

\[
H_3(t_1, t_2) = E[aI(T_{1j} > t_1, T_{2j} \geq t_2)I(D_2 \geq T_{2j})]
\]

and

\[
H_4(t_1, t_2) = E[aI(T_{1j} \geq t_1, T_{2j} \geq t_2)I(D_1 \geq T_{1j}, D_2 \geq T_{2j})],
\]

where $I(\cdot)$ is the usual indicator function.

Let $\Lambda(t_1, t_2) = (\Lambda_1(t_1, t_2), \Lambda_2(t_1, t_2), \Lambda_3(t_1, t_2))$ be the cumulative hazard rate vector, defined in Dabrowska (1988), where, for $j = 1, 2$ and 3,

\[
\Lambda_1(t_1, t_2) = \frac{P[T_{1j} \in dt_1, T_{2j} > t_2]}{P[T_{1j} \geq t_1, T_{2j} > t_2]},
\]

\[
\Lambda_2(t_1, t_2) = \frac{P[T_{1j} > t_1, T_{2j} \in dt_2]}{P[T_{1j} > t_1, T_{2j} \geq t_2]}
\]
and

\[ \Lambda_3(t_1, t_2) = \frac{P[T_{1j} \in dt_1, T_{2j} \in dt_2]}{P[T_{1j} \geq t_1, T_{2j} \geq t_2]} \]  

(6.9)

with \( \Lambda_1(0, t_2) = \Lambda_2(t_1, 0) = \Lambda_3(0, 0) = 0 \).

From (6.3), (6.4), (6.5) and (6.6), we obtain \( \Lambda_i(t_1, t_2) \), \( i = 1, 2, 3 \) as

\[ \Lambda_1(t_1, t_2) = -\int_0^{t_1} \frac{H_2(du, t_2)}{H_1(u^-, t_2)} \]  

(6.10)

\[ \Lambda_2(t_1, t_2) = -\int_0^{t_2} \frac{H_3(t_1, du)}{H_1(t_1, u^-)} \]  

(6.11)

and

\[ \Lambda_3(t_1, t_2) = \int_0^{t_1} \int_0^{t_2} \frac{H_4(du, dv)}{H_1(u^-, v^-)}. \]  

(6.12)

Then \( S(t_1, t_2) \) can be expressed in terms of \( H_i(t_1, t_2) \), \( i = 1, 2, 3 \) and 4, using (6.10), (6.11) and (6.12) (Dabrowska (1988)) as

\[ S(t_1, t_2) = S_1(t_1)S_2(t_2) \prod_{0 \leq u_1 \leq t_1, 0 \leq u_2 \leq t_2} [1 - L(du_1, du_2)], \]

where

\[ S_1(t_1) = \prod_{u_1 \leq t_1} (1 - \Lambda_1(du_1, 0)), \]

\[ S_2(t_2) = \prod_{u_2 \leq t_2} (1 - \Lambda_2(0, du_2)) \]
To develop a nonparametric estimator, the estimators of survivor function given in Wang and Chang (1999) and Pena et al. (2001) are extended to the paired setup. Suppose that \( n \) independent pair of individuals are available for the study. Let \((T_{1ji}, T_{2ji})\) be the time from the \((j-1)\)th to the \(j\)th event for the \(i\)th pair, \(j = 1, 2, \ldots\) and \(i = 1, 2, \ldots, n\). Let \((D_{1i}, D_{2i})\) be pair of censoring times corresponding to \(i\)th pair of individuals. Suppose that \((m_{1i}, m_{2i})\) is the pair of indices such that

\[
\sum_{j=1}^{m_{ki}-1} T_{kji} \leq D_{ki} \quad \text{and} \quad \sum_{j=1}^{m_{ki}} T_{kji} > D_{ki}, \quad k = 1, 2; \; i = 1, 2, \ldots n.
\]

When \(m_{ki} \geq 2\), the last recurrence time is not employed in the average to avoid the sampling bias, \(k = 1, 2\) and \(i = 1, 2, \ldots, n\). The recurrence times \((T_{11i}, T_{21i}), (T_{12i}, T_{22i}), \ldots, (T_{1m_{1i}i}, T_{2m_{2i}i}), i = 1, 2, \ldots, n\) are identically distributed conditional on \((m_{1i}, m_{2i}, D_{1i}, D_{2i})\).

Suppose now that

\[
\{(Y_{11i}, Y_{21i}), \ldots, (Y_{1m_{1i}i}, Y_{2m_{2i}i}) (\delta_{11i}, \delta_{21i}) \ldots (\delta_{1m_{1i},i}, \delta_{2m_{2i},i}), (D_{1i}, D_{2i})\}
\]

is the observed pairs of recurrence times along with censoring times for \(n\) pair of individuals with

\[
Y_{kji} = \begin{cases} 
T_{kji} & \text{if } j = 1, 2, \ldots, m_{ki} - 1 \\
T_{kmi}^+ & \text{if } j = m_{ki},
\end{cases}
\]

where \(T_{kmi}^+\) is the time from event \(m_{ki} - 1\) to the end of follow up for the \(k\)th individual in the \(i\)th pair and \(\delta_{kji} = I(Y_{kji} = T_{kji}), j = 1, 2, \ldots, m_{ki}, k = 1, 2 \) and \(i = 1, 2, \ldots n\). For \(i = 1, 2, \ldots, n\), let \(a_i = a(D_{1i}, D_{2i})\) be a positive valued function of \((D_{1i}, D_{2i})\) with \(E(a_i^2) < \infty\)
and \( m(i) = \min(m_{1i}, m_{2i}) \).

Now, the nonparametric estimators of \( H_i(t_1, t_2), i = 1, 2, 3 \) and 4 are obtained as

\[
\hat{H}_1(t_1, t_2) = \frac{\sum_{i=1}^{n} a_i \left[ \sum_{j=1}^{m(i)} I(Y_{1ji} > t_1, Y_{2ji} > t_2) \right]}{\sum_{i=1}^{n} a_i (m(i) + 1)} + \frac{\sum_{i=1}^{n} a_i \left[ I(D_{1i} - \sum_{j=1}^{m(i)} T_{1ji} > t_1, D_{2i} - \sum_{j=1}^{m(i)} T_{2ji} > t_2) \right]}{\sum_{i=1}^{n} a_i (m(i) + 1)},
\]

(6.13)

\[
\hat{H}_2(t_1, t_2) = \frac{\sum_{i=1}^{n} a_i \left[ \sum_{j=1}^{m(i)} I(Y_{1ji} > t_1, Y_{2ji} > t_2, \delta_{1ji} = 1) \right]}{\sum_{i=1}^{n} a_i (m(i) + 1)} + \frac{\sum_{i=1}^{n} a_i \left[ \sum_{j=1}^{m(i)} I(m_{2i} < m_{1i}) I(Y_{1ji} > t_1, D_{2i} - \sum_{j=1}^{m(i)} T_{2ji} > t_2, \delta_{1ji} = 1) \right]}{\sum_{i=1}^{n} a_i (m(i) + 1)},
\]

(6.14)

\[
\hat{H}_3(t_1, t_2) = \frac{\sum_{i=1}^{n} \sum_{j=1}^{m(i)} a_i \left[ \sum_{j=1}^{m(i)} I(Y_{1ji} > t_1, Y_{2ji} > t_2, \delta_{2ji} = 1) \right]}{\sum_{i=1}^{n} a_i (m(i) + 1)} + \frac{\sum_{i=1}^{n} a_i \left[ \sum_{j=1}^{m(i)} I(m_{1i} < m_{2i}) I(D_{1i} - \sum_{j=1}^{m(i)} T_{1ji} > t_1, Y_{2ji} > t_2, \delta_{2ji} = 1) \right]}{\sum_{i=1}^{n} a_i (m(i) + 1)},
\]

(6.15)

and

\[
\hat{H}_4(t_1, t_2) = \frac{\sum_{i=1}^{n} a_i \sum_{j=1}^{m(i)} I(Y_{1ji} > t_1, Y_{2ji} > t_2, \delta_{1ji} = \delta_{2ji} = 1)}{\sum_{i=1}^{n} a_i (m(i) + 1)}.
\]

(6.16)

Note that in (6.13), we have incorporated both recurrence time pairs and censored time pairs. The second factor in the numerator of (6.14) provides the information that first individual of the pair has a recurrence time and second individual has a censored time. Similar interpretation is true for (6.15). Substituting (6.13), (6.14), (6.15) and (6.16) in (6.10), (6.11) and (6.12), the
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Estimators of $\Lambda_i(t_1, t_2)$, $i = 1, 2, 3$ is given by

$$\hat{\Lambda}_1(t_1, t_2) = -\int_0^{t_1} \frac{\hat{H}_2(du, t_2)}{H_1(u-, t_2)},$$

(6.17)

$$\hat{\Lambda}_2(t_1, t_2) = -\int_0^{t_2} \frac{\hat{H}_3(t_1, du)}{H_1(t_1, u-)}$$

(6.18)

and

$$\hat{\Lambda}_3(t_1, t_2) = \int_0^{t_1} \int_0^{t_2} \frac{\hat{H}_4(du, dv)}{H_1(u-, v-)}$$

(6.19)

for $\hat{H}_1(t_1, t_2) > 0$ and are zero if $\hat{H}_1(t_1, t_2) = 0$. The estimator of $S(t_1, t_2)$ using the relationship given in Dabrowska (1988) is obtained as

$$\hat{S}(t_1, t_2) = \hat{S}_1(t_1)\hat{S}_2(t_2) \prod_{0 \leq u_1 \leq t_1, 0 \leq u_2 \leq t_2} \left[1 - \hat{L}(du_1, du_2)\right],$$

(6.20)

where

$$\hat{S}_1(t_1) = \prod_{u_1 \leq t_1} (1 - \hat{\Lambda}_1(du_1, 0)),$$

$$\hat{S}_2(t_2) = \prod_{u_2 \leq t_2} (1 - \hat{\Lambda}_2(0, du_2))$$

and

$$\hat{L}(du_1, du_2) = \frac{\hat{\Lambda}_1(du_1, u^-_2)\hat{\Lambda}_2(u^-_1, du_2) - \hat{\Lambda}_3(du_1, du_2)}{(1 - \hat{\Lambda}_1(du_1, u^-_2))(1 - \hat{\Lambda}_2(u^-_1, du_2))}.$$
form expression for the MSE (see Section 6.5). One can use the extension of the bootstrap procedure of resampling observed units developed in Efron(1981) to compute the MSE.

**Remark 6.1.** In the context of paired data without recurrence \((m_1 = m_2 = 1)\), the estimators (6.17), (6.18) and (6.19) are the modified versions of the estimators of cumulative hazard rate functions given in Dabrowska (1988).

### 6.4 Inverse Probability Weighted Estimator

In present setup also, the censoring times \((D_{1i}, D_{2i})\) are known for each pair. Further the \((m_{ki} + 1)\)-th observation is censored for each individual, \(k = 1, 2\) and \(i = 1, 2, \ldots, n\). Denoting \(H(t_1, t_2)\) as the common survivor function of \((Y_{1j}, Y_{2j})\), \(j = 1, 2, \ldots\), we have

\[
H(t_1, t_2) = S(t_1, t_2)G(t_1, t_2),
\]

which provides \(H(t_1, t_2) = \int_{t_1}^{\infty} \int_{t_2}^{\infty} G(u_1, u_2)S(du_1, du_2)\). Then \(S(t_1, t_2)\) is given by

\[
S(t_1, t_2) = \int_{t_1}^{\infty} \int_{t_2}^{\infty} \frac{H(du_1, du_2)}{G(u_1, u_2)}.
\]  

(6.21)

Based on independent and identically distributed paired sample of size \(n\), a simple estimator of \(S(t_1, t_2)\) is given by

\[
\hat{S}_e(t_1, t_2) = \frac{1}{\sum_{i=1}^{n} m_{(i)}} \sum_{i=1}^{n} \sum_{j=1}^{m_{(i)}} \frac{I(Y_{1ji} > t_1, Y_{2ji} > t_2)}{\hat{S}_G(Y_{1ji}, Y_{2ji})},
\]  

(6.22)

where

\[
\hat{S}_G(t_1, t_2) = \frac{1}{n} \sum_{i=1}^{n} I(D_{1i} > t_1, D_{2i} > t_2)
\]  

(6.23)

is the empirical survivor function of the censoring variables. The estimator (6.22) is referred as inverse probability weighted estimator for bivariate survivor function of recurrent events. The
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inverse probability weighted estimator is employed for the estimation of bivariate distribution function earlier by Burke (1988) and Dai and Bao (2009). Unlike generalized Dabrowska’s estimator, (6.22) is monotonic decreasing in \( t_k, k = 1, 2 \).

6.5 Asymptotic Properties

In this section, asymptotic properties of the estimators given in Section 6.3 are studied. Firstly, note that \( \hat{H}_i(t_1, t_2) \) is an unbiased estimators of \( H_i(t_1, t_2), i = 1, 2, 3, 4 \). Now we establish consistency of the estimators \( H_i(t_1, t_2), i = 1, 2, 3, 4 \). Suppose that \( b = (b_1, b_2) \) satisfies \( H(b_1, b_2) > 0 \). Let \( \| \cdot \|_{D^*} \) denote the supremum norm on set \( D^* = [0, b_1] \times [0, b_1] \).

**Theorem 6.1.** Under assumptions 1 and 2, \( \left\| \hat{H}_i(t_1, t_2) - H_i(t_1, t_2) \right\|_{D^*} \to 0 \) almost surely for \( (t_1, t_2) \in D^* \) and \( i = 1, 2, 3, 4 \).

*Proof.* The proof directly follows the definitions.

**Theorem 6.2.** Under assumptions 1 and 2, \( \left\| \hat{\Lambda}_i(t_1, t_2) - \Lambda_i(t_1, t_2) \right\|_{D^*} \to 0 \) almost surely for \( (t_1, t_2) \in D^* \) and \( i = 1, 2, 3 \).

*Proof.* For \( i = 1 \), we have

\[
\hat{\Lambda}_1(t_1, t_2) - \Lambda_1(t_1, t_2) = \int_0^{t_1} \left( \frac{H_2(du, t_2)}{H_1(u^-, t_2)} - \frac{\hat{H}_2(du, t_2)}{\hat{H}_1(u^-, t_2)} \right) \frac{1}{H_1(u^-, t_2)} \left( H_2(du, t_2) - \hat{H}_2(du, t_2) \right) 
+ \int_0^{t_1} \frac{\hat{H}_2(du, t_2)}{H_1(u^-, t_2)\hat{H}_1(u^-, t_2)} \left( \hat{H}_1(u^-, t_2) - H_1(u^-, t_2) \right)
\]

(6.24)
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Since \( \| \hat{H}_i(t_1, t_2) - H_i(t_1, t_2) \|_{D^*} \to 0 \) almost surely for \( i = 1, 2 \),
\( \| \hat{\Lambda}_1(t_1, t_2) - \Lambda_1(t_1, t_2) \|_{D^*} \to 0 \) almost surely.

The proof for \( i = 2, 3 \) is similar. \( \square \)

**Theorem 6.3.** Under assumptions 1 and 2, \( \| \hat{L}(t_1, t_2) - L(t_1, t_2) \|_{D^*} \to 0 \) almost surely for \( (t_1, t_2) \in D^* \).

**Proof.** The proof directly follows from Theorem 6.1 and Theorem 6.2. \( \square \)

**Theorem 6.4.** Under assumptions 1 and 2, \( \| \hat{S}(t_1, t_2) - S(t_1, t_2) \|_{D^*} \to 0 \) almost surely for \( (t_1, t_2) \in D^* \).

**Proof.** The proof is similar to that of Proposition 4.1 of Dabrowska (1988) by using Theorem 6.2 and Theorem 6.3. \( \square \)

To prove the asymptotic normality of the survivor function, the following lemmas may be used.

**Lemma 6.1.** Under assumptions 1 and 2, for all fixed \( (t_1, t_2) \in D^* \), \( \sqrt{n}(\hat{\Lambda}_k(t_1, t_2) - \Lambda_k(t_1, t_2)) \) converges in distribution to normal with mean zero and variance \( \sigma^2_k(t_1, t_2) \), \( k = 1, 2 \) and 3 which are given in (6.26), (6.27) and (6.28) respectively.

**Proof.** The result is proved for \( k = 1 \).

For \( k = 1 \),

\[
\hat{\Lambda}_1(t_1, t_2) - \Lambda_1(t_1, t_2) = \int_0^{t_1} \frac{1}{H_1(u-, t_2)} \left( H_2(du, t_2) - \hat{H}_2(du, t_2) \right) + \\
\int_0^{t_1} \frac{\hat{H}_2(du, t_2)}{H_1(u-, t_2)\hat{H}_1(u-, t_2)} \left( \hat{H}_1(u-, t_2) - H_1(u-, t_2) \right) \tag{6.25}
\]
Since \( \| \hat{H}_i(t_1, t_2) - H_i(t_1, t_2) \|_{D^*} \to 0 \) for \( i = 1, 2 \), (6.25) is asymptotically equal to

\[
\hat{\Lambda}_1(t_1, t_2) - \Lambda_1(t_1, t_2) \doteq \int_{0}^{t_1} \frac{1}{H_1(u^-, t_2)} \left( H_2(du, t_2) - \frac{1}{n} \sum_{i=1}^{n} \frac{a_i}{m^*_i} \sum_{j=1}^{m^*_i} I(Y_{1ji} = u, Y_{2ji} > t_2, \delta_{kji} = 1) \right) \\
+ \int_{0}^{t_1} \frac{H_2(du, t_2)}{H_1(u^-, t_2)^2} \left( \frac{1}{n} \sum_{i=1}^{n} \frac{a_i}{m^*_i} \sum_{j=1}^{m^*_i} I(Y_{1ji} = u, Y_{2ji} > t_2) - H_1(u^-, t_2) \right),
\]

which leads to

\[
\sqrt{n} \left( \hat{\Lambda}_1(t_1, t_2) - \Lambda_1(t_1, t_2) \right) \doteq \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \int_{0}^{t_1} \frac{1}{H_1(u^-, t_2)} H_2(du, t_2) \\
- \frac{a_i}{m^*_i} \sum_{j=1}^{m^*_i} I(Y_{1ji} = u, Y_{2ji} > t_2, \delta_{kji} = 1) \right) \\
+ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \int_{0}^{t_1} \frac{H_2(du, t_2)}{H_1(u^-, t_2)^2} \\
- \frac{a_i}{m^*_i} \sum_{j=1}^{m^*_i} I(Y_{1ji} > u, Y_{2ji} > t_2) - H_1(u^-, t_2) \right).
\]

For fixed \( (t_1, t_2) \in D^* \), by multivariate central limit theorem, the terms in the simple brackets of first and second integrals converges to a normal random variable with mean zero and variance
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as

\[
\sigma_1^2(t_1, t_2) = E \left[ \int_0^{t_1} \frac{H_2(du, t_2)}{H_1^2(u^-, t_2)} \frac{a_i}{m^*_i} \sum_{j=1}^{m^*_i} I(Y_{1ji} > u, Y_{2ji} > t_2) - \int_0^{t_1} \frac{1}{H_1(u^-, t_2)} \frac{a_i}{m^*_i} \sum_{j=1}^{m^*_i} I(Y_{1ji} = u, Y_{2ji} > t_2, \delta_{1ji} = 1) \right].
\]

(6.26)

The proof for \( k = 2 \) and 3 is similar and the expressions for variances are obtained as

\[
\sigma_2^2(t_1, t_2) = E \left[ \int_0^{t_2} \frac{H_3(t_1, du)}{H_1^2(t_1, u^-)} \frac{a_i}{m^*_i} \sum_{j=1}^{m^*_i} I(Y_{1ji} > t_1, Y_{2ji} > u) - \int_0^{t_2} \frac{1}{H_1(t_1, u^-)} \frac{a_i}{m^*_i} \sum_{j=1}^{m^*_i} I(Y_{1ji} > t_1, Y_{2ji} > u, \delta_{2ji} = 1) \right]
\]

(6.27)

and

\[
\sigma_3^2(t_1, t_2) = E \left[ \int_0^{t_1} \int_0^{t_2} \frac{H_4(du_1, du_2)}{H_1^2(u_1^-, u_2^-)} \frac{a_i}{m^*_i} \sum_{j=1}^{m^*_i} I(Y_{1ji} > u_1, Y_{2ji} > u_2) - \int_0^{t_1} \int_0^{t_2} \frac{1}{H_1(u_1^-, u_2^-)} \frac{a_i}{m^*_i} \sum_{j=1}^{m^*_i} I(Y_{1ji} = u_1, Y_{2ji} = u_2, \delta_{1ji} = \delta_{2ji} = 1) \right].
\]

(6.28)

Lemma 6.2. Under the assumptions 1 and 2, for all fixed \((t_1, t_2) \in D^*\), \(\sqrt{n}(\hat{L}(t_1, t_2) - L(t_1, t_2))\) converges to a normal random variable with mean zero and variance \(\sigma^2_{11}(t_1, t_2)\) as given in (6.31).

Proof. To prove this, consider

\[
\hat{L}(t_1, t_2) - L(t_1, t_2) = \frac{\hat{\Lambda}_1(t_1, t_2)\hat{\Lambda}_2(t_1, t_2) - \hat{\Lambda}_3(t_1, t_2)}{(1 - \hat{\Lambda}_1(t_1, t_2))(1 - \hat{\Lambda}_2(t_1, t_2))} - \frac{\Lambda_1(t_1, t_2)\Lambda_2(t_1, t_2) - \Lambda_3(t_1, t_2)}{(1 - \Lambda_1(t_1, t_2))(1 - \Lambda_2(t_1, t_2))}
\]
6.5. ASYMPTOTIC PROPERTIES

\[
\frac{\hat{L}(t_1, t_2) - L(t_1, t_2)}{\lambda_1(t_1, t_2) t(t_1, t_2) - \lambda_3(t_1, t_2)} = \frac{\hat{L}(t_1, t_2) - \lambda_3(t_1, t_2)}{(1 - \lambda_1(t_1, t_2))(1 - \lambda_2(t_1, t_2))} - \frac{\hat{L}(t_1, t_2) - \lambda_3(t_1, t_2)}{(1 - \lambda_1(t_1, t_2))(1 - \lambda_2(t_1, t_2))}.
\]

(6.29)

Since \( \left\| \lambda_i(t_1, t_2) - \lambda_i(t_1, t_2) \right\|_{D^*} \to 0 \) almost surely for \( i = 1, 2, 3 \), (6.29) is asymptotically equal to

\[
\hat{L}(t_1, t_2) - L(t_1, t_2) = \frac{\lambda_1(t_1, t_2) \lambda_2(t_1, t_2) - \lambda_3(t_1, t_2)}{(1 - \lambda_1(t_1, t_2))(1 - \lambda_2(t_1, t_2))} [\hat{L}(t_1, t_2) - \lambda_3(t_1, t_2)]
\]

+ \[ \frac{1}{(1 - \lambda_1(t_1, t_2))(1 - \lambda_2(t_1, t_2))} [\hat{L}(t_1, t_2) - \lambda_3(t_1, t_2)]. \]

(6.30)

Now

\[
\hat{L}(t_1, t_2) - \lambda_3(t_1, t_2) = \hat{L}(t_1, t_2) - \lambda_3(t_1, t_2)
\]

which is asymptotically equal to

\[
\hat{L}(t_1, t_2) - \lambda_3(t_1, t_2) = \hat{L}(t_1, t_2) - \lambda_3(t_1, t_2)
\]

Thus (6.30) is asymptotically equal to

\[
\hat{L}(t_1, t_2) - L(t_1, t_2) = \frac{\lambda_1(t_1, t_2) \lambda_2(t_1, t_2) - \lambda_3(t_1, t_2)}{(1 - \lambda_1(t_1, t_2))(1 - \lambda_2(t_1, t_2))} [\hat{L}(t_1, t_2) - \lambda_3(t_1, t_2)]
\]

\[ + \frac{1}{(1 - \lambda_1(t_1, t_2))(1 - \lambda_2(t_1, t_2))} [\hat{L}(t_1, t_2) - \lambda_3(t_1, t_2)]. \]
\[ -\Lambda_1(t_1, t_2) - \Lambda_1(t_1, t_2)(\hat{\Lambda}_2(t_1, t_2) - \Lambda_2(t_1, t_2)) - \Lambda_2(t_1, t_2) \\
(\hat{\Lambda}_1(t_1, t_2) - \Lambda_1(t_1, t_2)) + \frac{1}{(1 - \Lambda_1(t_1, t_2))(1 - \Lambda_2(t_1, t_2))} \\
[\Lambda_1(t_1, t_2)(\hat{\Lambda}_2(t_1, t_2) - \Lambda_2(t_1, t_2)) - (\hat{\Lambda}_3(t_1, t_2) - \Lambda_3(t_1, t_2))]. \]

By multivariate central limit theorem, \( \sqrt{n}(\hat{L}(t_1, t_2) - L(t_1, t_2)) \) converges to a normal distribution with mean zero and variance

\[ \sigma^2_{(1)}(t_1, t_2) = E \left[ \frac{\Lambda_1(t_1, t_2)\Lambda_2(t_1, t_2) - \Lambda_3(t_1, t_2)}{(1 - \Lambda_1(t_1, t_2))^2(1 - \Lambda_2(t_1, t_2))^2} \right] \left[ \Lambda_1(t_1, t_2)(\hat{\Lambda}_2(t_1, t_2) - \Lambda_2(t_1, t_2)) - (\hat{\Lambda}_3(t_1, t_2) - \Lambda_3(t_1, t_2)) \right] \]

\[ -\Lambda_1(t_1, t_2) - \Lambda_1(t_1, t_2)(\hat{\Lambda}_2(t_1, t_2) - \Lambda_2(t_1, t_2)) - \Lambda_2(t_1, t_2) \\
(\hat{\Lambda}_1(t_1, t_2) - \Lambda_1(t_1, t_2)) + \frac{1}{(1 - \Lambda_1(t_1, t_2))(1 - \Lambda_2(t_1, t_2))} \] 

\[ -\Lambda_2(t_1, t_2) - (\hat{\Lambda}_3(t_1, t_2) - \Lambda_3(t_1, t_2)) \right]^2. \] (6.31)

\[ \square \]

**Lemma 6.3.** Let

\[ S_3(t_1, t_2) = \prod_{0 \leq u_1 \leq t_1} \prod_{0 \leq u_2 \leq t_2} (1 - L(du_1, du_2)) \]

and

\[ \hat{S}_3(t_1, t_2) = \prod_{0 \leq u_1 \leq t_1} \prod_{0 \leq u_2 \leq t_2} (1 - \hat{L}(du_1, du_2)). \]

Under assumptions 1 and 2, for all fixed \((t_1, t_2) \in D^*\), \( \sqrt{n}(\hat{S}_3(t_1, t_2) - S_3(t_1, t_2)) \) converges to a normal random variable with mean zero and variance \( \sigma^2_{(3)}(t_1, t_2) \) as given in (6.32).
Proof. Using the approximation

\[ S_3(t_1, t_2) = \prod_{0 \leq u_1 \leq t_1} (1 - L(du_1, du_2)) \cong e^{-\sum_{u_1 \leq t_1} \sum_{u_2 \leq t_2} L(du_1, du_2)}, \]

the asymptotic normality of \( \sqrt{n} \left( \hat{S}_3(t_1, t_2) - S_3(t_1, t_2) \right) \), for fixed \((t_1, t_2) \in D^*\), follows from the functional delta method with variance as

\[ \sigma^2_{(3)}(t_1, t_2) = nS_3(t_1, t_2)^2 \sigma^2_{(1)}(t_1, t_2). \] (6.32)

Theorem 6.5. Under assumptions 1 and 2, for every fixed \((t_1, t_2) \in D^*\) and for large \(n\),

\[ \sqrt{n} \left( \hat{S}(t_1, t_2) - S(t_1, t_2) \right) \]

converges in distribution to normal with mean zero and variance \( \sigma^2(t_1, t_2) \) as given in (6.35).

Proof. Now

\[ \sqrt{n} \left( \hat{S}(t_1, t_2) - S(t_1, t_2) \right) = \sqrt{n} \left( \hat{S}_1(t_1) \hat{S}_2(t_2) \hat{S}_3(t_1, t_2) - S_1(t_1)S_2(t_2)S_3(t_1, t_2) \right) \]

\[ = \sqrt{n} \left( (\hat{S}_1(t_1) - S_1(t_1))\hat{S}_2(t_2)\hat{S}_3(t_1, t_2) \right) \]

\[ + S_1(t_1)\hat{S}_3(t_1, t_2)(\hat{S}_2(t_2) - S_2(t_2)) \]

\[ + S_1(t_1)S_2(t_2)(\hat{S}_3(t_1, t_2) - S_3(t_1, t_2)) \] \quad (6.33)

Since \( \left\| \hat{S}(t_1, t_2) - S(t_1, t_2) \right\|_{D^*} \to 0 \) almost surely, (6.33) is asymptotically equal to

\[ \sqrt{n} \left( \hat{S}(t_1, t_2) - S(t_1, t_2) \right) \cong \sqrt{n}S_2(t_2)S_3(t_1, t_2)(\hat{S}_1(t_1) - S_1(t_1)) \]

\[ + S_1(t_1)S_3(t_1, t_2)\sqrt{n}(\hat{S}_2(t_2) - S_2(t_2)) \]

\[ + \sqrt{n}S_1(t_1)S_2(t_2)(\hat{S}_3(t_1, t_2) - S_3(t_1, t_2)). \] (6.34)
Now, for fixed \((t_1, t_2) \in D^*\), \(\sqrt{n} \left( \hat{S}_k(t_1, t_2) - S_k(t_1, t_2) \right)\) converges in distribution to normal with mean zero and variance

\[
\sigma_{(t_k)}^{2(k)} = S_k^2(t_k) E \left[ \phi_i^{(k)}(t_k)^2 \right], \quad k = 1, 2,
\]

where \(\phi_i^{(k)}(t_k)\) is obtained as the identity (5) given in Wang and Chang(1999).

Since \(\sqrt{n} \left( \hat{S}_i(t_1, t_2) - S_i(t_1, t_2) \right)\) for \(i = 1, 2, 3\), converges to a normal random variable, the sum \(\sqrt{n} \left( \hat{S}(t_1, t_2) - S(t_1, t_2) \right)\) converges in distribution to normal with mean zero and variance as

\[
\sigma^2(t_1, t_2) = n E \left[ S_2(t_2) S_3(t_1, t_2) (\hat{S}_1(t_1) - S_1(t_1)) \right. \\
+ S_1(t_1) S_3(3)(t_1, t_2) (\hat{S}_2(t_2) - S_2(t_2)) \right. \\
+ S_1(t_1) S_2(t_2) (\hat{S}_3(t_1, t_2) - S_3(t_1, t_2)) \left. \right]^2.
\]

(6.35)

Remark 6.2. The results on strong consistency of the Campbell-Folder estimators given in Horvath (1983) can be used to prove the strong consistency of \(\hat{S}_e(t_1, t_2)\). The asymptotic normality of the estimator \(\hat{S}_e(t_1, t_2)\) can be established using the idea given in Dai and Bao (2009).

6.6 Data Analysis

In this section, the proposed inference procedures are applied to a recurrence data collected from 40 patients, who have undergone treatment during a one year period (Jan 1-Dec 31, 2000) for ‘pink eye’ disease on their eyes. This disease leads to redness and inflammation of the membranes covering the whites of the eyes and the membranes on the inner part of the eyelids. This disease can occur in people of any age. The leading cause of the disease is virus infection. The data is collected from a private hospital, ‘Devi Nursing Home, Tripunithura, Cochin’. The
data set includes two or more recurrence times. For the analysis, consider first two recurrence
times for the eyes. The data is given in Table 6.1. Note that the recurrence times of pair of eyes
have a common censoring time 52 weeks.

The variable $Y_{kji}$ represents $j$th observed gap time (in weeks) for $k$th eye of an $i$th individual,
$j = 1, 2, k = 1, 2$ and $i = 1, 2, \ldots, 40$. Since the censoring times of all individuals are constant,
we take $a_i = 1, i = 1, 2, \ldots, 40$ for computation of the estimate. The estimates of $S(t_1, t_2)$ for
different $(t_1, t_2)$ pair are computed using the procedure given in Section 6.3. The plot of the
estimates of $S(t_1, t_2)$ is presented in Figure 6.1. The estimates drastically decreases when $t_i$ is
larger than 7, $i = 1, 2$.

The estimates of marginal survival functions are computed and the plot of the estimates is given
in Figure 6.2. In Figure 6.2, darkened line denotes the estimate of marginal survival function
of $T_1$ and broken line represents that of $T_2$. From Figure 6.2, it follows that the probability
of survival of $T_2$ is larger than that of $T_1$ for most of the time points. The estimate $\hat{S}_{e}(t_1, t_2)$
is also computed. The plot of the estimates is given in Figure 6.3. The estimates of survival
probabilities are non increasing in $t_i, i = 1, 2$. 
Table 6.1: Data on recurrence times (in weeks) for pair of eyes

<table>
<thead>
<tr>
<th>$Y_{1i1}$</th>
<th>$Y_{12i}$</th>
<th>$Y_{21i}$</th>
<th>$Y_{22i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>19</td>
<td>33*</td>
<td>13</td>
<td>27</td>
</tr>
<tr>
<td>12</td>
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<td>16</td>
<td>33</td>
</tr>
<tr>
<td>16</td>
<td>10</td>
<td>8</td>
<td>18</td>
</tr>
<tr>
<td>18</td>
<td>34*</td>
<td>19</td>
<td>22</td>
</tr>
<tr>
<td>14</td>
<td>24</td>
<td>16</td>
<td>36*</td>
</tr>
<tr>
<td>4</td>
<td>18</td>
<td>16</td>
<td>36*</td>
</tr>
<tr>
<td>11</td>
<td>22</td>
<td>15</td>
<td>30</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>6</td>
<td>16</td>
</tr>
<tr>
<td>12</td>
<td>40*</td>
<td>16</td>
<td>36*</td>
</tr>
<tr>
<td>13</td>
<td>39*</td>
<td>9</td>
<td>26</td>
</tr>
<tr>
<td>8</td>
<td>21</td>
<td>14</td>
<td>29</td>
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<tr>
<td>7</td>
<td>16</td>
<td>2</td>
<td>15</td>
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<tr>
<td>10</td>
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<td>17</td>
<td>28</td>
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<tr>
<td>15</td>
<td>37*</td>
<td>16</td>
<td>26</td>
</tr>
<tr>
<td>5</td>
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<td>3</td>
<td>14</td>
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<td>9</td>
<td>21</td>
<td>6</td>
<td>19</td>
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<td>18</td>
<td>34*</td>
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<td>15</td>
<td>37*</td>
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<td>36*</td>
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<td>15</td>
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<tr>
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<td>18</td>
<td>34*</td>
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<td>41*</td>
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<td>12</td>
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<td>19</td>
<td>17</td>
<td>18</td>
<td>28</td>
</tr>
</tbody>
</table>

`*` denotes censored observations.
Figure 6.1: Plot of the estimates of $S(t_1, t_2)$.

Figure 6.2: Plot of the estimates of marginal survival functions.
6.7 Simulation Studies

In this section, simulation studies are conducted to evaluate the performance of the aforementioned inference procedures.

A set of values $\nu_1, \nu_2, \ldots, \nu_n$ for $n = 200$ is generated by the gamma distribution with parameters $(a, b)$ with $E(\nu) = ab$ and $V(\nu) = ab^2$. Two sets of $(a, b)$ are chosen $(a, b) = \left(\frac{2}{3}, \frac{3}{2}\right)$ and $(\frac{1}{2}, 2)$. Given $\nu_i$, the i.i.d. recurrence times are generated by a bivariate exponential distribution with independent marginal survival functions

$$S_j(t_j | \nu_i) = P(T_j > t_j | \nu_i) = e^{-\nu_i t_j}, \quad j = 1, 2, \quad i = 1, 2, \ldots n.$$  \hfill (6.36)

The true bivariate survivor function is thus obtained from (6.1) as

$$S(t_1, t_2) = (1 + bt_1 + bt_2)^{-a}.$$  \hfill (6.37)
The observation of the recurrence process is terminated by a common random censoring time \( U(0, 2) \). One thousand samples are generated for calculating the estimates of \( S(t_1, t_2) \). We compare the proposed estimate from censored data using \( a_i = D_{k_i} \) with the Dabrowska (1988) estimate (DE) from the pooled censored data \( (a_i = 1) \). The average of 1000 replicates for each estimate is computed. The average of \((m_{1i}, m_{2i})\) is \((2.51, 2.78)\). Table 6.2 provides the empirical bias, empirical standard error (ESE), bootstrap standard error (BSE) and coverage probability (CP) of the estimates. Here, the coverage probability of a confidence interval of \( \hat{S}(t_1, t_2) \) is the proportion of the time that the interval contains the true \( S(t_1, t_2) \). Note that bias from DE increases when the frailty variance increases.

<table>
<thead>
<tr>
<th>Estimator</th>
<th>((a, b))</th>
<th>((t_1, t_2))</th>
<th>Bias</th>
<th>ESE</th>
<th>BSE</th>
<th>CP</th>
</tr>
</thead>
<tbody>
<tr>
<td>GDE ((\frac{2}{3}, \frac{3}{2}))</td>
<td>((0.5, 0.8))</td>
<td>-0.0264</td>
<td>0.031</td>
<td>0.046</td>
<td>0.95</td>
<td></td>
</tr>
<tr>
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<td>((1, 1))</td>
<td>-0.0291</td>
<td>0.038</td>
<td>0.030</td>
<td>0.96</td>
<td></td>
</tr>
<tr>
<td></td>
<td>((1, 1.5))</td>
<td>0.0282</td>
<td>0.036</td>
<td>0.029</td>
<td>0.96</td>
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</tr>
<tr>
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<td>((1.5, 2))</td>
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<td>0.029</td>
<td>0.028</td>
<td>0.97</td>
<td></td>
</tr>
<tr>
<td>DE ((\frac{2}{3}, \frac{4}{2}))</td>
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<td>0.043</td>
<td>0.039</td>
<td>0.94</td>
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<td></td>
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<td>0.036</td>
<td>0.93</td>
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</tr>
<tr>
<td></td>
<td>((1, 1.5))</td>
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<td>0.038</td>
<td>0.039</td>
<td>0.94</td>
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<tr>
<td></td>
<td>((1.5, 2))</td>
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<td>0.033</td>
<td>0.036</td>
<td>0.95</td>
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<tr>
<td>GDE ((\frac{1}{2}, 2))</td>
<td>((0.5, 0.8))</td>
<td>-0.0245</td>
<td>0.044</td>
<td>0.044</td>
<td>0.95</td>
<td></td>
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<tr>
<td></td>
<td>((1, 1))</td>
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<td>0.036</td>
<td>0.035</td>
<td>0.96</td>
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<tr>
<td></td>
<td>((1, 1.5))</td>
<td>0.0262</td>
<td>0.039</td>
<td>0.038</td>
<td>0.96</td>
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<tr>
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<td>((1.5, 2))</td>
<td>0.0277</td>
<td>0.039</td>
<td>0.042</td>
<td>0.97</td>
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<tr>
<td>DE ((\frac{1}{2}, 2))</td>
<td>((0.5, 0.8))</td>
<td>-0.0434</td>
<td>0.049</td>
<td>0.051</td>
<td>0.93</td>
<td></td>
</tr>
<tr>
<td></td>
<td>((1, 1))</td>
<td>-0.0499</td>
<td>0.046</td>
<td>0.042</td>
<td>0.94</td>
<td></td>
</tr>
<tr>
<td></td>
<td>((1, 1.5))</td>
<td>0.0429</td>
<td>0.038</td>
<td>0.039</td>
<td>0.95</td>
<td></td>
</tr>
<tr>
<td></td>
<td>((1.5, 2))</td>
<td>0.0445</td>
<td>0.039</td>
<td>0.041</td>
<td>0.95</td>
<td></td>
</tr>
</tbody>
</table>

Further, to investigate the efficiency of the proposed estimate over the Dabrowska’s estimate under different censoring times \( U(0, 2), U(0, 4) \) and \( U(0, 6) \), simulated samples with the same
type of recurrence times having gamma frailty with \( a = b = 1 \) are employed. One thousand replications of the estimates for each censoring model are computed. The averages of \((m_{1i}, m_{2i})\) for the three censoring models are \((2.61, 2.88)\), \((3.11, 3.41)\) and \((4.55, 4.78)\). The empirical bias, ESE, BSE and CP of the two estimates at different time points are given in Table 6.3. The GDE is calculated using \( a_i = D_{ki} \). As expected, both empirical bias and ESE decrease when censoring time increases.

To find the optimal choice of \( a_i \), a simulation study is carried out by using the same recurrence times with gamma frailty \( a = b = 1 \). Now consider uniform \((0,4)\) and uniform \((0,8)\) as the common censoring distributions for \( D_k, k = 1, 2 \). As earlier, the empirical bias and variance of the proposed estimator are computed using one thousand replicates with \( a_i = 1 \), \( a_i = D_{ki} \) and \( a_i = D^2_{ki}, i = 1, 2, \ldots, n \). The results are given in Table 6.4. From Table 6.4, it follows that the estimate with \( a_i = D_{ki} \) outperforms the estimate with \( a_i = 1 \) and \( a_i = D^2_{ki} \) for most of the values of \((t_1, t_2)\). The study suggest that the weight \( a_i \) proportional to the observation time would be a good choice.
Table 6.3: Empirical bias, ESE, BSE and CP of the estimator of $S(t_1, t_2)$

<table>
<thead>
<tr>
<th>Estimator</th>
<th>censoring time</th>
<th>$(t_1, t_2)$</th>
<th>Bias</th>
<th>ESE</th>
<th>BSE</th>
<th>CP</th>
</tr>
</thead>
<tbody>
<tr>
<td>DE</td>
<td>$U(0, 2)$</td>
<td>(0.5, 0.8)</td>
<td>-0.0364</td>
<td>0.032</td>
<td>0.036</td>
<td>0.93</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1, 1)</td>
<td>0.0372</td>
<td>0.039</td>
<td>0.024</td>
<td>0.94</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1.1, 1.5)</td>
<td>0.0397</td>
<td>0.037</td>
<td>0.030</td>
<td>0.93</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1.5, 2)</td>
<td>-0.0305</td>
<td>0.029</td>
<td>0.029</td>
<td>0.95</td>
</tr>
<tr>
<td>DE</td>
<td>$U(0, 4)$</td>
<td>(0.5, 0.8)</td>
<td>0.0254</td>
<td>0.029</td>
<td>0.034</td>
<td>0.94</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1, 1)</td>
<td>-0.0271</td>
<td>0.021</td>
<td>0.022</td>
<td>0.93</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1.1, 1.5)</td>
<td>0.0285</td>
<td>0.020</td>
<td>0.021</td>
<td>0.94</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1.5, 2)</td>
<td>-0.0299</td>
<td>0.028</td>
<td>0.020</td>
<td>0.95</td>
</tr>
<tr>
<td>DE</td>
<td>$U(0, 6)$</td>
<td>(0.5, 0.8)</td>
<td>0.0220</td>
<td>0.026</td>
<td>0.029</td>
<td>0.93</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1, 1)</td>
<td>-0.0260</td>
<td>0.023</td>
<td>0.024</td>
<td>0.94</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1.1, 1.5)</td>
<td>-0.0270</td>
<td>0.021</td>
<td>0.020</td>
<td>0.93</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1.5, 2)</td>
<td>0.0200</td>
<td>0.020</td>
<td>0.019</td>
<td>0.93</td>
</tr>
<tr>
<td>GDE</td>
<td>$U(0, 2)$</td>
<td>(0.5, 0.8)</td>
<td>0.0100</td>
<td>0.025</td>
<td>0.015</td>
<td>0.95</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1, 1)</td>
<td>0.0190</td>
<td>0.024</td>
<td>0.014</td>
<td>0.94</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1.1, 1.5)</td>
<td>-0.0130</td>
<td>0.023</td>
<td>0.016</td>
<td>0.92</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1.5, 2)</td>
<td>0.0180</td>
<td>0.028</td>
<td>0.019</td>
<td>0.96</td>
</tr>
<tr>
<td>GDE</td>
<td>$U(0, 4)$</td>
<td>(0.5, 0.8)</td>
<td>-0.0091</td>
<td>0.019</td>
<td>0.010</td>
<td>0.95</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1, 1)</td>
<td>0.0099</td>
<td>0.018</td>
<td>0.012</td>
<td>0.95</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1.1, 1.5)</td>
<td>-0.0092</td>
<td>0.019</td>
<td>0.012</td>
<td>0.95</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1.5, 2)</td>
<td>0.0095</td>
<td>0.016</td>
<td>0.009</td>
<td>0.96</td>
</tr>
<tr>
<td>GDE</td>
<td>$U(0, 6)$</td>
<td>(0.5, 0.8)</td>
<td>0.0086</td>
<td>0.014</td>
<td>0.009</td>
<td>0.96</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1, 1)</td>
<td>-0.0087</td>
<td>0.016</td>
<td>0.010</td>
<td>0.95</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1.1, 1.5)</td>
<td>0.0080</td>
<td>0.018</td>
<td>0.011</td>
<td>0.96</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1.5, 2)</td>
<td>-0.0091</td>
<td>0.015</td>
<td>0.007</td>
<td>0.97</td>
</tr>
</tbody>
</table>
Table 6.4: Empirical bias, ESE, BSE and CP of the estimator \( \hat{S}(t_1, t_2) \)

<table>
<thead>
<tr>
<th>( a_i )</th>
<th>censoring</th>
<th>( (t_1, t_2) )</th>
<th>Bias</th>
<th>ESE</th>
<th>BSE</th>
<th>CP</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( U(0, 4) )</td>
<td>(0.5, 0.8)</td>
<td>-0.0381</td>
<td>0.0416</td>
<td>0.0421</td>
<td>0.93</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1, 1)</td>
<td>-0.0331</td>
<td>0.0426</td>
<td>0.0442</td>
<td>0.94</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1, 1.5)</td>
<td>-0.0346</td>
<td>0.0499</td>
<td>0.0423</td>
<td>0.94</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1.5, 2)</td>
<td>0.0358</td>
<td>0.0401</td>
<td>0.0434</td>
<td>0.93</td>
</tr>
<tr>
<td>( D_{ki} )</td>
<td>( U(0, 4) )</td>
<td>(0.5, 0.8)</td>
<td>-0.0261</td>
<td>0.0266</td>
<td>0.0276</td>
<td>0.96</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1, 1)</td>
<td>-0.0134</td>
<td>0.0287</td>
<td>0.0292</td>
<td>0.95</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1, 1.5)</td>
<td>-0.0187</td>
<td>0.0283</td>
<td>0.0299</td>
<td>0.96</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1.5, 2)</td>
<td>0.0199</td>
<td>0.0281</td>
<td>0.0311</td>
<td>0.96</td>
</tr>
<tr>
<td>( D_{ki}^2 )</td>
<td>( U(0, 4) )</td>
<td>(0.5, 0.8)</td>
<td>-0.0265</td>
<td>0.0400</td>
<td>0.0419</td>
<td>0.94</td>
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<tr>
<td></td>
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<td>(1, 1)</td>
<td>-0.0232</td>
<td>0.0435</td>
<td>0.0401</td>
<td>0.94</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1, 1.5)</td>
<td>-0.0261</td>
<td>0.0429</td>
<td>0.0420</td>
<td>0.95</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1.5, 2)</td>
<td>0.0211</td>
<td>0.0427</td>
<td>0.0418</td>
<td>0.94</td>
</tr>
<tr>
<td>1</td>
<td>( U(0, 8) )</td>
<td>(0.5, 0.8)</td>
<td>-0.0297</td>
<td>0.0415</td>
<td>0.0452</td>
<td>0.94</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1, 1)</td>
<td>-0.0298</td>
<td>0.0419</td>
<td>0.0448</td>
<td>0.94</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1, 1.5)</td>
<td>0.0309</td>
<td>0.0391</td>
<td>0.0421</td>
<td>0.93</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1.5, 2)</td>
<td>0.0319</td>
<td>0.0399</td>
<td>0.0388</td>
<td>0.94</td>
</tr>
<tr>
<td>( D_{ki} )</td>
<td>( U(0, 8) )</td>
<td>(0.5, 0.8)</td>
<td>-0.0161</td>
<td>0.0242</td>
<td>0.0261</td>
<td>0.95</td>
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<tr>
<td></td>
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<td>(1, 1)</td>
<td>-0.0123</td>
<td>0.0233</td>
<td>0.0257</td>
<td>0.96</td>
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<tr>
<td></td>
<td></td>
<td>(1, 1.5)</td>
<td>0.0167</td>
<td>0.0247</td>
<td>0.0249</td>
<td>0.96</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1.5, 2)</td>
<td>0.0155</td>
<td>0.0250</td>
<td>0.0276</td>
<td>0.97</td>
</tr>
<tr>
<td>( D_{ki}^2 )</td>
<td>( U(0, 8) )</td>
<td>(0.5, 0.8)</td>
<td>-0.0211</td>
<td>0.0399</td>
<td>0.0315</td>
<td>0.95</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1, 1)</td>
<td>-0.0221</td>
<td>0.0378</td>
<td>0.0325</td>
<td>0.94</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1, 1.5)</td>
<td>0.0201</td>
<td>0.0397</td>
<td>0.0318</td>
<td>0.94</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1.5, 2)</td>
<td>0.0208</td>
<td>0.0401</td>
<td>0.0338</td>
<td>0.95</td>
</tr>
</tbody>
</table>
6.8. CONCLUSION

To investigate the efficiency of the estimator \( \hat{S}(t_1, t_2) \), we generate 200 recurrence times from (6.36). The common censored times are generated from uniform \((0, b)\), where \(b\) is chosen as 4 and 8. The estimate (6.22) is calculated for various choices of \((t_1, t_2)\). One thousand replications of such estimate are computed. The empirical bias, ESE, BSE and CP of the estimates at different time points are given in Table 6.5. It may be noted that coverage probabilities of the estimator \( \hat{S}(t_1, t_2) \) is slightly larger than those of \( \hat{S}(t_1, t_2) \).

Table 6.5: Empirical bias, ESE, BSE and CP of the estimator \( \hat{S}(t_1, t_2) \)

<table>
<thead>
<tr>
<th>Censoring time</th>
<th>((t_1, t_2))</th>
<th>Bias</th>
<th>ESE</th>
<th>BSE</th>
<th>CP</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>((0.5,0.5))</td>
<td>0.022</td>
<td>0.0421</td>
<td>0.0424</td>
<td>0.94</td>
</tr>
<tr>
<td>(U(0, 4))</td>
<td>((1.0,1.0))</td>
<td>-0.028</td>
<td>0.0418</td>
<td>0.0435</td>
<td>0.95</td>
</tr>
<tr>
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<td>((1.0,1.5))</td>
<td>-0.023</td>
<td>0.0389</td>
<td>0.0465</td>
<td>0.95</td>
</tr>
<tr>
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<td>((1.5,2.0))</td>
<td>-0.027</td>
<td>0.0410</td>
<td>0.0441</td>
<td>0.94</td>
</tr>
<tr>
<td></td>
<td>((0.5,0.5))</td>
<td>0.018</td>
<td>0.0381</td>
<td>0.0401</td>
<td>0.95</td>
</tr>
<tr>
<td>(U(0, 8))</td>
<td>((1.0,1.0))</td>
<td>-0.019</td>
<td>0.0368</td>
<td>0.0411</td>
<td>0.96</td>
</tr>
<tr>
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<td>((1.0,1.5))</td>
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<td>0.0371</td>
<td>0.0405</td>
<td>0.97</td>
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<td>-0.020</td>
<td>0.0391</td>
<td>0.0395</td>
<td>0.96</td>
</tr>
</tbody>
</table>

6.8 Conclusion

In the present study, two nonparametric estimators for bivariate recurrent survivor function have been developed. The proposed estimators are the generalization of the well known Dabrowska’s estimator and Burke estimator for the bivariate survivor function. The asymptotic properties of the estimators are discussed. The optimal choice of \(a_i\) can be done using bootstrap procedure.