CHAPTER-III

ON WEAKLY SYMMETRIC $\varepsilon$ - TRANS-SASAKIAN MANIFOLDS.

The purpose of the paper is to continue the work of S.S.Shukla and D.D.Singh on $\varepsilon$-trans-Sasakian manifold, a new creation which is introduced by [1]. In fact we studied some of the properties of weakly symmetric $\varepsilon$-trans-Sasakian manifold and generalize some of the results of A.A Shaikh and S.K Hui [6] using weakly symmetric trans-Sasakian manifold i.e. for space like manifold [2]. We also studied some of the properties of $\varepsilon$-trans-Sasakian manifold.

3.1. Introduction.

Based on the work of K.L. Duggal [1], recently S.S.Shukla and D.D.Singh [2] have introduced the notion of $\varepsilon$-Trans-Sasakian manifold which is newly created by them and they studied the basic properties of this manifold. $\varepsilon$-Trans-Sasakian manifold is in the developing stage. In this paper we also consider $\varepsilon$-Trans-Sasakian manifold of type $(\alpha, \beta)$ study the properties of weakly symmetric $\varepsilon$-Trans-Sasakian manifold and generalize some of the results of A.A.Shaikh and S.K Hui [6]. We feel that there is a lot of scope for further research in this field.

The paper has been organized as follows. In section 2 preliminary results on $\varepsilon$-Trans-Sasakian manifold have been given. Section 3 deals with the main results of the paper wherein the significance of the $\xi$-

The content of this chapter is published in International Journal of Physical Sciences Ultra Scientist Vol. 23(1) M, pp 195-208 (2011).
Sectional Curvature of $\varepsilon$-Trans-Sasakian manifold of type $(\alpha, \beta)$ is introduced and used in some of the Theorems, also study the properties of weakly symmetric $\varepsilon$-Trans-Sasakian manifold and in fact many special cases of the theorems have been studied. Further an important technique for the evaluation of the smooth functions $\alpha$ and $\beta$ in terms of codifferentials of differentiable 2-form $\Phi$ and 1-form $\eta$ are given respectively.

**3.2. Preliminaries.** A $(2n+1)$-dimensional differentiable manifold $(M, g)$ is said to be an $\varepsilon$–almost contact metric manifold [1], if it admits a $(1, 1)$ tensor field $\varphi$, a structure vector field $\xi$, a 1-form $\eta$ and an indefinite metric $g$ such that.

(3.2.1) \[ \varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \]

(3.2.2) \[ g(\xi, \xi) = \varepsilon, \quad \eta(X) = \varepsilon g(X, \xi). \]

Using the above results one has,

(3.2.2) \[ \varphi \xi = 0 \quad \text{and} \quad \eta(\varphi X) = 0, \]

for any $C^\infty$ vector field $X$,

(3.2.3) \[ g(\varphi X, \varphi Y) = g(X, Y) - \varepsilon \eta(X) \eta(Y), \]

$X, Y \in \chi(M)$, where $\chi(M)$ is the set of all $C^\infty$ vector fields on $M$.

Replacing $Y$ by $\varphi Y$ in (3.2.3) we get

\[ g(\varphi X, \varphi^2 Y) = g(X, \varphi Y) - \varepsilon \eta(X) \eta(\varphi Y), \]
\[ g(\varphi X, -Y + \eta(Y) \xi) = g(X, \varphi Y) - \varepsilon \eta(X) 0, \]
\[ -g(\varphi X, Y) + \eta(Y) g(\varphi X, \xi) = g(X, \varphi Y) \]

(3.2.3) \[ g(\varphi X, Y) = -g(X, \varphi Y). \]

This shows that $\varphi$ is skew symmetric.

$\varepsilon$ is 1 or -1 according as $\xi$ is space like or time like and rank of $\varphi$ is $2n$. If
\( (3.2.4) \) \( \Phi(X, Y) = g(X, \phi Y), \)

for all \( X, Y \in \mathfrak{X}(M) \), where \( \Phi \) is the fundamental 2-form of the structure, then \( M(\phi, \xi, \eta, g, \epsilon) \) is called an \( \epsilon \)-almost contact metric manifold.

An \( \epsilon \)-almost contact metric manifold is called an \( \epsilon \)-trans-Sasakian Manifold if,

\[
(3.2.5) \quad (\nabla_X \phi) Y = \alpha \{ g(X, Y) \xi - \epsilon \eta(Y) X \} + \beta \{ g(\phi X, Y) \xi - \epsilon \eta(Y) \phi X \},
\]

for all \( X, Y \in \mathfrak{X}(M) \), where \( \nabla \) is the Levi-Civita connection with respect to \( g \).

An \( \epsilon \)-almost contact metric manifold is called an \( \epsilon \)-trans-Sasakian Manifold if and only if [2]

\[
(3.2.6) \quad \nabla_X \xi = \epsilon \{ -\alpha \phi X + \beta (X - \eta(X) \xi) \};
\]

for all \( X, Y \in \mathfrak{X}(M) \).

This section is devoted to basic results of \( \epsilon \)-Trans-Sasakian Manifold. For an \( \epsilon \)-trans-Sasakian Manifold, the following relations hold [2]

\[
(3.2.7) \quad (\nabla_X \eta) Y = -\alpha g(\phi X, Y) + \beta \{ g(X, Y) - \epsilon \eta(X) \eta(Y) \},
\]

for all \( X, Y \in \mathfrak{X}(M) \).

\[
(3.2.8) \quad R(X, Y) \xi = (\alpha^2 - \beta^2) \{ \eta(Y) X - \eta(X) Y \}
+ 2\alpha \beta \{ \eta(Y) \phi X - \eta(X) \phi Y \}
+ \epsilon \{ (Y \alpha) \phi X - (X \alpha) \phi Y + (Y \beta) \phi^2 X - (X \beta) \phi^2 Y \},
\]

\[
(3.2.9) \quad R(\xi, Y) X = (\alpha^2 - \beta^2) \{ \epsilon g(X, Y) \xi - \eta(X) Y \}
+ 2\alpha \beta \{ \epsilon g(\phi X, Y) \xi + \eta(X) \phi Y \}
+ \epsilon (X \alpha) \phi Y + \epsilon g(\phi X, Y) \text{grad} \alpha
- \epsilon g(\phi X, \phi Y) \text{grad} \beta + \epsilon X \beta (Y - \eta(Y) \xi)
\]
(3.2.10) \[ R(\xi, Y)\xi = \{\alpha^2 - \beta^2 - \varepsilon \xi^2\} \{ - Y - \eta(Y)\xi \} - \{2\alpha \beta + \varepsilon \xi \alpha\} \phi Y , \]
(3.2.11) \[ 2\alpha \beta + \varepsilon \xi \alpha = 0 , \]
for all \( X, Y \in \chi(M) \).

In an \( \varepsilon \) - trans-Sasakian Manifold of type \( (\alpha, \beta) \) if,
\[
\phi \text{grad} \alpha = (2n - 1)\text{grad} \beta ,
\]
then
(3.2.12) \[ \xi \beta = 0 , \]
(3.2.13) \[ \eta(R(X, Y)Z) = \varepsilon (\alpha^2 - \beta^2) \{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\} + 2\varepsilon \alpha \beta \{\eta(X)g(\phi Y, Z) - \eta(Y)g(\phi X, Z)\} + \{(X\alpha)g(\phi Y, Z) - (Y\alpha)g(\phi X, Z)\} + \{X\beta g(\phi^2 Y, Z) - (Y\beta)g(\phi^2 X, Z)\}, \]
(3.2.14) \[ \eta(R(X, Y)\xi) = 0 , \]
for all \( X, Y, Z \in \chi(M) \).

In an \( (2n+1) \) - dimensional \( \varepsilon \) - trans-Sasakian Manifold, we have
(3.2.15) \[ S(X, \xi) = \{2n(\alpha^2 - \beta^2) - \varepsilon \xi^2\} \eta(X) - \varepsilon (\phi X)\alpha - \varepsilon (2n - 1)X\beta , \]
(3.2.16) \[ Q\xi = \varepsilon [\{2n(\alpha^2 - \beta^2) - \varepsilon \xi^2\} \xi + \phi \text{grad} \alpha - (2n - 1)\phi \text{grad} \beta ] , \]
for any \( X \in \chi(M) \).

3.3. Some properties of \( \varepsilon \) - Trans-Sasakian Manifold

An \( \varepsilon \) - almost contact metric manifold is called an \( \varepsilon \) -trans-Sasakian manifold if,
\[
(\nabla_X \phi)Y = \alpha \{g(X, Y)\xi - \varepsilon \eta(Y)X\} + \beta \{g(\phi X, Y)\xi - \varepsilon \eta(Y)\phi X\}
\]
for any \( X, Y \in \chi(M) \).

We here note that if \( \varepsilon = 1 \), i.e. structure vector field \( \xi \) is space like, then an \( \varepsilon \) -trans-Sasakian manifold is usual trans-Sasakian manifold [3]. If \( \Phi \) is the fundamental 2-form of the \( \varepsilon \) -almost contact metric structure defined by (2.4), then we can state,

**Theorem 3.3.1.** In an \( \varepsilon \)- Trans-Sasakian Manifold we have,
\((\nabla_X \Phi)(Y, Z) = \varepsilon \{ \alpha \{ g(X, Z)\eta(Y) - \eta(Z)g(X, Y) \} + \beta \{ g(\varphi X, Z)\eta(Y) - \eta(Z)g(\varphi X, Y) \} \}, \)

where \( \Phi \) is the fundamental 2-form of the \( \varepsilon \)-almost contact metric structure defined by (2.4)

**Proof.** Consider,

\[
(3.3.1) \quad (\nabla_X \Phi)(Y, Z)
\]

\[
= \nabla_X (\Phi(Y, Z)) - \Phi(\nabla_X Y, Z) - \Phi(Y, \nabla_X Z)
\]

\[
= \nabla_X (g(Y, \varphi Z)) - \Phi(\nabla_X Y, Z) - \Phi(Y, \nabla_X Z)
\]

\[
= (\nabla_X g)(Y, \varphi Z) + g(\nabla_X Y, \varphi Z) + g(Y, \nabla_X \varphi Z)
\]

\[
= -g(\nabla_X Y, \varphi Z) - \Phi(Y, \nabla_X Z)
\]

\[
= (\nabla_X g)(Y, \varphi Z) + g(\nabla_X Y, \varphi Z)
\]

\[
+ g(Y, (\nabla_X \varphi)Z) + g(Y, \varphi \nabla_X Z),
\]

\[
- g(\nabla_X Y, \varphi Z) - \Phi(Y, \nabla_X Z)
\]

Canceling the opposite signed terms and using the fact that \( \nabla_X g = 0 \),

\( \Phi(Y, \nabla_X Z) = g(Y, \varphi \nabla_X Z) \),

in (3.3.1), we get

\[
(3.3.2) \quad (\nabla_X \Phi)(Y, Z) = g(Y, (\nabla_X \varphi)Z)
\]

\[
= g[Y, \alpha \{ g(X, Z)\xi - \varepsilon \eta(Z)X \} + \beta \{ g(\varphi X, Z)\xi - \varepsilon \eta(Z)\varphi X \}]
\]

Simplifying (3.3.2) by using the linearity property of \( g \) and \( \varepsilon g(Y, \xi) = \eta(Y) \), we get,

\[
(3.3.3)(\nabla_X \Phi)(Y, Z) = \alpha [g(X, Z)g(Y, \xi) - \varepsilon \eta(Z)g(Y, X)]
\]

\[
+ \beta [g(\varphi X, Z)g(Y, \xi) - \varepsilon \eta(Z)g(\varphi X, Y)]
\]

\[
= \alpha [g(X, Z)\varepsilon \eta(Y) - \varepsilon \eta(Z)g(Y, X)]
\]

\[
+ \beta [g(\varphi X, Z)\varepsilon \eta(Y) - \varepsilon \eta(Z)g(\varphi X, Y)]
\]

The proof of Theorem 3.1 follows from (3.3.3).
Now by taking $Z = \xi$, $Y = X$ in (3.3.3) of theorem 3.1 and simplifying, we get

$$\nabla_X \phi(X, \xi) = \epsilon [\alpha \{g(X, \xi)\eta(X) - \eta(\xi)g(X, X)\}$$

$$+ \beta \{g(\varphi X, \xi)\eta(X) - \eta(\xi)g(\varphi X, X)\}],$$

(3.3.4) \hspace{1cm} (\nabla_X \phi)(X, \xi) = -\epsilon \alpha,$$

where we have used the fact that $X$ is orthogonal to $\xi$, $g(\varphi X, X) = 0$, and $g(X, X) = 1$. Also from (3.2.7), we have

$$(\nabla_X \eta)Y = -\alpha g(\varphi X, Y) + \beta \{g(X, Y) - \epsilon \eta(X)\eta(Y)\}.$$

Putting $Y = X$ in the above equation, we get

$$(\nabla_X \eta)X = -\alpha g(\varphi X, X) + \beta \{g(X, X) - \epsilon \eta(X)\eta(X)\}$$

Applying the similar arguments as in (3.3.4), we get

(3.3.5) \hspace{1cm} (\nabla_X \eta)X = \beta$$

Hence we can state,

**Theorem 3.3.2.** In an $\epsilon$-trans-Sasakian Manifold, the smooth functions $\alpha$, $\beta$ are given by (3.3.4) and (3.3.5) respectively.

**Remark 3.3.1.** If $\phi$ and $\eta$ are known the smooth functions $\alpha$ and $\beta$ can be evaluated.

Now taking $\{e_i; i = 1, 2, 3, \ldots, 2n + 1\}$ as the orthonormal basis at each point of the tangent space so that,

$$(\nabla_{e_i} \phi)(e_i, \xi) = \epsilon [\alpha \{g(e_i, \xi)\eta(e_i) - \eta(\xi)g(e_i, e_i)\}$$

(3.3.6) \hspace{1cm} + \beta \{g(\varphi e_i, \xi)\eta(e_i) - \eta(\xi)g(\varphi e_i, e_i)\}],$$

Simplifying (3.3.6), we get

$$(\nabla_{e_i} \phi)(e_i, \xi) = -\delta \phi(\xi) = -2n \epsilon \alpha,$$

This simplifies to.
Also from (3.2.7), we have

\[(\nabla_{\chi} \eta)Y = -\alpha g(\phi X, Y) + \beta \{g(X, Y) - \epsilon \eta(X)\eta(Y)\}\]

Putting \(Y = X = e_{i}\) in the above, we get

\[(\nabla_{e_{i}} \eta)e_{i} = -\alpha g(e_{i}, e_{i}) + \beta \{g(e_{i}, e_{i}) - \epsilon \eta(e_{i})\eta(e_{i})\}\]

\[-\delta \eta = \beta \{(2n + 1) - \epsilon \}\]

(3.3.8) \hspace{1cm} \delta \eta = \beta \{\epsilon - (2n + 1)\}

Hence we can state,

**Theorem 3.3.3.** In an \((2n+1)\) - dimensional \(\epsilon\) - Trans-Sasakian Manifold the codifferentials of the fundamental two form \(\Phi\) and 1-form \(\eta\) satisfy (3.3.7) and (3.3.8) respectively.

**Remark 3.3.2.** If \(\Phi\) and \(\eta\) are known, the smooth functions \(\alpha\) and \(\beta\), can also be evaluated by using (3.3.7) and (3.3.8) respectively.

Next consider,

\[
d\eta(X,Y) = \frac{1}{2} \{X\eta(Y) - Y\eta(X) - \eta[X,Y]\}
\]

\[= \frac{1}{2} \{\nabla_{X} \eta(Y) - \nabla_{Y} \eta(X) - \eta[\nabla_{X} Y - \nabla_{Y} X]\}\]

\[= \frac{1}{2} \{(\nabla_{X} \eta)(Y) + \eta(\nabla_{X} Y) - (\nabla_{Y} \eta)(X)\}
\]

\[-\eta(\nabla_{Y} X) - \eta(\nabla_{X} Y) + \eta(\nabla_{Y} X)\]

(3.3.9) \hspace{1cm} d\eta(X,Y) = \frac{1}{2} \{(\nabla_{X} \eta)(Y) - (\nabla_{Y} \eta)(X)\}

Now substituting for \((\nabla_{X} \eta)Y\) in (3.9), we get

\[
d\eta(X,Y) = \frac{1}{2} \{-\alpha g(\phi X, Y) + \beta \{g(X, Y) - \epsilon \eta(X)\eta(Y)\}\}
\]

\[+ \alpha g(\phi Y, X) - \beta \{g(X, Y) - \epsilon \eta(X)\eta(Y)\}\]

Simplification gives
\[
\frac{1}{2} \{ -\alpha g(\varphi X, Y) + \alpha g(\varphi Y, X) \}
\]
\[
d\eta(X, Y) = \alpha \Phi(X, Y)
\]

This can be expressed as [5],

(3.3.10) \quad d\eta = \alpha \Phi

If \( \alpha \) is a non zero constant, \( \Phi \) is closed and one has [5] (cf.Blair D.E 1976) and using (3.2.5), then we get,

\[
(3.3.11) \quad g((\nabla_X \varphi) Y, Z) = d\eta(\varphi Y, X)\eta(Z) - d\eta(\varphi Z, X)\eta(Y)
\]
\[
= \alpha \Phi(\varphi Y, X)\eta(Z) - \alpha \Phi(\varphi Z, X)\eta(Y)
\]
\[
= \alpha \{ g(X, Y)\eta(Z) - \varphi g(X, Z)\eta(Y) \}
\]
\[
= \alpha \{ g(X, Y)g(Z, \xi) - \varphi g(X, Z)g(Y, \xi) \}
\]

which implies,

(3.3.12) \quad (\nabla_X \varphi) Y = \alpha \{ g(X, Y)\xi - \varphi X \eta(Y) \}

from (3.2.5) we have

\[
(\nabla_X \varphi) Y = \alpha \{ g(X, Y)\xi - \varphi X \eta(Y) \} + \beta \{ g(\varphi X, Y)\xi - \varphi \eta(Y)\varphi X \}
\]

Substituting for this from (3.3.12), we get

\[
\alpha \{ g(X, Y)\xi - \varphi X \eta(Y) \} = \alpha \{ g(X, Y)\xi - \varphi \eta(Y)X \} + \beta \{ g(\varphi X, Y)\xi - \varphi \eta(Y)\varphi X \}
\]

which on further simplification, we get

\[
\beta \{ g(\varphi X, Y)\xi - \varphi \eta(Y)\varphi X \} = 0
\]

From which we have,

(3.3.13) \quad \beta = 0.

Thus we can state,

**Theorem 3.3.4.** An \( \varepsilon \)- Trans –Sasakian manifold of type \( (\alpha, \beta) \) with \( \alpha \) a non zero constant is \( \varepsilon - \alpha \)-Sasakian Manifold.
Note. This theorem is proved in [2], however, we used technique of [4] to prove the theorem.

\( \xi \)-Sectional Curvature. The \( \xi \)-Sectional Curvature \( K(\xi, X) \) of an \( \varepsilon \)-trans-\( \xi \)-Sasakian manifold of type \((\alpha, \beta)\) for a unit vector field \( X \) orthogonal to \( \xi \) is given by

\[
K(\xi, X) = R(\xi, X, \xi, X)
\]

From (2.10) replacing \( Y = X \) we have,

\[
\begin{align*}
R(\xi, X) \xi &= \{\alpha^2 - \beta^2 - \varepsilon \xi \beta\} \{-X - \eta(X)\xi\} - \{2\alpha \beta + \varepsilon \xi \alpha\} \phi X \\
g(R(\xi, X) \xi, X) &= \{\alpha^2 - \beta^2 - \varepsilon \xi \beta\} \{-g(X, X) - \eta(X)g(\xi, X) - \{2\alpha \beta + \varepsilon \xi \alpha\} g(\phi X, X)
\end{align*}
\]

This under the above conditions simplifies to,

\[
K(\xi, X) = R(\xi, X, \xi, X) = -\{\alpha^2 - \beta^2 - \varepsilon \xi \beta\}
\]

Which is an expression for \( \xi \)-Sectional Curvature \( K(\xi, X) \) of an \( \varepsilon \)-trans-\( \xi \)-Sasakian manifold of type \((\alpha, \beta)\).

If \( \alpha^2 - \beta^2 - \varepsilon \xi \beta \neq 0 \), then \( \varepsilon \)-Trans-\( \xi \)-Sasakian manifold is of non-zero \( \xi \)-sectional curvature. Further if \( \alpha^2 - \beta^2 - \varepsilon \xi \beta = 0 \), then \( \varepsilon \)-Trans-\( \xi \)-Sasakian manifold is of zero \( \xi \)-sectional curvature. Thus we have

**Theorem 3.3.5.** In an \( \varepsilon \)-Trans-\( \xi \)-Sasakian manifold \( M \), the \( \xi \)-Sectional Curvature is given by (3.3.15).

From (3.3.15) we have the following remarks.

**Remarks.**

i) In an \( \varepsilon \)-\( \alpha \)-Sasakian Manifold the \( \xi \)-Sectional Curvature

\[
K(\xi, X) = -\alpha^2
\]

so that for an \( \varepsilon \)-Sasakian Manifold the \( \xi \)-sectional curvature is -1.

ii) In an \( \varepsilon \)-\( \beta \)-Kenmotsu Manifold the \( \xi \)-Sectional Curvature

\[
K(\xi, X) = \beta^2
\]

so that for an \( \varepsilon \)-Kenmotsu Manifold the \( \xi \)-sectional curvature is 1.
iii) In an $(\varepsilon)$-Cosymplectic Manifold the $\xi$-sectional curvature $K(\xi, X) = 0$.

3.4. Weakly Symmetric $\varepsilon$ - Trans –Sasakian manifold.

Definition 3.4.1. A non flat Riemannian manifold $(M^n, g)$ (n>2) is called weakly symmetric if its curvature tensor $R$ of type $(0, 4)$ satisfies the condition


for all vector fields $X, Y, Z, U, V \in \chi(M)$, $A, B$ and $D$ are associated 1-forms not simultaneously zero, $\nabla$ denotes the operator of the covariant differentiation with respect to the Riemannian metric $g$, $R$ is the Riemannian curvature of the manifold $M$.

Definition 3.4.2. An $\varepsilon$-Trans -Sasakian Manifold $(M^{2n+1}, g)$ (n>1) is said to be weakly symmetric if its Riemannian curvature $R$ of type $(0, 4)$ satisfies (4.1)

Let $\{\epsilon_i : i = 1, 2, 3, ..., 2n+1\}$ be an orthonormal basis of the tangent space $T_p(M)$ at point $p$ of the manifold. Then setting $Y = V = \epsilon_i$ in (4.1) and taking the summation over $i, 1 \leq i \leq 2n+1$, we get,


Putting $X = Z = U = \xi$ in (3.4.2), we get
By using (3.2.9), (3.2.15), equation (3.4.3) reduces to
\begin{equation}
(\nabla_\xi S)(\xi, \xi) = 2n \{A(\xi) + B(\xi) + D(\xi)\}\left(\alpha^2 - \beta^2 - \varepsilon \xi \beta\right)
\end{equation}

On the other hand,
\begin{equation}
(\nabla_\xi S)(\xi, \xi) = \nabla_\xi S(\xi, \xi) - S(\nabla_\xi \xi, \xi) - S(\xi, \nabla_\xi \xi)
= \nabla_\xi \{2n(\alpha^2 - \beta^2 - \varepsilon \xi \beta)\} - 2S(\nabla_\xi \xi, \xi)
\end{equation}
Using the fact that \(\nabla_\xi \xi = 0\) in (3.4.5), we get.
\begin{equation}
(\nabla_\xi S)(\xi, \xi) = 2n\{2\alpha \xi \alpha - 2\beta \xi \beta - \varepsilon \xi (\xi \beta)\}
\end{equation}

In view of (4.4) and (4.6) it follows that,
\begin{equation}
\{A(\xi) + B(\xi) + D(\xi)\}\left(\alpha^2 - \beta^2 - \varepsilon \xi \beta\right) = \{2\alpha \xi \alpha - 2\beta \xi \beta - \varepsilon \xi (\xi \beta)\}
\end{equation}
From (3.4.7), it follows that
\begin{equation}
A(\xi) + B(\xi) + D(\xi) = \frac{2\alpha \xi \alpha - 2\beta \xi \beta - \varepsilon \xi (\xi \beta)}{\alpha^2 - \beta^2 - \varepsilon \xi \beta},
\end{equation}
provided, \(\alpha^2 - \beta^2 - \varepsilon \xi \beta \neq 0\), hence we can state,

**Theorem 3.4.1.** In an weakly symmetric \(\varepsilon\)-trans-Sasakian Manifold \((M^{2n+1}, g)\) \((n>1)\) of non vanishing \(\xi\)-sectional curvature the relation (4.8) holds.

**Corollary 3.4.1.** In an weakly symmetric \(\varepsilon\)-Trans-Sasakian Manifold \((M^{2n+1}, g)\) \((n>1)\) of non vanishing \(\xi\)-sectional curvature the relation with \(\alpha\) and \(\beta\) as non zero constants the following relation holds,
\begin{equation}
A(\xi) + B(\xi) + D(\xi) = 0.
\end{equation}
Proof. Follows by taking \( \alpha \) and \( \beta \) as constants in (4.8) of Theorem 4.1 and also using the fact that for non vanishing \( \xi \)-sectional curvature \( \alpha^2 - \beta^2 \neq 0 \).

**Corollary 3.4.2.** In an weakly symmetric \( \varepsilon \)-Trans -Sasakian Manifold \((M^{2n+1}, g) \) \((n>1)\) of non vanishing \( \xi \)-sectional curvature the relation with \( \alpha \) as non zero constant the relation (3.4.9) holds.

**Proof.** As in theorem 4.1, it can also follow that weakly symmetric \( \varepsilon \)-Trans -Sasakian Manifold \((M^{2n+1}, g) \) \((n>1)\) of non vanishing \( \xi \)-sectional curvature, with \( \alpha \) as non zero constant is always \( \varepsilon \)-\( \alpha \)-Sasakian Manifold so that \( \beta = 0 \), hence from (3.4.8), proof follows.

**Remark.** There is no Cosymplectic \( \varepsilon \)-Trans -Sasakian Manifold \((M^{2n+1}, g) \) \((n>1)\) that LHS of (3.4.8) becomes an indeterminate.

Next substituting \( X = Z = \xi \) in (4.2), we get.

\[
(V \xi S)(\xi, U) = A(\xi)S(\xi, U) + B(\xi)S(\xi, U)
+ D(U)S(\xi, \xi) + B(R(\xi, \xi)U) + D(R(\xi, U)\xi)
\]

(3.4.10) \( (V \xi S)(\xi, U) = (A(\xi) + B(\xi))[\{2n(\alpha^2 - \beta^2) - \varepsilon \xi \beta\} \eta(U)
- \varepsilon(\phi U)\alpha - \varepsilon(2n - 1)U\beta}\}

\[+ 2n(\alpha^2 - \beta^2 - \varepsilon \xi \beta)D(U)\]

\[-(\alpha^2 - \beta^2 - \varepsilon(\xi \beta))D(U)\]

\[+ (\alpha^2 - \beta^2 - \varepsilon(\xi \beta))\eta(U)D(\xi)\]

Further simplifying (3.4.10) by using (3.4.8), we get
Now consider left hand side of (3.4.11),

\[(\nabla_{\xi} S)(\xi, U) = \nabla_{\xi} S(\xi, U) - S(\nabla_{\xi} \xi, U) - S(\xi, \nabla_{\xi} U)\]

\[= \nabla_{\xi} \{2n(\alpha^2 - \beta^2) - \varepsilon \xi \beta \} \eta(U)\]

\[- \varepsilon(\varphi U)\alpha - \varepsilon(2n - 1)U\beta\]

\[= \{2n(\alpha^2 - \beta^2) - \varepsilon \xi \beta \} \eta(\nabla_{\xi} U)\]

\[- \varepsilon(\varphi \nabla_{\xi} U)\alpha - \varepsilon(2n - 1)\nabla_{\xi} U\beta\}

Further simplifying (4.12) by taking covariant derivatives with respect to the structure vector field \(\xi\) we get,

\[(\nabla_{\xi} S)(\xi, U) = \{2n(2\alpha \xi \alpha - 2\beta \xi \beta) - \varepsilon \xi(\xi \beta)\} \eta(U)\]

\[- \varepsilon \varphi U(\xi \alpha) - \varepsilon(2n - 1)U(\xi \beta)\]

In view of (3.4.11) and (3.4.13), we get,

\[\{2n(2\alpha \xi \alpha - 2\beta \xi \beta) - \varepsilon \xi(\xi \beta)\} \eta(U) - \varepsilon \varphi U(\xi \alpha) - \varepsilon(2n - 1)U(\xi \beta)\]

\[= \{2n(2\alpha \xi \alpha - 2\beta \xi \beta) - \varepsilon \xi(\xi \beta)\} \eta(U)\]

\[- \varepsilon(\varphi U)\alpha - \varepsilon(2n - 1)U\beta\} + (2n - 1)(\alpha^2 - \beta^2 - \varepsilon \xi \beta)D(U)\]

Simplifying (3.4.14) for \(D(U)\), we get

\[D(U) = \frac{2n(2\alpha \xi \alpha - 2\beta \xi \beta) - \varepsilon \xi(\xi \beta)\} \eta(U)}{(2n - 1)(\alpha^2 - \beta^2 - \varepsilon \xi \beta)}\]
\[- \varepsilon \left( \frac{(2n-1)U(\xi \beta) + \varphi U(\xi \alpha)}{(2n-1)(\alpha^2 - \beta^2 - \varepsilon \xi \beta)} \right) \]

\[+ D(\xi) \left[ \frac{(2n-1)\{\alpha^2 - \beta^2\} \eta(U) - \varepsilon U \beta} - \varepsilon (\varphi U) \alpha \right] \]

\[- \frac{2\alpha \xi \alpha - 2\beta \xi \beta - \xi(\xi \beta)}{(2n-1)(\alpha^2 - \beta^2 - \varepsilon \xi \beta)^2} \cdot 2n(\alpha^2 - \beta^2 - \varepsilon \xi \beta) \eta(U) \]

\[- \varepsilon (\varphi U) \alpha - \varepsilon (2n-1) U \beta, \]

for any vector field \( U \) provided, \( \alpha^2 - \beta^2 - \varepsilon \xi \beta \neq 0 \).

Next substituting \( X = U = \xi \) in (3.4.2) and proceeding in a similar manner as above, we get,

\[(3.4.16) \quad B(Z) = \frac{2n(2\alpha \xi \alpha - 2\beta \xi \beta - \varepsilon \xi (\xi \beta)) \eta(Z)}{(2n-1)(\alpha^2 - \beta^2 - \varepsilon \xi \beta)} \]

\[- \varepsilon \left( \frac{(2n-1)Z(\xi \beta) + \varphi Z(\xi \alpha)}{(2n-1)(\alpha^2 - \beta^2 - \varepsilon \xi \beta)} \right) \]

\[+ B(\xi) \left[ \frac{(2n-1)\{\alpha^2 - \beta^2\} \eta(Z) - \varepsilon Z \beta} - \varepsilon (\varphi Z) \alpha \right] \]

\[- \frac{2\alpha \xi \alpha - 2\beta \xi \beta - \xi(\xi \beta)}{(2n-1)(\alpha^2 - \beta^2 - \varepsilon \xi \beta)^2} \cdot 2n(\alpha^2 - \beta^2 - \varepsilon \xi \beta) \eta(Z) \]

\[- \varepsilon (\varphi Z) \alpha - \varepsilon (2n-1) Z \beta, \]

for any vector field \( Z \) provided, \( \alpha^2 - \beta^2 - \varepsilon \xi \beta \neq 0 \). Hence we can state,

**Theorem 3.4.2.** In an weakly symmetric \( \varepsilon \)-Trans-Sasakian Manifold \((M^{2n+1}, g)\) \((n > 1)\) of non vanishing \( \xi \)-sectional curvature, the associated 1-forms \( D \) and \( B \) are given by (3.4.15) and (3.4.16) respectively.

Putting \( Z = U = \xi \) in (3.4.2), we get
\[(\nabla_x S)(\xi, \xi) = A(X)S(\xi, \xi) + B(\xi)S(X, \xi) \\
+ D(\xi)S(X, \xi) + B(R(X, \xi)\xi) \\
+ D(R(X, \xi)\xi)\]

\[(\nabla_x S)(\xi, \xi) = 2n(\alpha^2 - \beta^2 - \varepsilon(\xi\beta))A(X) \\
+ [B(\xi) + D(\xi)][\{2n(\alpha^2 - \beta^2) - \varepsilon \xi\beta\}]\eta(X) \\
- \varepsilon(\phi X)\alpha - \varepsilon(2n - 1)X\beta \]

\[-[\alpha^2 - \beta^2 - \varepsilon(\xi\beta)][\eta(X)\{B(\xi) + D(\xi)\} - B(X) - D(X)]\]

(3.4.17) \(\nabla_x S)(\xi, \xi) = 2n(\alpha^2 - \beta^2 - \varepsilon(\xi\beta))A(X) \\
+ [B(\xi) + D(\xi)][\{2n(\alpha^2 - \beta^2) - \varepsilon \xi\beta\}]\eta(X) \\
- \varepsilon(\phi X)\alpha - \varepsilon(2n - 1)X\beta \]

\[-[\alpha^2 - \beta^2 - \varepsilon(\xi\beta)][\eta(X)\{B(\xi) + D(\xi)\} \\
+ \{\alpha^2 - \beta^2 - \varepsilon(\xi\beta)\} \{B(X) + D(X)\}]\]

Now consider left hand side of (4.17)

\[(\nabla_x S)(\xi, \xi) = \nabla_x S(\xi, \xi) - S(\nabla_x \xi, \xi) - S(\xi, \nabla_x \xi)\]

\[(\nabla_x S)(\xi, \xi) = \nabla_x S(\xi, \xi) - 2S(\nabla_x \xi, \xi)\]

\[= \nabla_x \{2n(\alpha^2 - \beta^2 - \varepsilon(\xi\beta))\} \]

\[- 2S(\varepsilon \{-\alpha \phi X + \beta(X - \eta(X)\xi)\}, \xi) \]

\[= 2n\{2\alpha(X\alpha) - 2\beta(X\beta) - \varepsilon X(\xi\beta))\} \]

\[+ 2\varepsilon\alpha S(\phi X, \xi) - 2\varepsilon\beta S(X, \xi) + 2\varepsilon\beta\eta(X)S(\xi, \xi)\]
\((3.4.18)\) \((\nabla_X S)(\xi, \xi) = 2n\{2\alpha(X\alpha) - 2\beta(X\beta) - \varepsilon X(\xi\beta)\} \]
\[-2\alpha\{(\varphi^2 X)\alpha + (2n - 1)\varphi X\beta\} \]
\[-2\varepsilon\beta\{2n(\alpha^2 - \beta^2) - \varepsilon \xi \beta\}\eta(X) \]
\[-\varepsilon(\varphi X)\alpha - \varepsilon(2n - 1)X\beta \]
\[+ 4n\varepsilon\beta\eta(X)(\alpha^2 - \beta^2 - \varepsilon(\xi\beta)) \]

Hence \((3.4.17)\) and \((3.4.18)\) yield,

\((3.4.19)\)
\[2n\{2\alpha(X\alpha) - 2\beta(X\beta) - \varepsilon X(\xi\beta)\} \]
\[-2\alpha\{(\varphi^2 X)\alpha + (2n - 1)\varphi X\beta\} \]
\[-2\varepsilon\beta\{2n(\alpha^2 - \beta^2) - \varepsilon \xi \beta\}\eta(X) - \varepsilon(\varphi X)\alpha - \varepsilon(2n - 1)X\beta \]
\[+ 4n\varepsilon\beta\eta(X)(\alpha^2 - \beta^2 - \varepsilon(\xi\beta)) \]
\[= 2n(\alpha^2 - \beta^2 - \varepsilon(\xi\beta))A(X) \]
\[+ \{B(\xi) + D(\xi)\}(2n - 1)(\alpha^2 - \beta^2)\eta(X) \]
\[-\varepsilon(\varphi X)\alpha - \varepsilon(2n - 1)X\beta \]
\[+ \{\alpha^2 - \beta^2 - \varepsilon(\xi\beta)\}\{B(X) + D(X)\} \]

Now taking \(U = Z = X\) in \((3.4.15)\) and \((3.4.16)\) and adding we get,

\((3.4.20)\)
\[B(X) + D(X) \]
\[= 2\left[\frac{\{2n(2\alpha \xi \alpha - 2\beta \xi \beta) - \varepsilon \xi (\xi \beta)\}\eta(X)}{(2n - 1)(\alpha^2 - \beta^2 - \varepsilon \xi \beta)}\right] \]
\[-2\varepsilon\frac{\varphi X(\xi \alpha) + (2n - 1)X(\xi \beta)}{(2n - 1)(\alpha^2 - \beta^2 - \varepsilon \xi \beta)} \]
\[
\frac{B(\xi) + D(\xi)}{(2n-1)(\alpha^2 - \beta^2 - \epsilon \xi \beta)} \left[ (2n-1)(\alpha^2 - \beta^2) \right] \eta(X)
- \epsilon(\phi X)\alpha - \epsilon(2n-1)X\beta
- \frac{2\alpha \xi \alpha - 2\beta \xi \beta - \xi(\xi \beta)}{(2n-1)(\alpha^2 - \beta^2 - \epsilon \xi \beta)^2} \left[ 2n(\alpha^2 - \beta^2) - \epsilon \xi \beta \right] \eta(X)
- \epsilon(\phi X)\alpha - \epsilon(2n-1)X\beta
\]

Separating for the following expression from (4.20) we have,

(3.4.21) \[
\{B(\xi) + D(\xi)\} \left[ (2n-1)(\alpha^2 - \beta^2) \right] \eta(X)
- \epsilon(\phi X)\alpha - \epsilon(2n-1)X\beta
= \left( (2n-1)(\alpha^2 - \beta^2 - \epsilon \xi \beta) \right) \{B(X) + D(X)\}
- 2\{2n(2\alpha \xi \alpha - 2\beta \xi \beta - \epsilon \xi \beta)\} \eta(X)
+ 2\epsilon \{\phi X(\xi \alpha) + (2n-1)X(\xi \beta)\}
+ \frac{2\alpha \xi \alpha - 2\beta \xi \beta - \xi(\xi \beta)}{(2n-1)(\alpha^2 - \beta^2 - \epsilon \xi \beta)^2} \left[ 2n(\alpha^2 - \beta^2) - \epsilon \xi \beta \right] \eta(X)
- \epsilon(\phi X)\alpha - \epsilon(2n-1)X\beta
\]

Substituting for \{B(\xi) + D(\xi)\} \left[ (2n-1)(\alpha^2 - \beta^2) \right] \eta(X)
- \epsilon(\phi X)\alpha - \epsilon(2n-1)X\beta

From (3.4.21) in (3.4.19) and further after lengthy simplification finally we get,

(3.4.22)

\[
A(X) + B(X) + D(X) = \frac{2\alpha(X\alpha) - 2\beta(X\beta) - \epsilon X(\xi \beta)}{\alpha^2 - \beta^2 - \epsilon(\xi \beta)}
+ \frac{\alpha}{n} \left[ X\alpha - \eta(X)(\xi \alpha) - (2n-1)(\phi X)\beta \right]
+ \frac{\beta}{n} \left[ (\phi X)\alpha + (2n-1)(X\beta) - (\xi \beta)\eta(X) \right]
- \frac{\epsilon}{n} \left[ \phi X(\xi \alpha) + (2n-1)X(\xi \beta) \right]
\]

65
for any vector field $X$ provided $\alpha^2 - \beta^2 - \epsilon \xi \beta \neq 0$. This leads to the following,

**Theorem 3.4.3.** In a weakly symmetric $\epsilon$-Trans-Sasakian Manifold $(M^{2n+1}, g)$ $(n>1)$ of non vanishing $\xi$-sectional curvature, the sum of the associated 1-forms $A$, $B$, and $D$ is given by (3.4.22).

In particular, if $\varphi(\text{grad} \alpha) = \text{grad} \beta$, then $\xi \beta = 0$, and hence relation (3.4.22) reduces to the following form:

(3.4.23) \quad A(X) + B(X) + D(X)

\[
= \frac{2\alpha(X\alpha) - 2\beta(X\beta)}{\alpha^2 - \beta^2} + \frac{\alpha\{X\alpha - \eta(X)(\xi\alpha) - (2n-1)(\varphi X)\beta\}}{n(\alpha^2 - \beta^2)} + \frac{\beta\{(\varphi X)\alpha + (2n-1)(X\beta)\}}{n(\alpha^2 - \beta^2)} + \frac{4n\alpha(\xi\alpha)\eta(X) - \epsilon\varphi X(\xi\alpha)}{n(\alpha^2 - \beta^2)} - \frac{2\alpha(\xi\alpha)[2n(\alpha^2 - \beta^2)]\eta(X) - \epsilon(\varphi X)\alpha - \epsilon(2n-1)X\beta}{n\{\alpha^2 - \beta^2\}^2}
\]

for any vector field $X$ provided, $(\alpha^2 - \beta^2) \neq 0$. This leads to the following,

**Corollary 3.4.3.** In a weakly symmetric non cosymplectic $\epsilon$-trans-Sasakian manifold $(M^{2n+1}, g)$ $(n>1)$ of non vanishing $\xi$-sectional curvature satisfies the condition $\varphi(\text{grad} \alpha) = \text{grad} \beta$, then the sum of the associated 1-forms satisfies (3.4.23).
If \( \alpha = 1, \beta = 0 \), then (3.4.23) yields, \( A + B + D = 0 \). Thus we have the following corollary,

**Corollary 3.4.4.** There is no weakly symmetric \( \varepsilon \)-Sasakian manifold \((M^{2n+1}, g)\) (\(n>1\)), unless the sum of 1-forms is everywhere zero.

If \( \alpha = 0, \beta = 1 \), then (4.22) yields, \( A + B + D = 0 \). Thus we state the following.

**Corollary 3.4.5.** There is no weakly symmetric \( \varepsilon \)-Kenmotsu manifold \((M^{2n+1}, g)\) (\(n>1\)), unless the sum of 1-forms is everywhere zero.

If \( \alpha = 1, \beta = 0 \), then (3.4.22) yields,

\[
(3.4.24) \quad A(X) + B(X) + D(X) = \frac{2(X\alpha)}{\alpha} - \frac{1}{n} \left[ \phi^2 X\alpha + \varepsilon \varphi X(\xi\alpha) \right] - \frac{2}{n} \left[ \frac{\xi\alpha(2n\alpha^2 \eta(X) - \varepsilon \varphi \alpha)}{\alpha^3} \right] + \left[ \frac{4(\xi\alpha)\eta(X)}{\alpha} \right]
\]

Hence we can state,

**Corollary 3.4.6.** If an \( \varepsilon \)-\( \alpha \)-Sasakian manifold \((M^{2n+1}, g)\) (\(n>1\)) is weakly symmetric then the sum of the 1-forms satisfies (4.24).

In particular if \( \alpha \) is a non zero constant, then (4.24) yields,

\[
(3.4.25) \quad A + B + D = 0
\]

Hence we state,

**Corollary 3.4.7.** If an \( \varepsilon \)-\( \alpha \)-Sasakian manifold \((M^{2n+1}, g)\) (\(n>1\)) with \( \alpha \) non zero constant is weakly symmetric then the sum of the 1-forms satisfies (3.4.25).

If \( \alpha = 0 \), then (3.4.22) yields,

\[
(3.4.26) \quad A(X) + B(X) + D(X) = \frac{2\beta(X\beta) + \varepsilon X(\xi\beta)}{\beta^2 + \varepsilon (\xi\beta)} + \frac{\varepsilon \beta}{n} \left[ -\frac{2n\beta^2 \eta(X) + (2n-1)X\beta}{\beta^2 + \varepsilon (\xi\beta)} \right]
\]

67
\[ + \frac{1}{n} \left[ \frac{4n\beta \xi \beta + \varepsilon \xi \xi \beta}{\beta^2 + \varepsilon \xi \beta} \right] \eta(X) \]

\[ - \frac{1}{n} \left[ \frac{2\beta \xi \beta + \varepsilon \xi \xi \beta \{2n\beta^2 + \varepsilon \xi \beta\} \eta(X) + \varepsilon (2n - 1)X \xi \beta}{[\beta^2 + \varepsilon \xi \beta]^2} \right] \]

Hence we can state,

**Corollary 3.4.8.** If an \( \epsilon - \beta \)-Kenmotsu manifold \((M^{2n+1}, g)\) \((n > 1)\) is weakly symmetric then the sum of the 1-forms satisfies (3.4.26).

**Remark.** It is clear that if \( \xi \)-is a space like vector field of the structure, i.e. \( \epsilon = 1 \) then some of the results of [6] A.A. Shaikh and S.K. Hui i.e. Theorems 3.1.3.2 and 3.3 [216-218, pp] and the corollaries their under shall include as special cases of respective Theorems 4.1, 4.2, 4.3 and the corollaries their under.
References:


