CHAPTER - 2
FUZZY PRESENT WORTH APPROXIMATION OF CASH FLOW IN REPLACEMENT ANALYSIS

2.1 Introduction

Replacement analysis is a useful tool in decision making problem. This technique is used to determine an optimal replacement sequence. In most of the real world system, there are elements of uncertainty in process.

A firm or industry invests high amount of rupees for different machines and equipments. The invested amounts are barrowed from various banks. Day by day the interested accrued on the loan goes to high. The important task of financial experts is to estimate present worth of cash flow. It is important to decide which machine is to be replaced firstly than other. In replacement analysis use present worth of fuzzy cash flow for finding optimal replacement sequence of given machines.

Replacement analysis involves the uncertainty in cash flow. Cash flow is the future sum of invested rupees at the end of year. In cash flow the interest rates are vague in nature [29]. In real world, interest rates are varying from bank to bank and time to time.

Fuzzy set theory is a generalization of classical set theory. Since it provides us with mathematical tool for describing of vague or imprecise concept in financial investment decision, such as “approximately 1000”. For example 1000 would have membership of degree 1.0 (see fig.2.1).

If sufficient data is available probability theory is commonly used in cash flow. When the sufficient data is not available and probability information is not justified then decision maker use cash flow modeling
This theory is sufficiently applied to numerous areas, such as decision making, engineering and in investment analysis.

\[ \mu_{A}(\chi) \]

*Fig.2.1- Triangular Fuzzy number “approximately 1000”*

### 2.2 Replacement analysis

One of the most practical and topical area of engineering economics is replacement analysis. Mathematical models and analysis methods and analysis methods are used to determine the sequence of machine or asset for replacement purpose.

Replacement analysis is of great interest in uncertainty of cash flow. Cash flows are vague in nature. In this Chapter the cash flows are modeled as triangular fuzzy numbers (TFN’s). Use Kaufmann and Gupta [53] formula for fuzzy present worth of cash flow. The fuzzy numbers are based on the decision makers with his subjective judgments. Solution of this analysis is determined by various fuzzy ranking methods [5]. The ranking methods give the optimal sequence of machines for replacement.

Oakford [70] formulates the replacement problem as a network scheduling problem and solved for shortest path. In the replacement problem Savage [77] suggest that operating revenues and operating costs are increase linearly with time. The dynamic programming algorithm determines the minimum expected cost for the process [43]. The replacement decision is based on cash flows and interest rates of rupees
for given machines. The stochastic replacement model presented by Lohmann [60]. Uncertainty in replacement decision can be either deterministic or stochastic. In recent times, fuzzy set theory has surfaced in replacement decision [15], [53], [93].

2.3 Replacement model

In some replacement decision an asset or machine is required for a long period of time. In these cases the decision variables becomes the present worth approximation (PWA) of machine. The chosen replacement machine which corresponding to maximum present worth approximation [77].

The general deterministic t-time replacement cycle gives the present worth (PW) cost and cash flow at time ‘t’.

\[
R_t (i) = (I - S_t) + S_{t_i} + \sum_{i=1}^{t} R_t \left( \text{PW}_{t_i} \right)
\] (2.1)

Where;
I=Initial purchase price.
\(S_t\)=Salvage value at time ‘t’.
i = Interest rate.
\(R_t\)=Aggregate cash flow at end of period ‘t’.
\(\text{PW}_{t_i}\)=Present worth factor.

Two significant factors determining the optimal replacement cycle are aggregate cash flow at time period and interest rate. The aggregate cash flow and interest rate however are a source of considerable uncertainty and are modeled as triangular fuzzy numbers. These parameters are represented as fuzzy versions of their original counterpart by \(\tilde{R}_t\) and \(\tilde{\text{PW}}_{t_i}\) respectively.
Consider fuzzy cash flow present worth factor and interest rate, obtain the replacement cycle ‘t’ corresponding to \( F(R_t) \). The traditional model in equation (2.1) is manipulated to a proper representation such that all fuzzy numbers appear only once in the equation.

\[
F(R_t) = (1 - S_t) + S_{t_i} + \sum_{i=1}^{t} \left[ \tilde{R}_t \left( \tilde{P}_{W_{t_i}} \right) \right]
\]  

(2.2)

Where;

\( \tilde{R}_t \) = TFN representing “aggregate cash flow at end of period \( t_i \)”.  
\( \tilde{P}_{W_{t_i}} \) = TFN representing “present worth factor.

2.4 Fuzzy concept in cash flow analysis

Cash flows, the basic variable in replacement decisions, are used by managers and financial analysis. They measure the streams of money going in to particular organizations operation. Traditionally, cash flows are treated as either deterministic or stochastic. Errors in deterministic cash flow are skew the result of the analysis. Similarly, use of subjective probability distributions as the measure of uncertainty.

The fundamental types of uncertainty, non specificity, fuzziness and strife, are examined by Klir [55]. In replacement decisions various uncertainties are cash flow, purchase price, salvage value etc. are vague in nature. This is especially true when these variables are based on the natural language statement “approximately 1000”. Using fuzzy variables, we can represent this vagueness. These vague quantities will be represented using triangular fuzzy number (TFN’s).
A TFN is a fuzzy number $P = [a, b, c]$ with membership function:

$$
\mu_P(x) = \begin{cases} 
0 & \text{if } x < a \\
(x - a)/(b - a) & \text{if } a \leq x \leq b \\
(c - x)/(c - b) & \text{if } b \leq x \leq c \\
0 & \text{if } x > c 
\end{cases}
$$

### 2.5 Non-probabilistic methods in cash flow analysis

Lohmann [60] defines the triangular fuzzy numbers for solving fuzzy present worth problem, and use fuzzy numbers to develop fuzzy net future value (FNFV), also develop fuzzy equivalents to continuous interest payments, rate of interest. Buckley [15] provided an axiomatic development for financial mathematics, also he examined two classes of fuzzy quantities, compact fuzzy interval and invertible fuzzy interval.

Chiu and Park [29] use fuzzy numbers in cash flows analysis and provide a good survey of the major methods for ranking fuzzy projects. The cash flows are modeled as TFN’s and linear approximation to the product of two TFN’s is invested. The result formulation of fuzzy present worth (FPW) is:

$$
\tilde{PW} = \sum_{t=0}^{n} \left[ \frac{P_t}{\prod_{t=0}^t (1 + R_t)} \right] 
$$

Where;

$P_t =$ positive or negative TFN representing cash flow at the end of time $t$.

$n =$ number of evaluation period.

$R_t =$ non-negative TFN representing discount rate at the end of time $t$. 
2.6 Application of fuzzy operations in cash flow calculation

The general formula for calculating the present worth of cash flow (PW) is defined as;

\[ PW = \sum_{t=0}^{n} \left[ \frac{P_t}{\prod_{t=0}^{t} \left(1 + R_{t}^{l} \right)} \right] \]  

(2.4)

According to Kauffmann and Gupta [53], cash flow as well as the return rate (Interest rate) is considered as triangular fuzzy number. Hence the calculation of triangular fuzzy number that defines the present worth (PW) of the machines shall be given by;

\[ \sum_{t=0}^{n} \left[ \frac{\text{Max}\{P_{t0}, 0\}}{\prod_{t=0}^{t} \left(1 + R_{t0}^{l} \right)} + \frac{\text{Min}\{P_{t0}, 0\}}{\prod_{t=0}^{t} \left(1 + R_{t0}^{r} \right)} \right] + \sum_{t=0}^{n} \left[ \frac{\text{Max}\{P_{t1}, 0\}}{\prod_{t=0}^{t} \left(1 + R_{t1}^{l} \right)} + \frac{\text{Min}\{P_{t1}, 0\}}{\prod_{t=0}^{t} \left(1 + R_{t1}^{r} \right)} \right] \]

Where \( l(\alpha) \) and \( r(\alpha) \) are the left and right representations of TFN at each degree of membership.

Using the above equation and putting the values of \( \alpha = 0 \) and \( \alpha = 1 \) then the present worth approximation is;

\[ \text{PWA} = \sum_{t=0}^{n} \left[ \frac{\text{Max}\{P_{t0}, 0\}}{\prod_{t=0}^{t} \left(1 + R_{t0}^{l} \right)} + \frac{\text{Min}\{P_{t0}, 0\}}{\prod_{t=0}^{t} \left(1 + R_{t0}^{r} \right)} \right], \sum_{t=0}^{n} \left[ \frac{\text{Max}\{P_{t1}, 0\}}{\prod_{t=0}^{t} \left(1 + R_{t1}^{l} \right)} + \frac{\text{Min}\{P_{t1}, 0\}}{\prod_{t=0}^{t} \left(1 + R_{t1}^{r} \right)} \right] \]
This possibility of the cash flow during a certain period is positive or negative is represented in above equation. Since the interest rate is always positive, in case the cash flow is positive, we have to apply the fuzzy division that the use part $\text{Max}\{P,0\}$ and in case the flow is negative, we use other equation that consider part $\text{Min}\{P,0\}$.

### 2.7 Numerical Example

Let’s consider three machines, machines payments are barrowed from bank (for purchasing machine). The cash flow and interest rates are vague in nature therefore decision maker choose cash flow and interest rates in the form of triangular fuzzy numbers.

$P_1 = (100,110,130)$

$P_2 = (120,140,150)$

$P_3 = (110,130,140)$

and

$R_1 = (6\,\%\,\%, 7\,\%\,\%, 8\,\%\,\%)$

$R_2 = (6\,\%\,\%, 7\,\%\,\%, 9\,\%\,\%)$

$R_3 = (6\,\%\,\%, 8\,\%\,\%, 10\,\%\,\%)$

The Present Worth Approximation (PWA) factor of given three machines are;

$PWA_1 = \left(PWA_{11}, PWA_{12}, PWA_{13}\right)$

$PWA_2 = \left(PWA_{21}, PWA_{22}, PWA_{23}\right)$

$PWA_3 = \left(PWA_{31}, PWA_{32}, PWA_{33}\right)$

Where;

$PWA_{11} = \left(\frac{100}{1.07} + \frac{120}{1.07^2} + 110(1.07)(1.07)\right)$

$PWA_{11} \approx 287$
\[
PWA_{12} = (100/1.06) + [120/(1.06)(1.06)] + [110/(1.06)(1.06)(1.06)]
\]
\[
PWA_{12} = 293
\]

\[
PWA_{13} = (100/1.08) + [120/(1.08)(1.09)] + [110/(1.08)(1.09)(1.10)]
\]
\[
PWA_{13} = 310
\]

Then we get;
\[
PWA_1 = (287, 293, 310)
\]

And
\[
PWA_{21} = 321
\]
\[
PWA_{22} = 330
\]
\[
PWA_{23} = 338
\]

Then we get;
\[
PWA_2 = (321, 330, 338)
\]

Similarly
\[
PWA_{31} = 356
\]
\[
PWA_{32} = 366
\]
\[
PWA_{33} = 374
\]

Then we get;
\[
PWA_3 = (356, 366, 374)
\]

Present Worth Approximation for three machines are also in the from as TFN's they are summarized as follows

\[
PWA_1 = (287, 293, 310)
\]

\[
PWA_2 = (321, 330, 338)
\]

\[
PWA_3 = (356, 366, 374)
\]
The present worth approximation is depicted in below fig.2.2.

\[ \mu_{(P\text{WA})} \]

Fig. 2.2- Fuzzy PWA for three machines

2.8 Ranking methods for comparing given machines

One fundamental economic decision is the replacement of machines among given machines. We focus on the problem of replacement of machines under uncertainty in which the cash flow information is modeled as TFNs. Taking the result of the PWA formula and finding most dominant one.

There are a number of methods that are devised to rank TFNs. Most methods are tedious in graphic manipulation requiring complex mathematical calculation. Hence we propose methods they are:

1) Weighted Method

The weighted method compares the present worth by assigning relative weights to criteria that determine the preference of replace the given machines.

Since the present worth of a machine is described by three parameters the lowest possible present worth, most promising present worth and highest possible present worth. The comparison of machines is fundamentally depending on the basis of these parameters. Hence, as analogous to the dominance of triangular probability distribution, the average of three parameters serves as a criterion of comparison. Moreover, with a inherit nature of the largest possible estimate of present
worth, the most promising present worth is also considered as criterion. Thus, the evaluation of the present worth in the form of a TFN (a, b, c) is determined by assigning relative weights to each criterion.

\[ W = W_1 \left( a + b + c/3 \right) + W_2 b \quad (2.8) \]

Where \( W_1 \) and \( W_2 \) represent the relative weights of each criterion. By assigning \( W_1 \) equal to 1 then equation (2.8) can be reduced to;

\[ W = \left( a + b + c/3 \right) + W_2 b \quad (2.9) \]

The value of \( W \) should be determined by the nature and the magnitude of the “most promising” present worth. If the magnitude of the most present worth is important, a large number of \( W \), such as 0.3, is recommended otherwise, a smaller weight, such as 0.1, is recommended.

2) Chang’s Method

Chang [23] defines the mathematical expression of TFNs (a, b, c) as;

\[ C = (c - a)(a - b - c)/6 \quad (2.10) \]

3) Kaufmann and Gupta Method

Kaufmann and Gupta [53] suggest the three criteria for ranking TFNs with parameters (a, b, c). The dominance sequence is determined according to priority of;

1) Comparing the ordinary number.

\[ KG = \left( a + 2 \times b + c \right)/4 \quad (2.11) \]

2) Comparing the mode (the corresponding most promising value), b, of each TFN.

3) Comparing the range \( (c - a) \) of each case.

The preference of machines is determined by the amount of their ordinary numbers. The machine with the larger ordinary number is preferred. If the ordinary number are equal, the machine with the larger corresponding most promising value is preferred. If the
machine has the same ordinary number and most promising value, the
machine with the larger range is preferred.

2.9 Calculation of comparing machines for replacement

Methods of selecting machines using fuzzy ranking methods are as
follows;

2.9.1 Weighted Method

It is attributed to the main characteristics of a fuzzy number.

\[ W = W_1 (a + b + c/3) + W_2 b \]

Where \( W_1 \) and \( W_2 \) represents the relative weights of each criterion. By
assigning \( W_1 = 1 \) then

\[ W = (a + b + c/3) + W_2 b \]

Take \( W = 0.1 \)

From PWA of \( A_1, A_2 \) and \( A_3 \) the weighted values are:

\[ W(A_1) = (287 + 293 + 310/3) + 0.1(293) \]
\[ W(A_1) \square 326 \]

\[ W(A_2) = (321 + 330 + 338/3) + 0.1(330) \]
\[ W(A_2) \square 363 \]

\[ W(A_3) = (356 + 366 + 374/3) + 0.1(366) \]
\[ W(A_3) \square 402 \]

2.9.2 Chang’s Method

The range and fuzzy numbers are considered then

\[ C = (c - a)(a - b - c)/6 \]

The index values of three machines are:
C(A_1) = 3412
C(A_2) = 2802
C(A_3) = 3288

2.9.3 Kaufmann and Gupta Method

More emphasis is placed on the central value.

KG = \(a + 2 \times b + c\)/4

The index value is placed on the central value.

KG(A_1) \(\simeq\) 296
KG(A_2) \(\simeq\) 330
KG(A_3) \(\simeq\) 366

Table 2.1: Dominance sequence of three machines using different methods.

<table>
<thead>
<tr>
<th>Method</th>
<th>Index value</th>
<th></th>
<th></th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A_1</td>
<td>A_2</td>
<td>A_3</td>
<td></td>
</tr>
<tr>
<td>Weighted</td>
<td>326</td>
<td>363</td>
<td>402</td>
<td>3 &gt; 2 &gt; 1</td>
</tr>
<tr>
<td>Chang’s</td>
<td>3412</td>
<td>2802</td>
<td>3288</td>
<td>1 &gt; 3 &gt; 2</td>
</tr>
<tr>
<td>Kaufmann and Gupta</td>
<td>296</td>
<td>330</td>
<td>366</td>
<td>3 &gt; 2 &gt; 1</td>
</tr>
</tbody>
</table>

**Sequence of machine for replacement:** 3 > 2 > 1
2.10 Interpretation of result

In general, 1st machine has less approximate present worth (PWA) than that of machine 2nd and 3rd machine. Therefore replace the 3rd machine first than 2nd and finally 1st respectively i.e. Whenever Present Worth Approximation of a 3rd machine is high, then it is better to replace it. Because, the loan taken for the third machine from the bank and the interest accrued on the loan is comparatively high than that of other machines. Hence replacement of the third machine will promote the profit ratio of the firm/industry/corporation.

2.11 Concluding Remarks

In this Chapter we proposed an exact present worth formulation of cash flows that are modeled as triangular fuzzy numbers. The derivation of present worth formula involves complex arithmetic operations. With much less computational effort an approximate form of present worth is also derived.

The PWA formula of cash flow provides a general frame of fuzzy replacement analysis. Taking the result of the PW formula, the fuzzy replacement evaluation for given machines can be achieved by applying some fuzzy ranking methods on TFN’s. Several fuzzy ranking methods are selected and discussed. In numerical example, replacement of machine is evaluated and results are implemented by applying selected fuzzy ranking methods.