Chapter 2

Fuzzy Regularizable Graphs

2.1 Introduction

In this chapter, we introduce the notion of fuzzy regularizable graphs. We present a number of graphs which are fuzzy regularizable and also graphs which are not fuzzy regularizable. We show how to construct a fuzzy regularizable graph from an already existing fuzzy regularizable graph. We conclude the chapter by giving examples to show that the other way round does not work. The results of this chapter have been published in International Journal of Mathematics and Computation (IJMC), Vol. 4, No. S09, September 2009, P 105 – 110.

2.2 Fuzzy regularizable graphs

As mentioned in chapter 1, a fuzzy graph on a nonempty set $V$ is a pair $G = (\mu, \rho)$ where $\mu$ is a fuzzy subset of $V$ and $\rho$ is a fuzzy relation on $\mu$. We will further assume that $\rho$ is a symmetric function so that $\rho(u, v) = \rho(v, u)$ for all $u, v \in V$. If $u \in V$, we define the fuzzy degree of $u$ denoted by $fdeg(u)$ to be the sum of the membership grades of all edges which are incident on $u$. In other words, $fdeg(u) = \sum \rho(u, v)$ where the summation is taken over all $v \in V$ such that $v \neq u$.

Consider a crisp graph $G = (V, E)$. We say $G$ is fuzzy regularizable if it is possible to assign membership grades to the vertices and edges of $G$ so that $G$ becomes a fuzzy graph in such a way that $fdeg(u)$ is the same for all $u \in V$. In other words, the sum of the membership grades of all edges incident with every vertex is the same. $G$ is said to be complete fuzzy regularizable if it is possible to assign membership grades to the vertices and edges in such a way that $fdeg(u)$ is the same for all $u \in V$ besides satisfying the condition $\rho(u, v) = \mu(u) \wedge \mu(v)$ for all $u, v \in V$.

Proposition 2.2.1: Let $G$ be a regular graph. Then $G$ is fuzzy regularizable.

Proof: Since $G$ is a regular graph, the degree of each vertex in $G$ is the same say $r$. This means with every vertex, $r$ edges are incident. If we assign the same membership
grade $\alpha$ ($0 < \alpha < 1$) to each edge, then the sum of the membership grades of all edges incident with every vertex is $r\alpha$ proving that $G$ is fuzzy regularizable.

**Proposition 2.2.2:** The wheel $W_n$ ($n \geq 3$) is fuzzy regularizable.

**Proof:** We know the wheel $W_n$ has a cycle of length $n$. Let $v_1, v_2, \ldots, v_n$ be the vertices of this cycle. $W_n$ also has an additional vertex $v$ which is adjacent to all of $v_1, v_2, \ldots, v_n$. Thus the degree of $v$ is $n$ whereas the degree of each of the other vertices is 3. We will in fact prove that for any $r$ ($0 \leq r \leq 2n / (n - 1)$), it is possible to assign membership grades to the edges in such a way that the sum of the membership grades of all edges incident with each vertex is $r$. Assign a membership grade $r / n$ to each of the edges from $v$ to $v_i$ ($i = 1, 2, \ldots, n$). Then the sum of the membership grades of all edges incident with $v$ is $r / n + \ldots + r / n$ ($n$ times) which is nothing but $r$. Consider any other vertex $v_j$ ($1 < i < n$). Note that there are three edges which are incident with $v_j$. One is the edge between $v_i$ and $v$ which has been assigned the membership grade $r / n$. Other two edges are the edge between $v_i$ and $v_i - 1$ and the edge between $v_i$ and $v_i + 1$. (assume $v_{n+1}$ is $v_1$). Assign a membership grade of $(nr - r) / 2n$ to each of these edges. Then the sum of the membership grades of all edges incident with $v_i$ is $r / n + (nr - r) / 2n + (nr - r) / 2n$ which is nothing but $r$. This proves that $W_n$ is fuzzy regularizable.

For the wheel $W_3$, we have $r \leq 3$, for the wheel $W_4$, we have $r \leq 8 / 3$, for the wheel $W_5$, we have $r \leq 2.5$ and for the wheel $W_6$, we have $r \leq 2.4$. We see that as the value of $n$ increases, the value of $r$ decreases.

Let $G$ be a connected graph which has $n$ ($\geq 3$) vertices $v_1, v_2, \ldots, v_n$ with degree sequence $d_1, d_2, \ldots, d_n$ respectively. This means $d_i$ edges are incident with vertex $v_i$ ($i = 1, 2, \ldots, n$).

Without loss of generality, we can assume that $d_1 \geq d_2 \geq \ldots \geq d_n$. Suppose $G$ is fuzzy regularizable. We first note that $d_n \geq 2$. If $d_n = 1$, then there is only one edge incident with $v_n$. Assume that the membership grade on this edge is $\alpha$ and its other end vertex is $v_i$. Since $G$ is connected and $n \geq 3$, it follows that there should be at least one more edge which is incident with $v_i$. The sum of the membership grades of all edges incident with $v_n$ is $\alpha$ and since $G$ is fuzzy regularizable, the sum of the membership grades of all edges incident with $v_i$ (there are at least two such edges) should also be $\alpha$. 

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But the membership grade of just one edge between \( v_j \) and \( v_n \) is \( \alpha \). It is impossible and hence \( d_n \geq 2 \). We have thus proved the following.

**Theorem 2.2.1:** A connected fuzzy regularizable graph \( G \) with at least three vertices cannot have any pendant vertices (vertices of degree 1).

Since \( G \) is fuzzy regularizable, we can assign membership grades to the edges so that the sum of the membership grades of all edges incident with each vertex is the same say \( r \). We note that \( r \) must be less than or equal to the minimum degree \( d_n \) and hence \( r \) is less than or equal to every degree. Now \( d_1 \) edges are incident with the vertex \( v_1 \). Since the sum of the membership grades of all edges incident with \( v_1 \) is \( r \), it follows that there is at least one edge incident with \( v_1 \) (say the edge from \( v_1 \) to \( v_k \)) whose membership grade is less than or equal to \( r / d_1 \). There are \( d_k \) edges which are incident with the vertex \( v_k \). Let the membership grades of other edges be \( m_1, m_2, \ldots, m_{d_k - 1} \). Noting that each of these numbers is less than or equal to 1, we obtain

\[
d_k - 1 \geq m_1 + m_2 + \ldots + m_{d_k - 1} \geq r - r / d_1.
\]

Thus \( r (1 - 1 / d_1) \leq d_k - 1 \) so that \( r \leq d_1 (d_k - 1) / (d_1 - 1) \).

**Example:** For a graph, the degrees of whose vertices are 4, 2, 2, 2, 2,

we have \( r \leq 4 \times (2 - 1) / (4 - 1) = 4 / 3 \).

**2.3 Examples of graphs which are not fuzzy regularizable:**

1. Consider the graph shown below.

![Fig.2.1 Not a fuzzy regularizable graph](image)

Suppose the graph is fuzzy regularizable. Then we can assign membership grades to the edges in such a way that the sum of the membership grades of all edges incident with each vertex is the same say \( r \). Assign a membership grade \( \alpha > 0 \) to the edge \((a, b)\)
and a membership grade \( \beta > 0 \) to the edge \((a, c)\). Then the membership grade assigned to the edge \((c, d)\) is \( r - \beta \), the membership grade assigned to the edge \((b, d)\) is \( r - \alpha \) and the membership grade assigned to the edge \((a, d)\) is \( r - \alpha - \beta \). Since the sum of the membership grades of all edges incident with \(d\) is \(r\), it follows that \((r - \alpha) + (r - \beta) + (r - \alpha - \beta) = r\) from which we obtain \(r = \alpha + \beta\) so that the membership grade assigned to the edge \((a, d)\) is \( r - \alpha - \beta = 0 \), a contradiction. Hence the graph is not fuzzy regularizable.

2. Consider the graph given below.

![Fig. 2.2 Not a fuzzy regularizable graph](image)

Again, assume that the graph is fuzzy regularizable. Then we can assign membership grades to the edges in such a way that the sum of the membership grades of all edges incident with each vertex is the same say \(r\). Assign a membership grade \( \alpha > 0 \) to the edge \((a, b)\), a membership grade \( \beta > 0 \) to the edge \((a, e)\) and a membership grade \( \gamma > 0 \) to the edge \((c, g)\). Then the membership grade assigned to the edge \((b, c)\) is \( r - \alpha \), the membership grade assigned to the edge \((a, h)\) is \( r - \alpha - \beta \), the membership grade assigned to the edge \((a, h)\) is \( r - \alpha - \beta \), the membership grade assigned to the edge \((h, g)\) is \( \alpha + \beta \), the membership grade assigned to the edge \((c, g)\) is \( r - \alpha - \beta - \gamma \), the membership grade assigned to the edge \((f, e)\) is \( \alpha + \beta + \gamma \), the membership grade assigned to the edge \((e, d)\) is \( r - \alpha - 2\beta - \gamma \) and the membership grade assigned to the edge \((c, d)\) is \( \alpha + 2\beta + \gamma \). Since the sum of the membership grades of all edges incident with \(c\) is \(r\), it follows that \((r - \alpha) + \gamma + (\alpha + 2\beta + \gamma) = r\) from which we obtain \(\beta + \gamma = 0\). Since \(\beta > 0\) and \(\gamma > 0\), this contradiction proves that the graph is not fuzzy regularizable.
3. Consider the graph shown below.

![Graph Image](image)

Fig. 2.3 Not a fuzzy regularizable graph

Suppose the graph is fuzzy regularizable. Then we can assign membership grades to the edges in such a way that the sum of the membership grades of all edges incident with each vertex is the same say \( r \). Assign a membership grade \( \alpha > 0 \) to the edge \((v_1, v_2)\), a membership grade \( \beta > 0 \) to the edge \((v_3, v_9)\) a membership grade \( \gamma > 0 \) to the edge \((v_5, v_{11})\) and a membership grade \( \delta > 0 \) to the edge \((v_1, v_7)\). Then the membership grade assigned to the edge \((v_2, v_3)\) is \( r - \alpha \), the membership grade assigned to the edge \((v_3, v_4)\) is \( \alpha - \beta \), the membership grade assigned to the edge \((v_4, v_5)\) is \( r - \alpha + \beta \), the membership grade assigned to the edge \((v_5, v_6)\) is \( \alpha - \beta - \gamma \), the membership grade assigned to the edge \((v_6, v_7)\) is \( r - \alpha + \beta + \gamma \), the membership grade assigned to the edge \((v_7, v_8)\) is \( \alpha - \beta - \gamma - \delta \), the membership grade assigned to the edge \((v_8, v_9)\) is \( r - \alpha + \beta + \gamma + \delta \), the membership grade assigned to the edge \((v_9, v_{10})\) is \( \alpha - 2\beta - \gamma - \delta \), the membership grade assigned to the edge \((v_{10}, v_{11})\) is \( r - \alpha + 2\beta + \gamma + \delta \) and the membership grade assigned to the edge \((v_{11}, v_{12})\) is \( \alpha - 2\beta - 2\gamma - \delta \) and the membership grade assigned to the edge \((v_{12}, v_1)\) is \( r - \alpha + 2\beta + 2\gamma + \delta \). Since the sum of the membership grades of all edges incident with \( v_1 \) is \( r \), it follows that \( r - \alpha + 2\beta + 2\gamma + \delta + \delta + \delta + \alpha = r \) which means \( 2\beta + 2\gamma + 2\delta = 0 \) so that \( \beta + \gamma + \delta = 0 \). Since \( \beta > 0 \), \( \gamma > 0 \) and \( \delta > 0 \), this contradiction proves that the graph is not fuzzy regularizable.
**Theorem 2.3.2:** Let $G$ be a graph with $4n$ vertices. Assume that the vertices $v_1, v_3, \ldots, v_{4n-1}$ each has degree 3 and the remaining vertices $v_2, v_4, \ldots, v_{4n}$ each has degree 2. Apart from the edges connecting $v_1$ and $v_2$, $v_2$ and $v_3$, $v_{4n-1}$ and $v_{4n}$, there are edges connecting $v_{2i-1}$ and $v_{2i-1} + 2n$ ($i = 1, 2, 3, \ldots, n - 1$). Then $G$ is not fuzzy regularizable.

**Proof:** Suppose $G$ is fuzzy regularizable. Then the sum of the membership grades of all edges incident with each vertex is the same say $r$. Assume that the membership grade on the edge $(v_{2i-1}, v_{2i-1} + 2n)$ is $\beta_i > 0$ ($i = 1, 2, \ldots, n - 1$). Suppose the membership grade assigned to the edge $(v_1, v_2)$ is $\alpha > 0$. Then the membership grade assigned to the edge $(v_2, v_3)$ is $r - \alpha$, the membership grade assigned to the edge $(v_3, v_4)$ is $\alpha - \beta_2$, the membership grade assigned to the edge $(v_4, v_5)$ is $r - \alpha + \beta_2$, the membership grade assigned to the edge $(v_5, v_6)$ is $\alpha - \beta_2 - \beta_3$, the membership grade assigned to the edge $(v_6, v_7)$ is $r - \alpha + \beta_2 + \beta_3$, the membership grade assigned to the edge $(v_7, v_8)$ is $\alpha - \beta_2 - \beta_3 - \beta_4$, the membership grade assigned to the edge $(v_8, v_9)$ is $r - \alpha + \beta_2 + \beta_3 + \beta_4$ and so on. Generalizing, we obtain that the membership grade assigned to the edge $(v_{4n-1}, v_{4n})$ is $\alpha - \beta_2 - \beta_3 - \beta_4 \ldots - \beta_{2n}$ and the membership grade assigned to the edge $(v_{4n}, v_1)$ is $r - \alpha + \beta_2 + \beta_3 + \beta_4 + \ldots + \beta_{2n} + \alpha + \beta_1 = r$ from which it follows that $\beta_1 + \beta_2 + \beta_3 + \beta_4 + \ldots + \beta_{2n} = 0$, a contradiction since each $\beta_i > 0$. This contradiction proves that $G$ is not fuzzy regularizable.

**Theorem 2.3.3:** Consider a fuzzy regularizable graph $G_1$ whose vertices are $U_1, U_2, \ldots, U_s$ ($s > 1$) with degrees $u_1, u_2, \ldots, u_s$ respectively. Let $G$ be the graph obtained by introducing a new vertex $S$ and connecting $S$ with all the vertices $U_1, U_2, \ldots, U_s$. Then $G$ is fuzzy regularizable.

**Proof:** We note that the degree sequence of $G$ is $s, u_1 + 1, u_2 + 1, \ldots, u_s + 1$. Since $G_1$ is fuzzy regularizable, the sum of the membership grades of all edges incident with each vertex of $G_1$ is the same say $r$. Take $\alpha = r / (s - 1)$. Since $r > 0$ and $s > 1$, it follows that $\alpha > 0$. Clearly, $r \leq s - 1$ so that $\alpha \leq 1$. Assign the same membership grade $\alpha$ to the edge connecting $S$ with $U_i$ ($i = 1, 2, \ldots, s$). Then the sum of the membership grades of all edges incident with $S$ is $s\alpha$ (Note that $s$ edges are incident with $S$ and the membership grade of each one of them is $\alpha$). What happens to the sum of the
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membership grades of all edges incident with $U_i$? Previously, it was $r$ and now because of the new edge connecting $S$ with $U_i$ whose membership grade is $\alpha$, the sum becomes $r + \alpha = r + r / (s - 1) = r (1 +1 / (s - 1)) = r s / (s - 1) = sa$. Since it is true for every vertex $U_i$, it follows that $G$ is fuzzy regularizable.

Following examples show that removing a vertex and all edges incident with that vertex from a fuzzy regularizable graph may not yield a fuzzy regularizable graph.

1. Consider the graph $G$ given below.

![Fig. 2.4 Fuzzy regularizable graph](image)

The sum of the membership grades of all edges incident with each vertex is 1.2. Hence $G$ is fuzzy regularizable. However, the graph $G_1$ obtained from $G$ by removing $S$ and all edges incident with $S$ is not fuzzy regularizable which can be seen from the fact that the vertex $U_4$ is a pendant vertex in $G_1$. 
2. Consider the graph G given below.

![Fuzzy regularizable graph](image)

The sum of the membership grades of all edges incident with each vertex is 0.8. Hence G is fuzzy regularizable. However, the graph $G_1$ obtained from G by removing S and all edges incident with S is not fuzzy regularizable which can be easily seen.