Chapter 6

Integrated Density of States for Decaying Random Potentials
6.1 Introduction

There is a large body of work on the integrated density of states for ergodic potentials, however the integrated density of states is not known for each non-stationary random potentials. In this chapter, we investigate some bounds for the integrated density of states in the pure point regime for the case of decaying model as define below.

The random Schrödinger operator with decaying randomness is a random Hamiltonian $H^\omega$ on $\ell^2(\mathbb{Z}^d)$ given by

$$H^\omega = -\Delta + V^\omega, \quad \omega \in \Omega.$$  \hspace{1cm} (6.1.1)

$\Delta$ is the adjacency operator defined by (2.4.2). The random potential $V^\omega$ which is the multiplication operator on $\ell^2(\mathbb{Z}^d)$ by the sequence $\{a_n q_n(\omega)\}_{n \in \mathbb{Z}^d}$ defined by

$$V^\omega = \sum_{n \in \mathbb{Z}^d} a_n q_n(\omega) |\delta_n\rangle \langle \delta_n|,$$ \hspace{1cm} (6.1.2)

where $\{\delta_n\}_{n \in \mathbb{Z}^d}$ is the standard basis for $\ell^2(\mathbb{Z}^d)$. Here $\{a_n\}_{n \in \mathbb{Z}^d}$ is a sequence of positive real numbers such that $a_n \to 0$ as $|n| \to \infty$ and $\{q_n\}_{n \in \mathbb{Z}^d}$ are real valued i.i.d. random variables with an absolutely continuous probability distribution $\mu$ which has bounded density. We realize $q_n$ as $\omega(n)$ on $(\mathbb{R}^{\mathbb{Z}^d}, B_{\mathbb{R}^{\mathbb{Z}^d}}, \mathbb{P})$, $\mathbb{P} = \bigotimes \mu$ constructed via Kolmogorov theorem. We refer to this probability space as $(\Omega, \mathcal{B}, \mathbb{P})$ henceforth. For any $B \subset \mathbb{Z}^d$ we consider the orthogonal projection $\chi_B$ onto $\ell^2(B)$ and define the matrices

$$H_B^\omega = (\langle \delta_n, H^\omega \delta_m \rangle)_{n,m \in B}, \quad G^B(z; n, m) = \langle \delta_n, (H_B^\omega - z)^{-1} \delta_m \rangle, \quad G^B(z) = (H_B^\omega - z)^{-1}.$$ \hspace{1cm} (6.1.3)

$$G(z) = (H^\omega - z)^{-1}, \quad G(z; n, m) = \langle \delta_n, (H^\omega - z)^{-1} \delta_m \rangle, z \in \mathbb{C}^+.$$

We note that $H_B^\omega$ is the matrix

$$H_B^\omega = \chi_B H^\omega \chi_B : \ell^2(B) \to \ell^2(B), \text{ a.e. } \omega.$$
We note that by assumptions on $V^\omega$, the operators $H^\omega$ are self-adjoint a.e. $\omega$ and have a common core domain consisting of vectors of finite support. Let $\Lambda_L \subset \mathbb{Z}^d$ be the d-dimensional cube centered at origin as given in (4.1.2). We then assume that,

**Hypothesis 6.1.1.** 1. The measure $\mu$ is absolute continuous with density $\rho$ that satisfies

$$
\rho(x) = \begin{cases} 
0 & \text{if } |x| < 1 \\
\frac{1}{|x|^{\delta}} & \text{if } |x| \geq 1, \text{ for some } \delta > 1.
\end{cases}
$$

(6.1.4)

2. The sequence $a_n$ satisfy $0 < a_n \simeq |n|^{-\alpha}$, $\alpha > 0$.

3. The pair $(\alpha, \delta)$ is chosen such that $d - \alpha(\delta - 1) > 0$ holds. This implies that $\beta_L \to \infty$ as $L \to \infty$, where $\beta_L$ is given by

$$
\beta_L = \sum_{n \in \Lambda_L} a_n^{(\delta - 1)}
\simeq \sum_{n \in \Lambda_L} |n|^{-\alpha(\delta - 1)} \simeq O\left((2L + 1)^{d-\alpha(\delta - 1)}\right).
$$

(6.1.5)

**Remark 6.1.2.** We have taken an explicit $\rho(x)$ in (6.1.4) in order to simplify the calculations in the proofs of our results. Our results also hold for $\rho(x) = O\left(\frac{1}{|x|^\delta}\right)$, $\delta > 1$ as $|x| \to \infty$.

In [42], Kirsch-Krishna-Obermeit consider $H^\omega = -\Delta + V^\omega$ on $\ell^2(\mathbb{Z}^d)$ with the same $V^\omega$ as defined in (6.1.2). They showed that $\sigma(H^\omega) = \mathbb{R}$ and $\sigma_c(H^\omega) \subseteq [-2d, 2d]$ a.e. $\omega$, under some conditions on $\{a_n\}_{n \in \mathbb{Z}^d}$ and $\mu$ (The density of $\mu$ should not decay too fast at infinity and $a_n$ should not decay too fast). For the precise condition on $a_n$’s and $\mu$ we recall Definition 2.1 from [42], which is given as follows.

**Definition 6.1.3.** Let $\{a_n\}$ be a bounded, positive sequence on $\mathbb{R}$. Then, $\{a_n\} - \text{supp } \mu$ is defined by

$$
\{a_n\} - \text{supp } \mu := \left\{ x \in \mathbb{R} : \sum_n \mu\left(a_n^{-1}(x - \epsilon, x + \epsilon)\right) = \infty \ \forall \ \epsilon > 0 \right\}.
$$

(6.1.6)
We call a probability measure \( \mu \) asymptotically large with respect to \( a_n \) if 
\[ \{ a_{kn} \} - \text{supp } \mu = \mathbb{R}, \text{ for all } k \in \mathbb{Z}^+. \]

To show the existence of point spectrum outside \([-2d, 2d]\) they verified Simon-Wolf criterion [56, Theorem 12.5] by showing exponential decay of the fractional moment of the Green function [42, Lemma 3.2]. The decay is valid for \(|n - m| > 2R \) with energy \( E \in \mathbb{R} \setminus [-2d, 2d] \) and is given by

\[
\mathbb{E}^\omega (|G^A (E + \iota \epsilon : n, m)|^s) \leq D_{P(n, m)} e^{-c \left( \frac{|n-m|}{2} \right)}, \quad E \in \mathbb{R} \setminus [-2d, 2d], \tag{6.1.7}
\]

where \( \epsilon > 0, \ 0 < s < 1, \ c \) is a positive constant and \( R \in \mathbb{Z}^+ \). Here, \( D_{P(n, m)} \) is a constant independent of \( E \) and \( \epsilon \), but polynomially bounded in \(|n| \) and \(|m| \).

Jaksic-Last showed in [35, Theorem 1.2] that for \( d \geq 3 \), if \( a_n \simeq |n|^{-\alpha} \quad \alpha > 1 \) then there is no singular spectrum inside \((-2d, 2d)\) of \( H^\omega \).

In the second section of this chapter we describe the the spectrum of \( H^\omega \) using [42, Theorem 2.7]. We then show that the average spacing of eigenvalues of \( H^\omega_{\Lambda_L} \) close to the energy \( E \in \mathbb{R} \setminus [-2d, 2d] \) is of order \( \beta_L^{-1} \), whereas those close to \( E \in [-2d, 2d] \) have average spacing of the order \( \frac{1}{(2L+1)d} \). This shows that the eigenvalues of \( H^\omega_{\Lambda_L} \) are more densely distributed inside \([-2d, 2d] \), the continuous spectrum of \( H^\omega \), than in the pure point regime i.e., outside \([-2d, 2d] \).

As we move to state our main results of this chapter, we define the following:

\[
N^\omega_L (E) = \# \{ j : E_j \leq E, \ E_j \in \sigma (H^\omega_{\Lambda_L}) \}, \tag{6.1.8}
\]

\[
\tilde{N}^\omega_L (E) = \# \{ j : E_j \geq E, \ E_j \in \sigma (H^\omega_{\Lambda_L}) \}, \tag{6.1.9}
\]

\[
\gamma_L (.) = \frac{1}{\beta_L} \sum_{n \in \Lambda_L} \mathbb{E}^\omega (\langle \delta_n, E_{H^\omega_{\Lambda_L}} (.) \delta_n \rangle). \tag{6.1.10}
\]

Our main results are as follows:
Theorem 6.1.4. If $E < -2d$ and $\epsilon = -2d - E > 0$ then, we have

$$\frac{1}{(\delta - 1)(4d + \epsilon)^{\delta - 1}} \leq \lim_{L \to \infty} \frac{1}{\beta L} \mathbb{E}^{\omega}(N_{L}^{\omega}(E)) \leq \lim_{L \to \infty} \frac{1}{\beta L} \mathbb{E}^{\omega}(\tilde{N}_{L}^{\omega}(E)) \leq \frac{1}{(\delta - 1)\epsilon^{\delta - 1}}.$$ 

For $E = 2d + \epsilon > 2d$ we have

$$\frac{1}{(\delta - 1)(4d + \epsilon)^{\delta - 1}} \leq \lim_{L \to \infty} \frac{1}{\beta L} \mathbb{E}^{\omega}(\tilde{N}_{L}^{\omega}(E)) \leq \lim_{L \to \infty} \frac{1}{\beta L} \mathbb{E}^{\omega}(\tilde{N}_{L}^{\omega}(E)) \leq \frac{1}{(\delta - 1)\epsilon^{\delta - 1}}.$$ 

Now we investigate the average number of eigenvalues of $H_{\Lambda}^{\omega}$ inside $[-2d, 2d]$, which can be given as follows:

**Corollary 6.1.5.** For any interval $(M_{1}, M_{2}) \supseteq [-2d, 2d]$ we have

$$\lim_{L \to \infty} \frac{1}{(2L + 1)d} \mathbb{E}^{\omega}(\# \{\sigma(H_{\Lambda}^{\omega}) \cap (M_{1}, M_{2})\}) = 1. \quad (6.1.11)$$

**Corollary 6.1.6.** If $M_{1} < -2d$ and $M_{2} > 2d$ then, we have

$$\lim_{L \to \infty} \gamma_{L}((\infty, M_{1}] \cup [M_{2}, \infty)) \leq \frac{1}{\delta - 1} \left[\frac{1}{(-2d - M_{1})^{\delta - 1}} + \frac{1}{(M_{2} - 2d)^{\delta - 1}}\right]. \quad (6.1.12)$$

For any interval $I \subseteq \mathbb{R} \setminus [-2d, 2d]$ with length $|I| > 4d$ there is a constant $C_{I} > 0$ such that

$$\lim_{L \to \infty} \gamma_{L}(I) \geq C_{I} > 0. \quad (6.1.13)$$

**Corollary 6.1.7.** Let $M_{1} < -2d$ and $M_{2} > 2d$ and $\gamma_{L} \mid_{(M_{1}, M_{2})^{c}}$ denote the restriction of $\gamma_{L}$ to $\mathbb{R} \setminus (M_{1}, M_{2})$. The sequence of measure $\{\gamma_{L} \mid_{(M_{1}, M_{2})^{c}}\}_{L}$ admits a subsequence which converges vaguely to a non-trivial measure, say $\gamma$.

The above theorem gives estimates for the average of $N_{L}^{\omega}(E)$ and $\tilde{N}_{L}^{\omega}(E)$, but we can also get a point-wise estimate of the above quantities which is given by following theorem.
Theorem 6.1.8. If \( d \geq 2 \), \( 0 < \alpha < \frac{1}{2} \) and \( 1 < \delta < \frac{1}{2\alpha} \) then for almost all \( \omega \)

\[
\frac{1}{(\delta - 1)(2d - E)^{(\delta - 1)}} \leq \lim_{L \to \infty} \frac{1}{\beta_L} N_L^\omega(E) \leq \lim_{L \to \infty} \frac{1}{\beta_L} \tilde{N}_L^\omega(E) \leq \frac{1}{(\delta - 1)(-2d - E)^{(\delta - 1)}} \text{ for } E < -2d,
\]

\[
\frac{1}{(\delta - 1)(2d + E)^{(\delta - 1)}} \leq \lim_{L \to \infty} \frac{1}{\beta_L} \tilde{N}_L^\omega(E) \leq \lim_{L \to \infty} \frac{1}{\beta_L} \tilde{N}_L^\omega(E) \leq \frac{1}{(\delta - 1)(E - 2d)^{(\delta - 1)}} \text{ for } E > 2d.
\]

In [22], Figotin-Germinet-Klein-Müller studied the Anderson Model on \( L^2(\mathbb{R}^d) \) with decaying random potentials given by

\[
H_\omega = -\Delta + \lambda \gamma_\alpha V_\omega \text{ on } L^2(\mathbb{R}^d),
\]

where \( \lambda > 0 \) is the disorder parameter and \( \gamma_\alpha \) is the envelope function

\[
\gamma_\alpha(x) := (1 + |x|^2)^{-\frac{\alpha}{2}}, \quad \alpha \geq 0.
\]

They assumed that the density of the single site distribution is in \( L^\infty(\mathbb{R}^d) \) and has compact support. They showed that if \( \alpha \in (0, 2) \) then \( H_\omega \) has infinitely many eigenvalues in \(( -\infty, 0 ) \) a.e. \( \omega \). In [22, Theorem 3], they gave the bound for \( N_\omega(E) \), \( E < 0 \) (number of eigenvalues of \( H_\omega \) below \( E \)) in terms of density of states for the stationary (i.i.d. case) Model.

In [28], Gordon-Jakšić-Molchanov-Simon studied the Model given by

\[
H_\omega = -\Delta + \sum_{n \in \mathbb{Z}^d} (1 + |n|^\alpha) q_n(\omega), \quad \alpha > 0 \text{ on } \ell^2(\mathbb{Z}^d),
\]

where \( \{q_n\} \) are i.i.d. random variables uniformly distributed on \([0, 1]\). They showed that if \( \alpha > d \) then \( H_\omega \) has discrete spectrum a.e. \( \omega \). For the case when \( \alpha \leq d \) they construct a strictly decreasing sequence \( \{a_k\}_{k \in \mathbb{N}} \) of positive numbers such that if \( \frac{d}{k} \geq \alpha > \frac{d}{k+1} \) then for a.e. \( \omega \) we have the following:

(i) \( \sigma(H_\omega) = \sigma_{pp}(H_\omega) \) and eigenfunctions of \( H_\omega \) decay exponentially,
(ii) \( \sigma_{ess}(H^\omega) = [a_k, \infty) \) and
(iii) \#\( \sigma_{disc}(H^\omega) < \infty \).

They also showed that
(a) If \( \frac{d}{k} > \alpha > \frac{d}{k+1} \) and \( E \in (a_j, a_{j-1}) \), \( 1 \leq j \leq k \), then
\[
\lim_{L \to \infty} \frac{N_L^\omega(E)}{L^{d-j\alpha}} = N_j(E)
\]
exists for a.e. \( \omega \) and is a non random function.

(b) If \( \alpha = \frac{d}{k} \) and \( E \in (a_j, a_{j-1}) \), \( 1 \leq j < k \) the above is valid. If \( E \in (a_k, a_{k-1}) \) then
\[
\lim_{L \to \infty} \frac{N_L^\omega(E)}{\ln L} = N_k(E)
\]
exists for a.e. \( \omega \) and is a non random function.

In this chapter, we essentially show that for decaying potentials the confinement length is \((2L+1)^d \) inside \([-2d, 2d] \) and \( \beta_L \) outside \([-2d, 2d] \). On the other hand, for the growing potentials (as in [28]), the confinement length is a function of energy.

### 6.2 On the pure point and continuous spectrum

In this section, we work out the spectrum of \( H^\omega \) under the Hypothesis 6.1.1. Let \( x < 0 \) and \( \epsilon > 0 \) such that \( x + \epsilon < 0 \) then, for large enough \( |n| \geq M \) we have \( a_n^{-1}(x + \epsilon) \leq -1 \) since \( a_n^{-1} \to \infty \) as \( |n| \to \infty \). Therefore, we have, for \( |n| \geq M \)
\[
\mu\left( \frac{1}{a_n(x - \epsilon, x + \epsilon)} \right) = \int_{a_n^{-1}(x-\epsilon)}^{a_n^{-1}(x+\epsilon)} \rho(t) dt
\]
\[
= a_n^{(\delta-1)} \int_{x-\epsilon}^{x+\epsilon} \frac{1}{|t|^\delta} dt.
\]

Hence,
\[
\sum_{n \in \mathbb{Z}^d} \mu\left( \frac{1}{a_n(x - \epsilon, x + \epsilon)} \right) \geq \int_{x-\epsilon}^{x+\epsilon} \frac{1}{|t|^\delta} dt \sum_{|n| \geq M} a_n^{(\delta-1)} = \infty,
\]
(6.2.1)
since $\beta_L = \sum_{n \in \Lambda_L} a_n^{(\delta-1)} \to \infty$ as $L \to \infty$ (using 6.1.5).

For $x > 0$, a similar calculation as above will give

$$\sum_{n \in \mathbb{Z}^d} \mu \left( \frac{1}{a_n} (x - \epsilon, x + \epsilon) \right) = \infty, \quad \epsilon > 0. \quad (6.2.2)$$

Now let $\epsilon > 0$. Since $a_n^{-1} \to \infty$ as $|n| \to \infty$, there exists an $M$ such that $a_n^{-1} \epsilon > 1$ for $|n| \geq M$. So, we have

$$\sum_{n \in \mathbb{Z}^d} \mu \left( \frac{1}{a_n} (-\epsilon, \epsilon) \right) \geq \sum_{|n| \geq M} \mu (-a_n^{-1} \epsilon, a_n^{-1} \epsilon)$$

$$= 2 \sum_{|n| \geq M} \int_{1}^{a_n^{-1} \epsilon} \frac{1}{t^\delta} dt$$

$$= \frac{2}{\delta - 1} \sum_{|n| \geq M} (1 - \epsilon^{1-\delta} a_n^{\delta-1}).$$

Since, $\frac{2}{\delta - 1} \sum_{n \in \Lambda_L} (1 - \epsilon^{1-\delta} a_n^{\delta-1}) \approx \frac{2}{\delta - 1} [(2L + 1)^d - (2L + 1)^{d-\alpha(\delta-1)}] \to \infty$ as $L \to \infty$, it follows that

$$\sum_{n \in \mathbb{Z}^d} \mu \left( \frac{1}{a_n} (-\epsilon, \epsilon) \right) = \infty. \quad (6.2.3)$$

If $0 < \epsilon_1 < \epsilon_2$ then, we have

$$\mu \left( a_n^{-1} (x - \epsilon_1, x + \epsilon_1) \right) \leq \mu \left( a_n^{-1} (x - \epsilon_2, x + \epsilon_2) \right) \quad \forall \ x \in \mathbb{R}.$$  

Now, using the above inequality together with (6.2.1), (6.2.2) and (6.2.3) we have, for all $\epsilon > 0$,

$$\sum_{n \in \mathbb{Z}^d} \mu \left( a_n^{-1} (x - \epsilon, x + \epsilon) \right) = \infty, \quad \forall \ x \in \mathbb{R}. \quad (6.2.4)$$

Then, using (6.2.4) from [42, Definition 2.1] we see that

$$M = \cap_{k \in \mathbb{Z}^+} (a_{kn} - \text{supp } \mu) = \mathbb{R}. $$

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Therefore, [42, Corollary 2.5] and [42, Theorem 2.3] will give the following description about the spectrum of $H^\omega$.

$$
\sigma_{\text{ess}}(H^\omega) = [-2d, 2d] + \mathbb{R} = \mathbb{R} \text{ and } \sigma_c(H^\omega) \subseteq [-2d, 2d] \text{ a.e } \omega.
$$

### 6.3 Proof of main results

**Proof of Theorem 6.1.4.**

Define

$$
A_{L,\pm}^\omega = \pm 2d + \sum_{n \in \Lambda_L} a_n q_n(\omega) P_{h_n}.
$$

and

$$
\begin{align*}
N_{\pm, L}^\omega(E) &= \#\{j : E_j \leq E, E_j \in \sigma(A_{L,\pm}^\omega)\}, \\
N_L^\omega(E) &= \#\{j : E_j \leq E, E_j \in \sigma(H_{\Lambda_L}^\omega)\}.
\end{align*}
$$

Since $\sigma(\Delta) = [-2d, 2d]$, following operator inequality

$$
A_{L,-}^\omega \leq H_{\Lambda_L}^\omega \leq A_{L,+}^\omega.
$$

(6.3.1)

is obvious, with

$$
H_{\Lambda_L}^\omega = \chi_{\Lambda_L} \Delta \chi_{\Lambda_L} + \sum_{n \in \Lambda_L} a_n q_n(\omega) P_{h_n}.
$$

A simple application of the min-max principle [36, Theorem 6.44] shows that

$$
N_{+, L}^\omega(E) \leq N_L^\omega(E) \leq N_{-, L}^\omega(E).
$$

(6.3.2)

Now, the spectrum $\sigma(A_{L,\pm}^\omega)$ of $A_{L,\pm}^\omega$ consists of only eigenvalues and is given by

$$
\sigma(A_{L,\pm}^\omega) = \{n \in \Lambda_L : \pm 2d + a_n q_n(\omega)\}.
$$
Let $E < -2d$ with $E = -2d - \epsilon$, for some $\epsilon > 0$. Then,

$$\begin{align*}
N_{\omega, L}(E) &= \# \{ n \in \Lambda_L : -2d + a_n q_n(\omega) \leq -2d - \epsilon \} \\
&= \# \{ n \in \Lambda_L : q_n(\omega) \in (-\infty, -a_n^{-1}\epsilon] \} \\
&= \sum_{n \in \Lambda_L} \chi_{\{ \omega : q_n(\omega) \in (-\infty, -a_n^{-1}\epsilon] \}},
\end{align*}$$

(6.3.3)

Since $q_n$ are i.i.d, if we take expectation of both sides of (6.3.3) we get

$$\begin{align*}
\mathbb{E}^\omega (N_{\omega, L}(E)) &= \sum_{n \in \Lambda_L} \mu(-\infty, -a_n^{-1}\epsilon] \\
&= \sum_{n \in \Lambda_L} \int_{-\infty}^{-a_n^{-1}\epsilon} \rho(x) dx.
\end{align*}$$

(6.3.4)

Since $a_n^{-1} \to \infty$ as $|n| \to \infty$ and $\epsilon > 0$, there exists an $M \in \mathbb{N}$ such that

$$a_n^{-1} \epsilon > 1, \quad -a_n^{-1} \epsilon < -1 \quad \forall |n| > M.$$

Therefore, from (6.3.3) we get, for large enough $L$,

$$\begin{align*}
\mathbb{E}^\omega (N_{\omega, L}(E)) &= \sum_{n \in \Lambda_L} \int_{-\infty}^{-a_n^{-1}\epsilon} \rho(x) dx \\
&= \sum_{n \in \Lambda_L, |n| > M} \int_{-\infty}^{-a_n^{-1}\epsilon} \rho(x) dx + \sum_{n \in \Lambda_L, |n| \leq M} \int_{-\infty}^{-1} \rho(x) dx.
\end{align*}$$

(6.3.5)

(6.3.6)

Since $\# \{ n \in \mathbb{Z}^d : |n| \leq M \} \leq (2M + 1)^d$, we have

$$\begin{align*}
\sum_{n \in \Lambda_L, |n| \leq M} \int_{-\infty}^{-1} \rho(x) dx &\leq (2M + 1)^d \int_{-\infty}^{-1} \rho(x) dx \\
&= (2M + 1)^d \int_{-\infty}^{-1} \frac{1}{|x|^\delta} dx \\
&= \frac{(2M + 1)^d}{(\delta - 1)^{\delta - 1}}, \quad \delta > 1 \text{ is given.}
\end{align*}$$

(6.3.7)
If we take $\beta_L = \sum_{n \in \Lambda_L} a_n^{(\delta-1)}$ then, $\beta_L \to \infty$ as $L \to \infty$ and we have, from (6.3.7),

$$\lim_{L \to \infty} \frac{1}{\beta_L} \sum_{n \in \Lambda_L, |n| \leq M} \int_{-\infty}^{-1} \rho(x) dx = 0. \quad (6.3.8)$$

Now,

$$\sum_{n \in \Lambda_L, |n| > M} \int_{-\infty}^{-a_n^{-1} \epsilon} \rho(x) dx = \sum_{n \in \Lambda_L, |n| > M} a_n^{-1} \int_{-\infty}^{-\epsilon} \rho(a_n^{-1} t) dt \quad (6.3.9)$$

$$= \sum_{n \in \Lambda_L, |n| > M} a_n^{(\delta-1)} \int_{-\infty}^{-\epsilon} \frac{1}{|t|^\delta} dt$$

$$= \frac{\epsilon^{1-\delta}}{\delta-1} \sum_{n \in \Lambda_L, |n| > M} a_n^{(\delta-1)}, \delta > 1.$$

This equality gives,

$$\lim_{L \to \infty} \frac{1}{\beta_L} \sum_{n \in \Lambda_L, |n| > M} \int_{-\infty}^{-a_n^{-1} \epsilon} \rho(x) dx = \frac{\epsilon^{1-\delta}}{\delta-1}. \quad (6.3.10)$$

Then, using (6.3.8) and (6.3.10) in (6.3.5), we get

$$\lim_{L \to \infty} \frac{1}{\beta_L} \mathbb{E}^\omega(N_{-L}^\omega(E)) = \frac{\epsilon^{1-\delta}}{\delta-1} = \frac{1}{(\delta-1) \epsilon^{(\delta-1)}} > 0. \quad (6.3.11)$$

A similar calculation with $\mathbb{E}^\omega(N_{+L}^\omega(E))$ gives,

$$\lim_{L \to \infty} \frac{1}{\beta_L} \mathbb{E}^\omega(N_{+L}^\omega(E)) = \frac{(4d+1)^{1-\delta}}{\delta-1} = \frac{1}{(\delta-1)(4d+1)^{\delta-1}} > 0. \quad (6.3.12)$$

Now, using (6.3.11) and (6.3.12) from (6.3.2), we conclude the inequality

$$\frac{1}{(\delta-1)(4d+1)^{\delta-1}} \leq \lim_{L \to \infty} \frac{1}{\beta_L} \mathbb{E}^\omega(N_L^\omega(E)) \leq \lim_{L \to \infty} \frac{1}{\beta_L} \mathbb{E}^\omega(N_L^\omega(E)) \leq \frac{1}{(\delta-1) \epsilon^{(\delta-1)}},$$

(6.3.13)
If we define
\[
\tilde{N}_\omega^{\pm,L}(E) = \# \{ j : E_j \geq E, \ E_j \in \sigma(A_{\omega}^{L\pm}) \}, \quad \tilde{N}_\omega^{\pm}(E) = \# \{ j : E_j \geq E, \ E_j \in \sigma(H_{\omega}^{\tilde{\gamma}}) \}
\] (6.3.14)
then the Min-max theorem and (6.3.1) together give
\[
\tilde{N}_-^{\omega,L}(E) \leq \tilde{N}_L^{\omega}(E) \leq \tilde{N}_+^{\omega,L}(E).
\] (6.3.15)

If \( E = 2d + \epsilon > 2d \), for some \( \epsilon > 0 \), a similar calculation results in
\[
\frac{1}{(\delta - 1)(4d + \epsilon)(\delta - 1)} \leq \lim_{L \to \infty} \frac{1}{\beta_L} \mathbb{E}^{\omega}(\tilde{N}_L^{\omega}(E)) \leq \lim_{L \to \infty} \frac{1}{\beta_L} \mathbb{E}^{\omega}(\tilde{N}_L^{\omega}(E)) \leq \frac{1}{(\delta - 1)\epsilon^{(\delta - 1)}}.
\] (6.3.16)
The inequalities (6.3.13) and (6.3.16) together prove the Theorem 6.1.4.

**Proof of Corollary 6.1.5:**
Since \( H_{\omega}^{\tilde{\gamma}} \) is a matrix of order \( (2L + 1)^d \), we have \( \# \sigma(H_{\omega}^{\tilde{\gamma}}) = (2L + 1)^d \). If \( M_1 < -2d \) and \( M_2 > 2d \) then,
\[
\# \left\{ \sigma(H_{\omega}^{\tilde{\gamma}}) \cap (-\infty, M_1) \right\} + \# \left\{ \sigma(H_{\omega}^{\tilde{\gamma}}) \cap (M_1, M_2) \right\} + \# \left\{ \sigma(H_{\omega}^{\tilde{\gamma}}) \cap [M_2, \infty) \right\} = (2L+1)^d.
\] (6.3.17)
Since
\[
\frac{1}{(2L+1)^d} \mathbb{E}^{\omega} \left\{ \sigma(H_{\omega}^{\tilde{\gamma}}) \cap (-\infty, M_1) \right\} = \frac{\beta_L}{(2L+1)^d} \mathbb{E}^{\omega}(N_{L}^{\omega}(M_1)),
\] (6.3.18)
and from (6.3.13) and Hypothesis 6.1.1 we have
\[
\lim_{L \to \infty} \frac{1}{\beta_L} \mathbb{E}^{\omega}(N_{L}^{\omega}(M_1)) < \infty, \quad \text{and} \quad \lim_{L \to \infty} \frac{\beta_L}{(2L+1)^d} = 0.
\]
the following limit holds
\[
\lim_{L \to \infty} \frac{1}{(2L + 1)^d} \mathbb{E}^\omega \{ \sigma(H_{A_L}^\omega) \cap (-\infty, M_1] \} = 0. \tag{6.3.19}
\]

Similarly, using (6.3.16) we get
\[
\lim_{L \to \infty} \frac{1}{(2L + 1)^d} \mathbb{E}^\omega \{ \sigma(H_{A_L}^\omega) \cap [M_2, \infty) \} = 0. \tag{6.3.20}
\]

Using the inequalities (6.3.17), (6.3.19) and (6.3.20), we see that for any interval \((M_1, M_2)\) containing \([-2d, 2d]\)
\[
\lim_{L \to \infty} \frac{1}{(2L + 1)^d} \mathbb{E}^\omega \left( \# \{ \sigma(H_{A_L}^\omega) \cap (M_1, M_2) \} \right) = 1.
\]

\[\square\]

**Corollary 6.1.6:**

If \(M_1 < -2d\) then from (6.1.10) we have
\[
\gamma_L(-\infty, M_1] = \frac{1}{\beta_L} \mathbb{E}^\omega \left( Tr \left( E_{H_{A_L}}(-\infty, M_1] \right) \right) \tag{6.3.21}
\]
\[
= \frac{1}{\beta_L} \mathbb{E}^\omega \left( N_{A_L}^\omega(M_1) \right). \text{ (using (6.1.8)).}
\]

This equality together with (6.3.13) gives
\[
\lim_{L \to \infty} \gamma_L(-\infty, M_1] \leq \frac{1}{(\delta - 1)(-2d - M_1)^{\delta - 1}} \quad \text{using } \epsilon = -2d - M_1. \tag{6.3.22}
\]

Similarly, for \(M_2 > 2d\), using (6.3.16), we get
\[
\lim_{L \to \infty} \gamma_L[M_2, \infty) \leq \frac{1}{(\delta - 1)(M_2 - 2d)^{\delta - 1}} \quad \text{using } \epsilon = M_2 - 2d. \tag{6.3.23}
\]

Now, (6.3.22) and (6.3.23) together prove (6.1.12).
Let \( J = [E_1, E_2] \subset (-\infty, -2d) \) with \(|J| > 4d\), set \( E_1 = -2d - \epsilon_1 \), \( E_2 = -2d - \epsilon_2 \) such that \( \epsilon_1 - \epsilon_2 > 4d \). Then,

\[
\gamma_L(J) = \frac{1}{\beta_L} E^\omega(N^\omega_L(E_2)) - \frac{1}{\beta_L} E^\omega(N^\omega_L(E_1))
\geq \frac{1}{\beta_L} E^\omega(N^\omega_{\gamma L}(E_2)) - \frac{1}{\beta_L} E^\omega(N^\omega_{\gamma L}(E_1)) \quad \text{(using (6.3.2)).}
\]

Therefore, (6.3.12) and (6.3.11) give (6.1.13), namely

\[
\lim_{L \to \infty} \gamma_L(J) \geq \frac{1}{(\delta - 1)} \left[ \frac{1}{(4d + \epsilon_2)(\delta - 1)} - \frac{1}{\epsilon_1(\delta - 1)} \right] > 0.
\]

Similar result holds even when \( J \subset (2d, \infty) \) with \(|J| > 4d\).

\[\square\]

**Proof of Corollary 6.1.7:**

From (6.1.12) we have

\[
\sup_L \gamma_L((-\infty, M_1] \cup [M_2, \infty)) < \infty.
\]

(6.3.25)

We write \( \mathbb{R} \setminus (M_1, M_2) = \bigcup_n A_n \), countable union of compact sets. Now, \( \gamma_L \mid_{A_n} \) (restriction of \( \gamma_L \) to \( A_n \)) admits a weakly convergence subsequence by Banach-Alaoglu Theorem. Then, by a diagonal argument we select a subsequence of \( \{\gamma_L\} \) which converges vaguely to a non-trivial measure, say \( \gamma \) on \( \mathbb{R} \setminus (M_1, M_2) \).

The non-triviality of \( \gamma \) is given by the fact that if \( J \subset \mathbb{R} \setminus (M_1, M_2) \) is an interval such that \( 4d < |J| < \infty \) then from (6.1.13) we get

\[
\inf_L \gamma_L(J) > 0.
\]

\[\square\]

Before we proceed to the proof of Theorem 6.1.8, we state the following lemma.

**Lemma 6.3.1.** Let \( \{X_n\} \) be sequence of random variables on a probability space \((\Omega, \mathcal{B}, \mathbb{P})\)
satisfying

\[ \sum_{n=1}^{\infty} P(\omega : |X_n(\omega) - X(\omega)| > \epsilon) < \infty, \quad \epsilon > 0. \]

Then \( X_n \xrightarrow{n \to \infty} X \) a.e. \( \omega \).

The proof follows from the Borel-Cantelli lemma 1.1.7.

**Proof of Theorem 6.1.8:**

Let \( E = -2d - \epsilon \) for some \( \epsilon > 0 \) and define

\[ X_n(\omega) := \chi_{\{\omega : q_n(\omega) \leq -a_n^{-1} \epsilon\}}. \] (6.3.26)

Since \( \{q_n\}_n \) are i.i.d., \( \{X_n\} \) is a sequence of independent random variables. Now, from (6.3.3) we have

\[ N_{-L}(E) = \sum_{n \in \Lambda_L} X_n(\omega). \] (6.3.27)

We want to prove the following:

\[ \lim_{L \to \infty} \frac{N_{-L}(E) - \mathbb{E}^{\omega}(N_{-L}(E))}{\beta_L} = 0 \quad a.e \ \omega. \] (6.3.28)

In view of Lemma 6.3.1, in order to prove the above equation, it is enough to show that

\[ \sum_{L=1}^{\infty} P\left( \omega : \left| \frac{N_{-L}(E) - \mathbb{E}^{\omega}(N_{-L}(E))}{\beta_L} \right| > \eta \right) < \infty \quad \forall \ \eta > 0. \] (6.3.29)

Now, using Chebyshev’s inequality we get

\[ \sum_{L=1}^{\infty} P\left( \omega : \left| \frac{N_{-L}(E) - \mathbb{E}^{\omega}(N_{-L}(E))}{\beta_L} \right| > \eta \right) \leq \sum_{L=1}^{\infty} \frac{1}{\eta^2 \beta_L^2} \mathbb{E}^{\omega} \left( N_{-L}(E) - \mathbb{E}^{\omega}(N_{-L}(E)) \right)^2. \] (6.3.30)
We proceed to estimate the RHS of the above inequality.

\[
\mathbb{E}^\omega \left( N_{-L}^\omega (E) - \mathbb{E}^\omega (N_{-L}^\omega (E)) \right)^2 = \mathbb{E}^\omega \left( \sum_{n \in \Lambda} (X_n(\omega) - \mathbb{E}^\omega (X_n(\omega)))^2 \right) \\
= \sum_{n \in \Lambda} \mathbb{E}^\omega \left( (X_n(\omega) - \mathbb{E}^\omega (X_n(\omega)))^2 \right) \quad (X_n \text{ are independent}) \\
\leq \sum_{n \in \Lambda} \mathbb{E}^\omega (X_n^2) \\
= \sum_{n \in \Lambda} \mathbb{E}^\omega (X_n) (X_n^2 = X_n) \\
= \mathbb{E}^\omega (N_{-L}^\omega (E)) \quad (\text{using (6.3.27)}).
\]

Now using the above estimate in (6.3.30) we get,

\[
\sum_{L=1}^{\infty} \mathbb{P} \left( \omega : \left| N_{-L}^\omega (E) - \mathbb{E}^\omega (N_{-L}^\omega (E)) \right| > \eta \right) \leq \frac{1}{\eta^2} \sum_{L=1}^{\infty} \frac{1}{\beta_L^2} \mathbb{E}^\omega (N_{-L}^\omega (E)) \quad (6.3.31) \\
= \frac{1}{\eta^2} \sum_{L=1}^{\infty} \frac{1}{\beta_L} \mathbb{E}^\omega (N_{-L}^\omega (E)) \\
\leq C \sum_{L=1}^{\infty} \frac{1}{\beta_L} \quad (\text{using (6.3.11)}) \\
\leq \sum_{L=1}^{\infty} \frac{1}{L^{d-\alpha(\delta-1)}} \quad (\text{using (6.1.5)}).
\]

As we have assumed in the theorem that \(0 < \alpha < \frac{1}{2}, \; 1 < \delta < \frac{1}{2\alpha} \) and \(d \geq 2\), we have \(d - \alpha(\delta - 1) > 1\). Thus, (6.3.29) follows from (6.3.31).

Therefore, from (6.3.28), for a.e. \(\omega\), we have

\[
\lim_{L \to \infty} \frac{1}{\beta_L} N_{-L}^\omega (E) = \lim_{L \to \infty} \frac{1}{\beta_L} \mathbb{E}^\omega (N_{-L}^\omega (E)) \quad (6.3.32) \\
= \frac{1}{(\delta - 1) \epsilon^{(\delta-1)}} \quad (\text{using (6.3.11)}) \\
= \frac{1}{(\delta - 1)(-2d - E)^{\delta-1}} \quad (E = -2d - \epsilon).
\]

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A similar calculation gives, for a.e. \( \omega \),

\[
\lim_{L \to \infty} \frac{1}{\beta_L} N^\omega_{\pm,L}(E) = \lim_{L \to \infty} \frac{1}{\beta_L} \mathbb{E}^\omega (N^\omega_{\pm,L}(E)) = \frac{1}{(\delta - 1)((4d + \epsilon)^{(\delta - 1)})} \quad (\text{using} \ (6.3.12))
\]

\[
= \frac{1}{(\delta - 1)(2d - E)^{(\delta - 1)}} \quad (E = -2d - \epsilon).
\]

The inequalities (6.3.32), (6.3.33) together with (6.3.2) give, for \( E < -2d \) for a.e. \( \omega \),

\[
\frac{1}{(\delta - 1)(2d - E)^{(\delta - 1)}} \leq \lim_{L \to \infty} \frac{1}{\beta_L} N^\omega_{\pm,L}(E) \leq \lim_{L \to \infty} \frac{1}{\beta_L} N^\omega_{\pm,L}(E) \leq \frac{1}{(\delta - 1)(-2d - E)^{(\delta - 1)}}.
\]

(6.3.34)

For \( E > 2d \) we compute \( \tilde{N}^\omega_{\pm,L}(E) \) (as in (6.3.14)) exactly in the same way as give above. Thus, we can prove that, for a.e. \( \omega \),

\[
\lim_{L \to \infty} \frac{1}{\beta_L} \tilde{N}^\omega_{\pm,L}(E) = \lim_{L \to \infty} \frac{1}{\beta_L} \mathbb{E}^\omega (\tilde{N}^\omega_{\pm,L}(E)) = \frac{1}{(\delta - 1)(E - 2d)^{(\delta - 1)}}
\]

and

\[
\lim_{L \to \infty} \frac{1}{\beta_L} \tilde{N}^\omega_{\pm,L}(E) = \lim_{L \to \infty} \frac{1}{\beta_L} \mathbb{E}^\omega (\tilde{N}^\omega_{\pm,L}(E)) = \frac{1}{(\delta - 1)(2d + E)^{(\delta - 1)}}.
\]

These equalities, together with (6.3.15) give the following. For \( E > 2d \), a.e. \( \omega \),

\[
\frac{1}{(\delta - 1)(2d + E)^{(\delta - 1)}} \leq \lim_{L \to \infty} \frac{1}{\beta_L} \tilde{N}^\omega_{\pm,L}(E) \leq \lim_{L \to \infty} \frac{1}{\beta_L} \tilde{N}^\omega_{\pm,L}(E) \leq \frac{1}{(\delta - 1)(E - 2d)^{(\delta - 1)}}.
\]

(6.3.35)

We conclude the chapter with the following comment:

It will be interesting to investigate the non-randomness of

\[
\frac{1}{E - 2d - \epsilon} N^L_\pm(\omega), \ E > 2d
\]
\((\text{or } \frac{1}{L(\alpha - \alpha_0 - 1)\tilde{N}_L^\alpha, \ E < -2d) \text{ as } L \to \infty.\) Once we have the existence of this limit it will be easy to investigate the eigenvalue statistics in the pure point regime for the decaying model.